Interior feedback stabilization of wave equations with dynamic boundary delay

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Outline

color code : **Yellow : dynamic boundary conditions** , **red : time delay**

Consider the damped wave equation, with **dynamic boundary conditions** and **time delay** :

$$\begin{cases} u_{tt} - \Delta u + a u_t = 0, & x \in \Omega,\ t > 0\,, \\ u(x,t) = 0, & x \in \Gamma_0,\ t > 0\,, \\ u_{tt}(x,t) = -\dfrac{\partial u}{\partial \nu}(x,t) - \mu u_t(x, t - \tau) & x \in \Gamma_1,\ t > 0\,, \\ u(x,0) = u_0(x) & x \in \Omega\,, \\ u_t(x,0) = u_1(x) & x \in \Omega\,, \\ u_t(x, t-\tau) = f_0(x, t-\tau) & x \in \Gamma_1,\ t \in (0,\tau)\,, \end{cases} \tag{1}$$

where $u = u(x,t)$, $t \geq 0$, $x \in \Omega$ which is a bounded regular domain of $\mathbb{R}^N$, ($N \geq 1$), $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $mes(\Gamma_0) > 0$, $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \varnothing$, a, $\mu > 0$ and u_0 , u_1, f_0 are given functions. Moreover, $\tau > 0$ represents the time delay

Questions to be asked :

1. Existence, uniqueness and global existence?
2. Is the stationary solution $u = 0$ stable and what is the rate of the decay?

$$\begin{cases} u_{tt} - \Delta u + a u_t = 0, & x \in \Omega,\ t > 0\,, \\ u(x,t) = 0, & x \in \Gamma_0,\ t > 0\,, \\ u_{tt}(x,t) = -\dfrac{\partial u}{\partial \nu}(x,t) - \mu u_t(x,t)) & x \in \Gamma_1,\ t > 0\,, \\ u(x,0) = u_0(x) & x \in \Omega\,, \\ u_t(x,0) = u_1(x) & x \in \Omega\,, \end{cases}$$

- Longitudinal vibrations in a homogeneous bar in which there are viscous effects, and spring-mass system, Pellicer and Sola-Morales, 90's
- Artificial boundary condition for unbounded domain : transparent and absorbing, and a lot of mix between these two types, Majda-Enquist 80's,
- Ω is an exterior domain of $\mathbb{R}^3$ in which homogeneous fluid is at rest except for sound waves. Each point of the boundary is subjected to small normal displacements into the obstacle. This type of dynamic boundary conditions are known as **acoustic boundary conditions**, Beale , 80's
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Datko [Dat91], showed that solutions of :

$$\begin{cases} w_{tt} - w_{xx} - a w_{xxt} = 0, & x \in (0,1),\ t > 0, \\ w(0,t) = 0,\ \ w_x(1,t) = -k w_t(1, t - \tau), & t > 0, \end{cases}$$

a , k , $\tau > 0$ become unstable for an arbitrarily small values of τ and any values of a and k. Datko et *al* [DLP86] treated the following one dimensional problem:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + 2a u_t(x,t) + a^2 u(x,t) = 0, & 0 < x < 1,\ \ t > 0, \\ u(0,t) = 0, & t > 0, \\ u_x(1,t) = -k u_t(1, t - \tau), & t > 0, \end{cases} \tag{2}$$

If $k \dfrac{e^{2a} + 1}{e^{2a} - 1} < 1$ then the delayed feedback system is stable for all sufficiently small delays. If $k \dfrac{e^{2a} + 1}{e^{2a} - 1} > 1$, then there exists a dense open set D in $(0, \infty)$ such that for each $\tau \in D$, system (2) admits exponentially unstable solutions.

[Dat91] R. Datko. Two questions concerning the boundary control of certain elastic systems. *J. Differential Equations*, 92(1):27–44, 1991.

[DLP86] R. Datko, J. Lagnese, and M. P. Polis. An example on the effect of time delays in boundary feedback stabilization of wave equations. *SIAM J. Control Optim.*, 24(1):152–156, 1986.

Recently, Ammari et *al* [ANP10] have treated the $N-$dimensional wave equation

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + a u_t(x,t-\tau) = 0, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \Gamma_0,\ t > 0, \\ \dfrac{\partial u}{\partial \nu}(x,t) = -k u(x,t), & x \in \Gamma_1,\ t > 0, \\ u(x,0) = u_0(x) & x \in \Omega\,, \\ u_t(x,0) = u_1(x) & x \in \Omega\,, \\ u_t(x,t-\tau) = f_0(x,t-\tau) & x \in \Gamma_1,\ t \in (0,\tau)\,, \end{cases} \tag{3}$$

Under the usual geometric condition on the domain Ω, they showed an exponential stability result, provided that the delay coefficient a is sufficiently small.

[ANP10] K. Ammari, S. Nicaise, and C. Pignotti, Feedback boundary stabilization of wave equations with interior delay, *Systems Control Lett.*, 59 (2010), pp. 623–628.

Nicaise and Pignotti,[NP06], examined a system of wave equation with a linear boundary damping term with a delay:

$$\begin{cases} u_{tt} - \Delta u = 0, & x \in \Omega,\ t > 0\,, \\ u(x,t) = 0, & x \in \Gamma_0,\ t > 0\,, \\ \dfrac{\partial u}{\partial \nu}(x,t) = \mu_1 u_t(x,t) + \mu_2 u_t(x, t - \tau) & x \in \Gamma_1,\ t > 0\,, \\ u(x,0) = u_0(x), & x \in \Omega\,, \\ u_t(x,0) = u_1(x) & x \in \Omega\,, \\ u_t(x, t-\tau) = g_0(x, t-\tau) & x \in \Omega, \tau > 0\,, \end{cases} \tag{4}$$

and proved under the assumption $\mu_2 < \mu_1$ that null stationary state is exponentially stable. They also proved instability if this condition fails.
They also studied [NP08, NVF09], internal feedback, time-varying delay and distributed delay.

[NP06] S. Nicaise and C. Pignotti. Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. *SIAM J. Control Optim.*, 45(5):1561–1585, 2006.

[NP08] S. Nicaise and C. Pignotti. Stabilization of the wave equation with boundary or internal distributed delay. *Diff. Int. Equs.*, 21(9-10):935–958, 2008.

[NVF09] S. Nicaise, J. Valein, and E. Fridman. Stabilization of the heat and the wave equations with boundary time-varying delays. *DCDS-S.*, S2(3):559–581, 2009.

Main results

1. Problem (1) has a unique global solution.
2. *A shifted related problem* has quadratic polynomial decay to zero.
3. Numerical experiments show at least an exponential decay.

Outline

First we reformulate the boundary delay problem, then by a semigroup approach and using the Lumer-Phillips' theorem we will prove the global existence.

Notations

- $H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) / \; u_{\Gamma_0} = 0\}$
  γ_1 the trace operator from $H^1_{\Gamma_0}(\Omega)$ on $L^2(\Gamma_1)$
  $H^{1/2}(\Gamma_1) = \gamma_1\big(H^1_{\Gamma_0}(\Omega)\big)$.
- $E(\Delta, L^2(\Omega)) = \{u \in H^1(\Omega)$ such that $\Delta u \in L^2(\Omega)\}$
  For $u \in E(\Delta, L^2(\Omega))$, $\dfrac{\partial u}{\partial \nu} \in H^{-1/2}(\Gamma_1)$ and we have Green's formula:

$$\int_\Omega \nabla u(x) \nabla v(x) dx = \int_\Omega -\Delta u(x) v(x) dx + \left\langle \frac{\partial u}{\partial \nu}; v \right\rangle_{\Gamma_1} \forall v \in H^1_{\Gamma_0}(\Omega),$$

where $\langle .; . \rangle_{\Gamma_1}$ means the duality pairing $H^{-1/2}(\Gamma_1)$ and $H^{1/2}(\Gamma_1)$.

As in [NP06], we add the new variable:

$$z(x,\rho,t) = u_t(x, t-\tau\rho), \; x \in \Gamma_1, \; \rho \in (0,1), \; t > 0. \tag{5}$$

Then, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \text{ in } \Gamma_1 \times (0,1) \times (0,+\infty). \tag{6}$$

Therefore, problem (1) is equivalent to:

$$\begin{cases} u_{tt} - \Delta u + a u_t = 0, & x \in \Omega, \; t > 0 \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & x \in \Gamma_1, \rho \in (0,1), \; t > 0 \\ u(x,t) = 0, & x \in \Gamma_0, \; t > 0 \\ u_{tt}(x,t) = -\dfrac{\partial u}{\partial \nu}(x,t) - \mu z(x,1,t) & x \in \Gamma_1, \; t > 0 \\ z(x,0,t) = u_t(x,t) & x \in \Gamma_1, t > 0 \\ u(x,0) = u_0(x) & x \in \Omega \\ u_t(x,0) = u_1(x) & x \in \Omega \\ z(x,\rho,0) = f_0(x, -\tau\rho) & x \in \Gamma_1, \; \rho \in (0,1) \end{cases} \tag{7}$$

Let $V := (u, u_t, \gamma_1(u_t), z)^T$; then V satisfies the problem:

$$\begin{cases} V'(t) = (u_t, u_{tt}, \gamma_1(u_{tt}), z_t)^T = \mathcal{A}V(t), \quad t > 0, \\ V(0) = V_0, \end{cases} \tag{8}$$

where $'$ denotes the derivative with respect to time t, $V_0 := (u_0, u_1, \gamma_1(u_1), f_0(., -.\tau))^T$ and the operator $\mathcal{A}$ is defined by:

$$\mathcal{A}\begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} = \begin{pmatrix} v \\ \Delta u - av \\ -\dfrac{\partial u}{\partial \nu} - \mu z(., 1) \\ -\dfrac{1}{\tau} z_\rho \end{pmatrix}$$

Energy space:

$$\mathcal{H} = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1) \times L^2(\Gamma_1) \times L^2(0,1),$$

$\mathcal{H}$ is a Hilbert space with respect to the inner product

$$\left\langle V, \tilde{V} \right\rangle_{\mathcal{H}} = \int_\Omega \nabla u . \nabla \tilde{u} dx + \int_\Omega v \tilde{v} dx + \int_{\Gamma_1} w \tilde{w} d\sigma + \xi \int_{\Gamma_1} \int_0^1 z \tilde{z} d\rho d\sigma$$

for $V = (u, v, w, z)^T$, $\tilde{V} = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{z})^T$ and ξ **defined later**.
The domain of $\mathscr{A}$ is the set of $V = (u, v, w, z)^T$ such that:

$$(u, v, w, z)^T \in \left(H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega)\right) \times H^1_{\Gamma_0}(\Omega) \times L^2(\Gamma_1) \times L^2\left(\Gamma_1; H^1(0,1)\right), \quad (9)$$

$$w = \gamma_1(v) = z(.,0) \text{ on } \Gamma_1. \quad (10)$$

Definition of the "shifted" operator

Let us finally define $\boldsymbol{\xi^\star = \mu\tau}$. For all $\xi > \xi^\star$ we also define $\boldsymbol{\mu_1 = \dfrac{\xi}{2\tau} + \dfrac{\mu}{2}}$ and

$$\mathscr{A}_{\mathbf{d}} = \mathscr{A} - \mu_1 \, \boldsymbol{I}.$$

Existence result

Theorem 1

Let $V_0 \in \mathcal{H}$*, then there exists a unique solution* $V \in C(\mathbb{R}_+; \mathcal{H})$ *of problem (8). Moreover, if* $V_0 \in \mathcal{D}(\mathcal{A})$*, then*

$$V \in C(\mathbb{R}_+; \mathcal{D}(\mathcal{A})) \cap C^1(\mathbb{R}_+; \mathcal{H}).$$

To prove Theorem 1, we first prove that there exists a unique solution $V \in C(\mathbb{R}_+; \mathscr{H})$ of the shifted problem:

$$\begin{cases} V'(t) = \mathscr{A}_d V(t), & t > 0, \\ V(0) = V_0, \end{cases} \tag{11}$$

Then as $\mathscr{A} = \mathscr{A}_d + \mu_1 I$, there will exist $V \in C(\mathbb{R}_+; \mathscr{H})$ solution of problem (8). In order to prove the existence and uniqueness of the solution of problem (11) we use the semigroup approach and the Lumer-Phillips' theorem.

Sketch of the proof. First step : $\mathcal{A}_d$ is dissipative 1

Let $V = (u, v, w, z)^T \in \mathcal{D}(\mathcal{A})$, we have:

$$\begin{aligned}\langle \mathscr{A} V, V\rangle_{\mathscr{H}} = & \int_\Omega \nabla u.\nabla v dx + \int_\Omega v\Delta u dx - \int_\Omega a|v(x)|^2 dx \\ & + \int_{\Gamma_1} w\left(-\frac{\partial u}{\partial \nu} - \mu z(\sigma, 1)\right) d\sigma - \frac{\xi}{\tau}\int_{\Gamma_1}\int_0^1 z z_\rho d\rho d\sigma.\end{aligned}$$

Using Green's formula and the compatibility condition (10) gives:

$$\langle \mathscr{A} V, V\rangle_{\mathscr{H}} = -\int_\Omega a|v(x)|^2 dx - \mu\int_{\Gamma_1} z(\sigma,1) w d\sigma - \frac{\xi}{\tau}\int_{\Gamma_1}\int_0^1 z_\rho z d\rho dx. \qquad (12)$$

But from the compatibility condition (10), we get:

$$-\frac{\xi}{\tau}\int_{\Gamma_1}\int_0^1 z_\rho z\, d\rho\, d\sigma = \frac{\xi}{2\tau}\int_{\Gamma_1}\left(v^2 - z^2(\sigma,1,t)\right) d\sigma .$$

$$\begin{aligned}\langle \mathscr{A} V, V\rangle_{\mathscr{H}} = & -\int_\Omega a|v(x)|^2 dx - \frac{\xi}{2\tau}\int_{\Gamma_1}\int_0^1 z^2(\sigma,1,t) d\sigma + \frac{\xi}{2\tau}\int_{\Gamma_1} |v|^2(\sigma) d\sigma \\ & -\mu\int_{\Gamma_1} z(\sigma,1) w d\sigma .\end{aligned} \qquad (13)$$

Young's inequality gives :

$$-\int_{\Gamma_1} v(\sigma,t)z(\sigma,1)\,d\sigma \leq \frac{1}{2}\int_{\Gamma_1} z^2(\sigma,1)\,d\sigma + \frac{1}{2}\int_{\Gamma_1} v^2(\sigma,t)d\sigma$$

Using the definition of $\mu_1 = \dfrac{\xi}{2\tau} + \dfrac{\mu}{2}$ gives :

$$\langle \mathscr{A}V, V\rangle_{\mathscr{H}} + \int_\Omega a|v(x)|^2 dx - \mu_1 \int_{\Gamma_1} |v(\sigma)|^2 d\sigma + \left(\frac{\xi}{2\tau} - \frac{\mu}{2}\right)\int_{\Gamma_1} z^2(\sigma,1)d\sigma \leq 0\,. \quad (14)$$

$\mathscr{A}_d = \mathscr{A} - \mu_1 I$ is dissipative

As $\xi^\star = \mu\tau$ we have $\forall \xi > \xi^\star$, $\left(\dfrac{\xi}{2\tau} - \dfrac{\mu}{2}\right) > 0$. We finally get:

$$\left\langle \left(\mathscr{A} - \mu_1 I\right)V, V\right\rangle_{\mathscr{H}} \leq 0\,. \quad (15)$$

Thus the operator $\mathscr{A}_d = \mathscr{A} - \mu_1 I$ is dissipative.

For $F = (f_1, f_2, f_3, f_4)^T \in \mathscr{H}$, $V = (u, v, w, z)^T \in \mathscr{D}(\mathscr{A})$ solution of

$$(\lambda I - \mathscr{A})\, V = F,$$

which is:

$$\begin{aligned} \lambda u - v &= f_1, &(16)\\ \lambda v - \Delta u + av &= f_2, &(17)\\ \lambda w + \frac{\partial u}{\partial \nu} + \mu z(.,1) &= f_3, &(18)\\ \lambda z + \frac{1}{\tau} z_\rho &= f_4. &(19) \end{aligned}$$

To find $V = (u, v, w, z)^T \in \mathcal{D}(\mathcal{A})$ solution of the system (16), (17), (18) and (19), we proceed as in [NP06], with two major changes:

1. the dynamic condition on Γ_1 which adds an unknown and an equation,
2. the presence of $v = u_t$ in this dynamic boundary condition.

We suppose u is determined with the appropriate regularity and we find V.

Outline

Theorem 2 (Polynomial decay)

Let $\xi > \xi^*$. Then there exists a constant $C > 0$ such that, for all $V_0 \in \mathscr{D}(\mathscr{A}_d)$, the semigroup $e^{t\mathscr{A}_d}$ satisfies the following estimate

$$\left\| e^{t\mathscr{A}_d} V_0 \right\|_{\mathscr{H}} \leq \frac{C}{\sqrt{t}} \left\| V_0 \right\|_{\mathscr{D}(\mathscr{A}_d)}, \forall t > 0. \tag{20}$$

Lemma 3 (Asymptotic behavior of the spectrum)

A C_0 semigroup $e^{t\mathcal{L}}$ of contractions on a Hilbert space $\mathcal{H}$ satisfies

$$\left\| e^{t\mathcal{L}} U_0 \right\|_{\mathcal{H}} \leq \frac{C}{t^{\frac{1}{\theta}}} \left\| U_0 \right\|_{\mathcal{D}(\mathcal{L})}$$

for some constant $C > 0$ and for $\theta > 0$ if and only if

$$\rho(\mathcal{L}) \supset \{ i\beta | \beta \in \mathbb{R} \} \equiv i\mathbb{R}, \tag{21}$$

and

$$\limsup_{|\beta| \to \infty} \frac{1}{\beta^\theta} \left\| (i\beta I - \mathcal{L})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} < \infty, \tag{22}$$

where $\rho(\mathcal{L})$ denotes the resolvent set of the operator $\mathcal{L}$.

Remark 1

In view of this theorem we need to identify the spectrum of $\mathscr{A}_d$ lying on the imaginary axis. Unfortunately, as the embedding of $L^2\left(\Gamma_1, H^1(0,1)\right)$ into $L^2\left(\Gamma_1 \times (0,1)\right) = L^2\left(\Gamma_1 \times L^2(0,1)\right)$ is not compact, $\mathscr{A}_d$ has not a compact resolvent. Therefore its spectrum $\sigma(\mathscr{A}_d)$ does not consist only of eigenvalues of $\mathscr{A}_d$. We have then to show that :

1. if β is a real number, then $i\beta I - \mathscr{A}_d$ is injective and
2. if β is a real number, then $i\beta I - \mathscr{A}_d$ is surjective.

It is the objective of the two following lemmas.

Lemma 4

If β is a real number, then $i\beta$ is not an eigenvalue of $\mathscr{A}_d$.

$$\mathscr{A}_d Z = i\beta Z \ \Rightarrow Z = 0$$

$\mathscr{A}_d Z = i\beta Z$ if and only if

$$\begin{aligned}
(i\beta + \mu_1)u - v &= 0, & (23)\\
(i\beta + \mu_1)v - \Delta u + av &= 0, & (24)\\
(i\beta + \mu_1)w + \frac{\partial u}{\partial \nu} + \mu z(.,1) &= 0, & (25)\\
(i\beta + \mu_1)z + \frac{1}{\tau} z_\rho &= 0\ . & (26)
\end{aligned}$$

By taking the inner product with Z and using (15), we get:

$$\Re\left(< \mathscr{A}_d Z, Z >_{\mathscr{H}}\right) \leq -\int_\Omega a\,|v(x)|^2\,dx - \left(\frac{\xi}{2\tau} - \frac{\mu}{2}\right)\int_{\Gamma_1} |z(\sigma,1)|^2\,d\sigma. \qquad (27)$$

Thus we firstly obtain that:

$$v = 0 \text{ and } z(.,1) = 0.$$

Next, according to (23), we have $v = (i\beta + \mu_1)\,u$. Thus we have $u = 0$; since $w = \gamma_1(v) = z(.,0)$, we obtain also $w = 0$ and $z(.,0) = 0$. Moreover as z satisfies (26) by integration, we obtain:

$$z(.,\rho) = z(.,0)\,e^{-\tau(i\beta+\mu_1)\rho}.$$

But as $z(.,0) = 0$, we finally have $z = 0$.
Thus the only solution is the trivial one.

Lemma 5

Let $\xi > \xi^*$. If β is a real number, then $i\beta$ belongs to the resolvent set $\rho(\mathscr{A}_d)$ of $\mathscr{A}_d$.

In view of Lemma 4 it is enough to show that $i\beta I - \mathscr{A}_d$ is surjective.

Lemma 6

The resolvent operator of $\mathscr{A}_d$ satisfies condition (22) for $\theta = 2$.

Suppose that condition (22) is false with $\theta = 2$. By the Banach-Steinhaus Theorem, there exists a sequence of real numbers $\beta_n \to +\infty$ and a sequence of vectors $Z_n = (u_n, v_n, w_n, z_n)^t \in \mathscr{D}(\mathscr{A}_d)$ with $\|Z_n\|_{\mathscr{H}} = 1$ such that

$$||\beta_n^2(i\beta_n I - \mathscr{A}_d)Z_n||_{\mathscr{H}} \to 0 \quad \text{as} \quad n \to \infty, \tag{28}$$

i.e.,

$$\beta_n\left((i\beta_n + \mu_1)u_n - v_n\right) \equiv f_n \to 0 \quad \text{in} \quad H^1_{\Gamma_0}(\Omega), \tag{29}$$

$$\beta_n\left(i\beta_n v_n - \Delta u_n + (\mu_1 + a)v_n\right) \equiv g_n \to 0 \quad \text{in} \quad L^2(\Omega), \tag{30}$$

$$\beta_n\left((i\beta_n + \mu_1)w_n + \frac{\partial u_n}{\partial \nu} + \mu z_n(.,1)\right) \equiv h_n \to 0 \quad \text{in} \quad L^2(\Gamma_1), \tag{31}$$

$$\beta_n\left((i\beta_n + \mu_1)z_n + \frac{1}{\tau}\partial_\rho z_n\right) \equiv k_n \to 0 \quad \text{in} \quad L^2(\Gamma_1 \times (0,1)) \tag{32}$$

since $\beta_n \leq \beta_n^2$.

Our goal is to derive from (28) that $||Z_n||_{\mathscr{H}}$ converges to zero, thus there is a contradiction.

We first notice that we have

$$||\beta_n^2(i\beta_n I - \mathscr{A}_d)Z_n||_{\mathscr{H}} \geq |\Re\left(\langle \beta_n^2(i\beta_n I - \mathscr{A}_d)Z_n, Z_n\rangle_{\mathscr{H}}\right)|. \tag{33}$$

Then, by (27) and (28),

$$\beta_n\, v_n \to 0, \quad \text{in } L^2(\Omega), \quad \beta_n\, z_n(.,1) \to 0, \quad \text{in } L^2(\Gamma_1), \tag{34}$$

$$u_n \to 0,\, \Delta u_n \to 0 \ \text{ in } L^2(\Omega) \Rightarrow u_n \to 0 \ \text{ in } H^1_{\Gamma_0}(\Omega)\ . \tag{35}$$

This further leads, by (31) and the trace theorem, to

$$w_n \to 0 \ \text{ in } L^2(\Gamma_1). \tag{36}$$

Moreover, since $Z_n \in \mathcal{D}(\mathscr{A}_d)$, we have, by (36),

$$z_n(.,0) \to 0 \ \text{ in } L^2(\Gamma_1). \tag{37}$$

We have

$$z_n(.,\rho) = z_n(.,0)\, e^{-(i\beta_n+\mu_1)\tau\rho} + \int_0^\rho e^{-(i\beta_n+\mu_1)\tau(\rho-s)}\, \frac{\tau\, k_n(s)}{\beta_n}\, ds. \tag{38}$$

Which implies, according to (38), (37) and (32), that

$$z_n \to 0 \ \text{ in } L^2(\Gamma_1 \times (0,1))$$

and clearly contradicts $\|Z_n\|_{\mathscr{H}} = 1$.

Outline

$\Omega = (0,1)\,,\ \Gamma_0 = \{0\}\,,\ \Gamma_1 = \{1\}$

To solve numerically problem (1), we have to consider its equivalent formulation, namely problem (41), which writes in the present case:

$$\begin{cases} u_{tt} - u_{xx} + a\, u_t = 0, & x \in (0,1),\ \ t > 0\,, \\ u(0,t) = 0, & t > 0\,, \\ u_{tt}(1,t) = -u_x(1,t) - \mu z(x,1,t), & t > 0\,, \\ \tau z_t(1,\rho,t) + z_\rho(1,\rho,t) = 0, & \rho \in (0,1)\,,\ t > 0\,, \\ z(1,0,t) = u_t(1,t) & t > 0\,, \\ u(x,0) = u_0(x) & x \in (0,1)\,, \\ u_t(x,0) = u_1(x) & x \in (0,1)\,, \\ z(1,\rho,0) = f_0(1,-\tau\rho) & \rho \in (0,1)\ . \end{cases} \tag{39}$$

Implicit Euler method for the time discretisation and finite difference (centered) for the space discretisaiton. No CFL is needed. Without the control term au_t, the delay term and with Dirichlet boundary conditions, the energy is conserved.

$$E(t) = \left\| \left(u(.,t), u_t(.,t), u_t(1,t), z(1,.,t) \right)^T \right\|_{\mathscr{H}}$$

For every simulations the numerical parameters are the following:

$$\tau = 2\,,\; \xi = 2\xi^\star\,,\; \Delta x = \frac{1}{20}\,,\; \Delta\rho = \frac{1}{20}\,,\; \Delta t = 0.1$$
$$u_0(x) = u_1(x) = xe^{10x}\,,\; f_0(1,\rho) = e^{\rho}e^{10}\;.$$

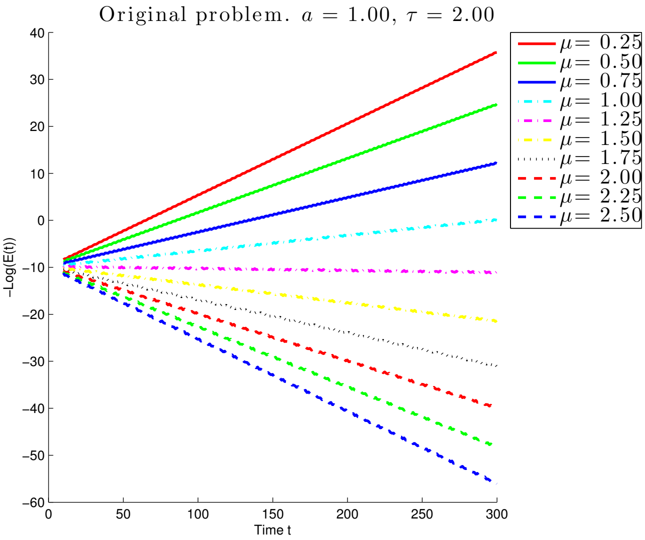

Figure: Energy (in -log scale) versus time: influence of μ. Original problem

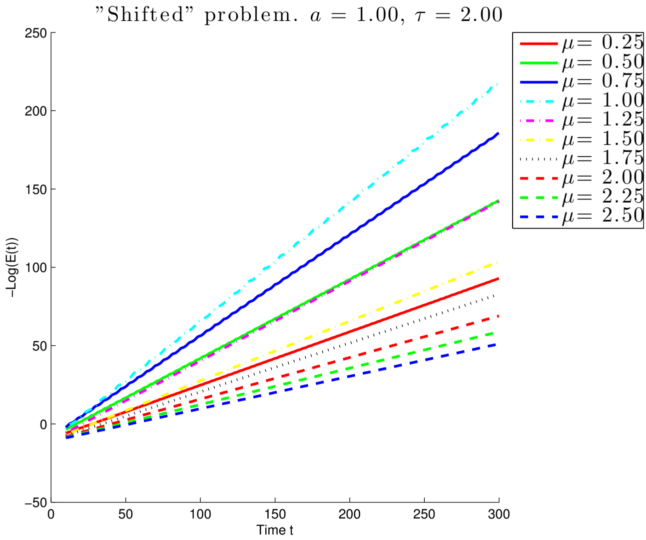

Figure: Energy (in -log scale) versus time: influence of μ. "Shifted" problem

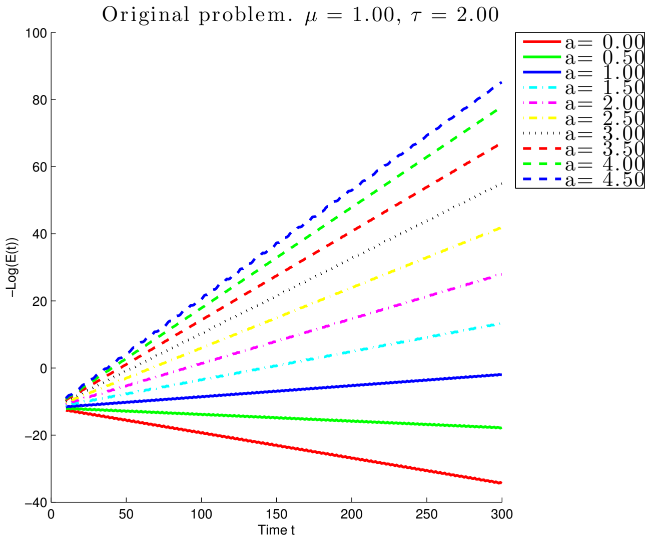

Figure: Energy (in -log scale) versus time: influence of a. Original problem

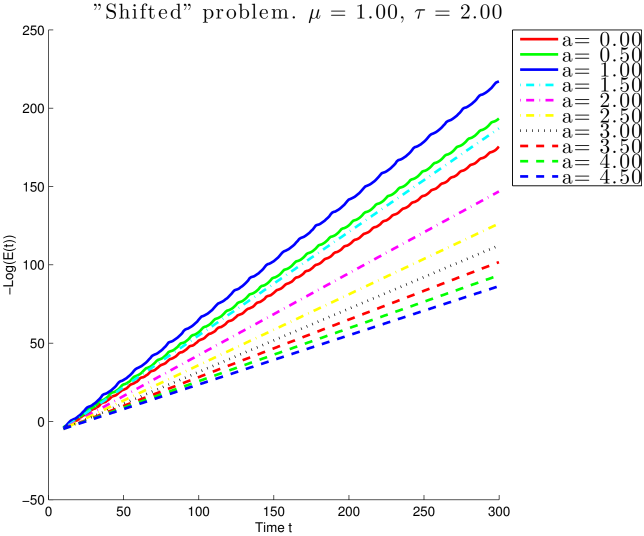

Figure: Energy (in -log scale) versus time: influence of a. "Shifted" problem

Outline

Outline

Changing the damping law

Let us consider now the same system as (1) but with a Kelvin-Voigt damping:

$$\begin{cases} u_{tt} - \Delta u + a\Delta u_t = 0, & x \in \Omega,\ t > 0\,, \\ u(x,t) = 0, & x \in \Gamma_0,\ t > 0\,, \\ u_{tt}(x,t) = -\dfrac{\partial u}{\partial \nu}(x,t) - a\dfrac{\partial u_t}{\partial \nu}(x,t)\mu u_t(x,t-\tau) & x \in \Gamma_1,\ t > 0\,, \\ u(x,0) = u_0(x) & x \in \Omega\,, \\ u_t(x,0) = u_1(x) & x \in \Omega\,, \\ u_t(x,t-\tau) = f_0(x,t-\tau) & x \in \Gamma_1,\ t \in (0,\tau)\,, \end{cases} \tag{40}$$

$$\begin{cases} u_{tt} - \Delta u + a\Delta u_t = 0, & x \in \Omega,\ t > 0 \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & x \in \Gamma_1, \rho \in (0,1)\,,\ t > 0 \\ u(x,t) = 0, & x \in \Gamma_0,\ t > 0 \\ u_{tt}(x,t) = -\dfrac{\partial u}{\partial \nu}(x,t) - a\dfrac{\partial u_t}{\partial \nu}(x,t) - \mu z(x,1,t) & x \in \Gamma_1\,,\ t > 0 \\ z(x,0,t) = u_t(x,t) & x \in \Gamma_1, t > 0 \\ u(x,0) = u_0(x) & x \in \Omega \\ u_t(x,0) = u_1(x) & x \in \Omega \\ z(x,\rho,0) = f_0(x,-\tau\rho) & x \in \Gamma_1,\ \rho \in (0,1) \end{cases} \tag{41}$$

Let the operator $\mathscr{A}_{kv}$ defined by:

$$\mathscr{A}_{kv}\begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} = \begin{pmatrix} v \\ \Delta u + a\,\Delta v \\ -\dfrac{\partial u}{\partial \nu} - a\dfrac{\partial v}{\partial \nu} - \mu z\,(.,1) \\ -\dfrac{1}{\tau} z_\rho \end{pmatrix}.$$

The domain of $\mathscr{A}_{kv}$ is the set of $V = (u, v, w, z)^T$ such that:

$$(u, v, w, z)^T \in \left(H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega)\right) \times H^1_{\Gamma_0}(\Omega) \times L^2(\Gamma_1) \times L^2\left(\Gamma_1; H^1(0,1)\right), \quad (42)$$

$$\frac{\partial v}{\partial \nu} \in L^2(\Gamma_1), \quad (43)$$

$$w = \gamma_1(v) = z(.,0) \text{ on } \Gamma_1. \quad (44)$$

Poincaré's like constant

For $c \in \mathbb{R}$, we define:

$$C_\Omega(c) = \inf_{u \in H^1_{\Gamma_0}(\Omega)} \frac{\|\nabla u\|_2^2 + c\|u\|_{2,\Gamma_1}^2}{\|u\|_2^2} \tag{45}$$

$C_\Omega(c)$ is the first eigenvalue of the operator $-\Delta$ under the Dirichlet-Robin boundary conditions:

$$\begin{cases} u(x) = 0, & x \in \Gamma_0 \\ \dfrac{\partial u}{\partial \nu}(x) + cu(x) = 0 & x \in \Gamma_1 \end{cases} . \tag{46}$$

It exists a unique $c^\star < 0$ such that:

$$C_\Omega(c^\star) = 0 \; . \tag{47}$$

Existence and exponential stability

In the following, we fix $\xi = \mu\tau$ in the norm.

Theorem 7

Suppose that a and μ satisfy the following assumption:

$$\mu < |c^{\star}| a. \tag{48}$$

Then, the operator $\mathscr{A}_{kv}$ generates a C_0 semigroup of contractions on $\mathscr{H}$. We have, in particular, if $V_0 \in \mathscr{H}$, then there exists a unique solution $V \in C(\mathbb{R}_+; \mathscr{H})$ of problem (40). Moreover, if $V_0 \in \mathscr{D}(\mathscr{A}_{kv})$, then

$$V \in C(\mathbb{R}_+; \mathscr{D}(\mathscr{A}_{kv})) \cap C^1(\mathbb{R}_+; \mathscr{H}).$$

Moreover the semigroup operator $e^{t\mathscr{A}_{kv}}$ is exponential stable on $\mathscr{H}$. We have the following result.

Theorem 8

Suppose that the assumption (48) is satisfied. Then, there exist $C, \omega > 0$ such that for all $t > 0$ we have

$$\left\| e^{t\mathscr{A}_{kv}} \right\|_{\mathcal{L}(\mathscr{H})} \leq C\, e^{-\omega t}.$$

Lemma 9

A C_0 semigroup $e^{t\mathcal{L}}$ of contractions on a Hilbert space $\mathcal{H}$ satisfies, for all $t > 0$,

$$||e^{t\mathcal{L}}||_{\mathcal{L}(\mathcal{H})} \leq Ce^{-\omega t}$$

for some constant $C, \omega > 0$ if and only if

$$\rho(\mathcal{L}) \supset \left\{ i\beta \;\middle|\; \beta \in \mathbb{R} \right\} \equiv i\mathbb{R}, \tag{49}$$

and

$$\limsup_{|\beta| \to \infty} \|(i\beta I - \mathcal{L})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty, \tag{50}$$

where $\rho(\mathcal{L})$ denotes the resolvent set of the operator $\mathcal{L}$.

Outline

Numerical results

The constant $c^\star$ must satisfy:

$$\begin{cases} u_{xx} = 0\,, & x \in (0,1), \\ u(0) = 0\,, & u_x(1) + c^\star u(1) = 0\ . \end{cases}$$

Thus we obtain $c^\star = -1$.

Numerical results

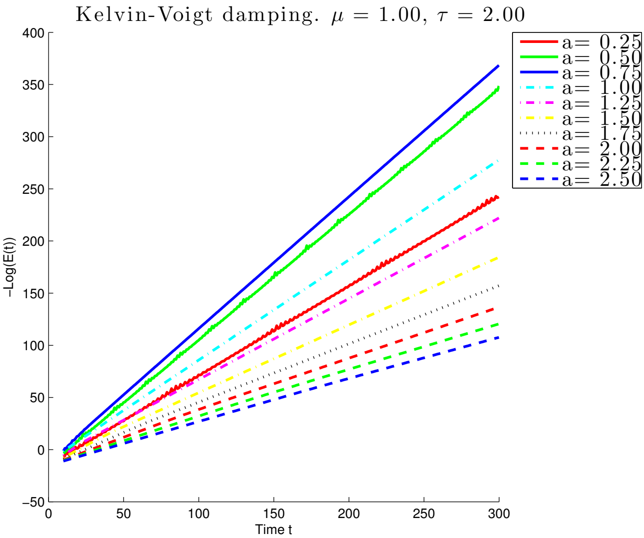

Figure: Influence of a.

Numerical results

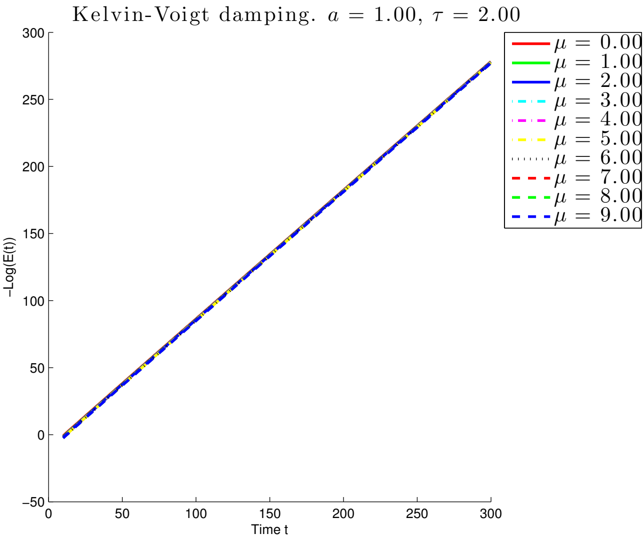

Figure: Influence of *mu*.

To be done

Sharper estimate and instability result?

1. Can we estimate the rate of decay with respect to the parameters a, μ, τ in 1D and $a, \mu, \tau, meas(\Omega), meas(\Gamma_1)$ in multi-D?
2. Instability result for $\xi \leq \xi^\star$?
3. For the Kelvin-Voigt damping, instability if $\mu \geq |c^\star| a$?

[AG2016] K. Ammari and S. Gerbi. Interior feedback stabilization of wave equations with dynamic boundary delay *Submitted*

Thank you for your attention