Exponential decay for solutions to semilinear damped wave equation

Stéphane Gerbi

LAMA, Université Savoie Mont-Blanc, Chambéry, France

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Ecole doctorale des Sciences et Techniques de l'Information, Université de Monastir

Joint work with Belkacem Said-Houari, Alhosn University, Abu Dhabi, UAE

Outline of the talk

Consider the damped wave equation:

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \mu u_t = u|u|^{p-2} & x \in \Omega, \ t > 0 \\ u(x,t) = 0, & x \in \partial\Omega, \ t > 0 \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) & x \in \Omega \quad . \end{cases} \tag{1}$$

in a bounded regular domain $\Omega \subset \mathbb{R}^N$, $p \geq 2$ and ω, μ are positive constants.
Questions to be asked :

1. Local existence,
2. Global existence,
3. Starting in the stable manifold, what is the decay rate of the solution to $u = 0$?

[GS06] *F. Gazzola, M. Squassina*. Global solutions and finite time blow up for damped semilinear wave equations. Annales de l'Institut Henri Poincaré, Analyse Non linéaire. *Vol 23, pp 185-207, 2006.*

Definition 1

For $T > 0$, we denote

$$Y_T = \left\{ \begin{array}{ll} u \in C^0\left([0,T], H_0^1(\Omega)\right) \cap & C^1\left([0,T], L^2(\Omega)\right) \cap C^2\left([0,T], H^{-1}(\Omega)\right) \\ & u_t \in L^2\left([0,T], L^2(\Omega)\right) \end{array} \right\}$$

Given $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$, a function $u \in Y_T$ is a local solution to (1), if $u(0) = u_0,\ u_t(0) = u_1$ and

$$\int_\Omega u_{tt}\phi dx + \int_\Omega \nabla u \nabla \phi dx + \omega \int_\Omega \nabla u_t \nabla \phi dx + \mu \int_\Omega u_t \phi dx = \int_\Omega |u|^{p-2} u \phi dx,$$

for any function $\phi \in H_0^1(\Omega)$ and a.e. $t \in [0, T]$.

Let us first define the Sobolev critical exponent $\bar{p}$ as:

$$\bar{p} = \begin{cases} \dfrac{2N}{N-2}, & \text{for } \omega > 0 \text{ and } N \geq 3 \\ \dfrac{2N-2}{N-2}, & \text{for } \omega = 0 \text{ and } N \geq 3 \end{cases} \quad \text{and } \bar{p} = \infty, \text{ if } N = 1, 2 \quad .$$

We first state a local existence theorem whose proof is given by Gazzola and Squassina [GS06, Theorem 3.1].

Theorem 2

Assume $2 < p \leq \bar{p}$. Let $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$. Then there exist $T > 0$ and a unique solution of (1) over $[0, T]$ in the sense of definition 1.

[GS06] *F. Gazzola, M. Squassina*. Global solutions and finite time blow up for damped semilinear wave equations. Annales de l'Institut Henri Poincaré, Analyse Non linéaire. *Vol 23, pp 185-207, 2006.*

Definition 3

Let $2 < p \leq \bar{p}$, $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$. We denote u the solution of (1). We define:

$$T_{max} = \sup\Big\{ T > 0\,,\ u = u(t) \textit{ exists on } [0, T] \Big\}$$

Since the solution $u \in Y_T$ (the solution is "enough regular"), let us recall that if $T_{max} < \infty$, then

$$\lim_{\substack{t \to T_{max} \\ t < T_{max}}} \|\nabla u\|_2 + \|u_t\|_2 = +\infty \quad .$$

If $T_{max} < \infty$, we say that the solution of (1) blows up and that T_{max} is the blow up time.
If $T_{max} = \infty$, we say that the solution of (1) is global.

We define the following functions:

$$\begin{aligned} I(t) &= I(u(t)) = \|\nabla u\|_2^2 - \|u\|_p^p, && (2) \\ J(t) &= J(u(t)) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p, && (3) \\ E(t) &= E(u(t)) = J(t) + \frac{1}{2}\|u_t\|_2^2 && (4) \end{aligned}$$

The potential well depth is defined as: $d = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \max_{\lambda \geq 0} J(\lambda u)$.

"Nehari manifold": $\mathcal{N} = \left\{u \in H_0^1(\Omega)\setminus\{0\};\ I(t) = 0\right\}$.

$$\mathcal{N}^+ = \left\{u \in H_0^1(\Omega);\ I(t) > 0\right\} \cup \{0\} \text{ and } \mathcal{N}^- = \left\{u \in H_0^1(\Omega); I(t) < 0\right\}.$$

The *stable* set $\mathcal{W}$ and *unstable* set $\mathcal{U}$ are defined respectively as:

$$\mathcal{W} = \left\{u \in H_0^1(\Omega); J(t) \leq d\right\} \cap \mathcal{N}^+ \text{ and } \mathcal{U} = \left\{u \in H_0^1(\Omega); J(t) \leq d\right\} \cap \mathcal{N}^-.$$

We have :

$$d = \min_{u \in \mathcal{N}} J(u).$$

As it was remarked by Gazzola and Squassina, this alternative characterization of d shows that

$$\beta = \text{dist}(0, \mathcal{N}) = \min_{u \in \mathcal{N}} \|\nabla u\|_2 = \sqrt{\frac{2dp}{p-2}} > 0 \quad .$$

We denote by C_* the best constant in the Poincaré-Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ defined by:

$$C_*^{-1} = \inf \left\{ \|\nabla u\|_2 : u \in H_0^1(\Omega), \|u\|_p = 1 \right\} \quad .$$

For $p < \bar{p}$, the embeddig is compact and C_* is related to the potential depth by:

$$d = \frac{p-2}{2p} \, C_*^{-2p/(p-2)} \quad .$$

Theorem 4

Assume $2 < p \leq \bar{p}$*. Let* $u_0 \in \mathcal{N}^+$ *and* $u_1 \in L^2(\Omega)$*. Moreover, assume that* $E(0) < d$*. Then :*

- $T_{max} = \infty$
- *there exist two positive constants* $\widehat{C}$ *and* ξ *independent of* t *such that:*

$$0 < \|\nabla u(t)\|_2^2 + \|u_t\|_2^2 \leq \widehat{C} e^{-\xi t}, \; \forall\, t \geq 0.$$

Under the same conditions, Gazzola and Squassina obtained :

- $T_{max} = \infty$
- there exists $C > 0$ such that:

$$0 < \|\nabla u(t)\|_2^2 + \|u_t\|_2^2 \leq \frac{C}{t}, \; \forall\, t > 0.$$

Remark 1

- *Multiplying (1) by u_t, integrating over Ω and using integration by parts we obtain:*
$$\frac{dE(t)}{dt} = -\omega\|\nabla u_t\|_2^2 - \mu\|u_t\|_2^2 \text{ , } \forall t \geq 0.$$
Thus the function E is decreasing along the trajectories.
- *If there exists $t_0 \in [0, T)$ such that*
$$E(t_0) < d$$
the same result stays true. It is the reason why we choose $t_0 = 0$.
- *The condition $E(0) < d$ is equivalent to the inequality:*
$$C_*^p \left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}} < 1$$

Lemma 5

Assume $2 \leq p \leq \bar{p}$. Let $u_0 \in \mathcal{N}^+$ and $u_1 \in L^2(\Omega)$. Moreover, assume that $E(0) < d$. Then $u(t,.) \in \mathcal{N}^+$ for each $t \in [0,T)$.

proof: After some calculations, and using the remarks, we easily show that under the condition : $E(0) < d$, we obtain :

$$\|u(t)\|_p^p < \|\nabla u(t)\|_2^2 \quad \forall t \in [0,T).$$

Hence $\|\nabla u\|_2^2 - \|u\|_p^p > 0$, $\forall t \in [0,T)$. This shows that

$$u(t,.) \in \mathcal{N}^+, \; \forall t \in [0,T) \quad .$$

Lemma 6

Assume $2 < p \leq \bar{p}$. Let $u_0 \in \mathcal{N}^+$ and $u_1 \in L^2(\Omega)$. Moreover, assume that $E(0) < d$. Then the solution of the problem (1) is global in time.

For $\varepsilon > 0$, to be chosen later, we define

$$L(t) = E(t) + \varepsilon \int_\Omega u_t u dx + \frac{\varepsilon\omega}{2} \|\nabla u\|_2^2 \quad . \tag{5}$$

It is straightforward to see that $L(t)$ and $E(t)$ are equivalent in the sense that there exist two positive constants β_1 and $\beta_2 > 0$ depending on ε such that for $t \geq 0$

$$\beta_1 E(t) \leq L(t) \leq \beta_2 E(t). \tag{6}$$

By taking the time derivative of the function L defined above in equation (5), using problem (1), and performing several integration by parts, we get:

$$\begin{aligned} \frac{dL(t)}{dt} &= -\omega\|\nabla u_t\|_2^2 - \mu\|u_t\|_2^2 + \varepsilon\|u_t\|_2^2 - \varepsilon\|\nabla u\|_2^2 \\ &\quad +\varepsilon\|u\|_p^p - \varepsilon\mu \int_\Omega u_t\, u dx \quad . \end{aligned} \tag{7}$$

By using Young's inequality, we obtain, for any $\delta > 0$

$$\int_\Omega u_t u dx \leq \frac{1}{4\delta}\|u_t\|_2^2 + \delta\|u\|_2^2 \quad . \tag{8}$$

Thus we first have (since $\|u(t)\|_p^p < \|\nabla u(t)\|_2^2 \quad \forall t > 0$):

$$\begin{aligned} \frac{dL(t)}{dt} \leq\ & -\omega\|\nabla u_t\|_2^2 + \left(\varepsilon\left(\frac{\mu}{4\delta}+1\right)-\mu\right)\|u_t\|_2^2 \\ & +\varepsilon\left(\mu C_*^2\delta + \underbrace{C_*^p\left(\frac{2p}{(p-2)}E(0)\right)^{\frac{p-2}{2}} - 1}_{<0}\right)\|\nabla u\|_2^2 \quad . \end{aligned} \tag{9}$$

We may find $\eta > 0$, which depends only on δ, such that:

$$\frac{dL(t)}{dt} \leq -\omega\|\nabla u_t\|_2^2 + \left(\varepsilon\left(\frac{\mu}{4\delta}+1\right)-\mu\right)\|u_t\|_2^2 - \varepsilon\eta\|\nabla u\|_2^2$$

Consequently, using the definition of the energy (4), for any positive constant M, we obtain:

$$\begin{aligned}\frac{dL(t)}{dt} &\leq -M\varepsilon E(t) + \left(\varepsilon\left(\frac{\mu}{4\delta} + 1 + \frac{M}{2}\right) - \mu\right)\|u_t\|_2^2 - \omega\|\nabla u_t\|_2^2 \\ &+\varepsilon\left(\frac{M}{2} - \eta\right)\|\nabla u\|_2^2 \quad . \end{aligned} \tag{10}$$

Now, choosing $M \leq 2\eta$, and ε small enough such that

$$\left(\varepsilon\left(\frac{\mu}{4\delta} + 1 + \frac{M}{2}\right) - \mu\right) < 0 \quad ,$$

inequality (10) becomes:

$$\frac{dL(t)}{dt} \leq -M\varepsilon E(t),\ \forall t \geq 0.$$

Using the equivalence between E and L we finally get the differential equation ($\xi = -M\varepsilon/\beta_2$):

Differential inequality

$$\frac{dL(t)}{dt} \leq -\xi L(t)\,, \quad \forall t \geq 0 \quad .$$

$$L(t) \leq Ce^{-\xi t}\,, \quad \forall t \geq 0 \quad .$$

Consequently, using the equivalence between L and E, it exists $\widehat{C} > 0$, such that:

$$E(t) \leq \widehat{C}e^{-\xi t}\,, \quad \forall t \geq 0 \quad .$$

Remark 2

- *Note that we can obtain the same results as in Theorem 4 in the case of the absence of the strong damping i.e.* $\omega = 0$*, by taking the following Lyapunov function:*

$$L(t) = E(t) + \varepsilon \int_\Omega u_t u dx$$

- *It is clear that the following problem:*

$$\begin{cases} u_{tt} - \mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) - \omega \Delta u_t + \mu u_t = u|u|^{p-2} & x \in \Omega,\ t > 0 \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0 \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & x \in \Omega \quad . \end{cases}$$

could be treated with the same method and we obtained also an exponential decay of the solution if the initial condition is in the positive Nehari manifold and its energy is lower that the potential well depth.

Thank you for your attention