Existence and exponential stability of the damped wave equation with a dynamic boundary condition and a delay term.

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Outline

1. Introduction
2. Well-posedness of the problem : existence and uniqueness.
   - Setup and notations
   - Semigroup formulation : existence and uniqueness.
3. Asymptotic behavior
4. Some remarks

Consider the damped wave equation, with **dynamic boundary conditions** and **time delay**:

$$
\begin{cases}
u_{tt} - \Delta u - \alpha \Delta u_t = 0, & x \in \Omega,\ t > 0\,, \\
u(x,t) = 0, & x \in \Gamma_0,\ t > 0\,, \\
u_{tt}(x,t) = -\left(\dfrac{\partial u}{\partial \nu}(x,t) + \dfrac{\alpha \partial u_t}{\partial \nu}(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau) \right) & x \in \Gamma_1,\ t > 0\,, \\
u(x,0) = u_0(x) & x \in \Omega\,, \\
u_t(x,0) = u_1(x) & x \in \Omega\,, \\
u_t(x, t-\tau) = f_0(x, t-\tau) & x \in \Gamma_1,\ t \in (0,\tau)\,,
\end{cases} \tag{1}
$$

where $u = u(x,t)$, $t \geq 0$, $x \in \Omega$ which is a bounded regular domain of $\mathbb{R}^N$, $(N \geq 1)$, $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $mes(\Gamma_0) > 0$, $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \varnothing$, $\alpha, \mu_1, \mu_2 > 0$ and u_0 , u_1, f_0 are given functions. Moreover, $\tau > 0$ represents the time delay

Questions to be asked :

1. Existence, uniqueness and global existence?
2. Is the stationary solution $u = 0$ stable and what is the rate of the decay?

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- Longitudinal vibrations in a homogeneous bar in which there are viscous effects, and spring-mass system, Pellicer and Sola-Morales, 90's
- Artificial boundary condition for unbounded domain : transparent and absorbing, and a lot of mix between these two types, Majda-Enquist 80's,
- Ω is an exterior domain of $\mathbb{R}^3$ in which homogeneous fluid is at rest except for sound waves. Each point of the boundary is subjected to small normal displacements into the obstacle. This type of dynamic boundary conditions are known as **acoustic boundary conditions**, Beale , 80's
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In the absence of delay, and with a nonlinear source terms, Gerbi and Said-Houari [GS2008, GS2011] showed the local existence, an exponential decay when the initial energy is small enough, an exponential growth when the initial energy is large enough and a blow-up phenomenon for linear boundary conditions ($m = 2$)

$$\begin{cases} u_{tt} - \Delta u - \alpha \Delta u_t = |u|^{p-2} u, & x \in \Omega, \ t > 0 \\ u(x,t) = 0, & x \in \Gamma_0, \ t > 0 \\ u_{tt}(x,t) = -\left[\dfrac{\partial u}{\partial \nu}(x,t) + \dfrac{\alpha \partial u_t}{\partial \nu}(x,t) + r |u_t|^{m-2} u_t(x,t) \right] & x \in \Gamma_1, \ t > 0 \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) & x \in \Omega \quad . \end{cases}$$

[GS2008] *S. Gerbi and B. Said-Houari*, **Local existence and exponential growth for a semilinear damped wave equation with dynamic boundary conditions**. Advances in Differential Equations *Vol. 13, No 11-12, pp. 1051-1074, 2008.*

[GS2011] *S. Gerbi and B. Said-Houari*, **Asymptotic stability and blow up for a semilinear damped wave equation with dynamic boundary conditions**. Nonlinear Analysis: Theory, Methods & Applications *Vol. 74, pp. 7137-7150, 2011.*

Datko [Dat91], showed that solutions of :

$$\begin{cases} w_{tt} - w_{xx} - a w_{xxt} = 0, & x \in (0,1),\ t > 0, \\ w(0,t) = 0,\ \ w_x(1,t) = -k w_t(1, t - \tau), & t > 0, \end{cases}$$

a , k , $\tau > 0$ become unstable for an arbitrarily small values of τ and any values of a and k. Datko et *al* [DLP86] treated the following one dimensional problem:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + 2a u_t(x,t) + a^2 u(x,t) = 0, & 0 < x < 1,\ \ t > 0, \\ u(0,t) = 0, & t > 0, \\ u_x(1,t) = -k u_t(1, t - \tau), & t > 0, \end{cases} \tag{2}$$

If $k \dfrac{e^{2a}+1}{e^{2a}-1} < 1$ then the delayed feedback system is stable for all sufficiently small delays. If $k \dfrac{e^{2a}+1}{e^{2a}-1} > 1$, then there exists a dense open set D in $(0, \infty)$ such that for each $\tau \in D$, system (2) admits exponentially unstable solutions.

[Dat91] R. Datko. **Two questions concerning the boundary control of certain elastic systems.** *J. Differential Equations*, 92(1):27–44, 1991.

[DLP86] R. Datko, J. Lagnese, and M. P. Polis. **An example on the effect of time delays in boundary feedback stabilization of wave equations.** *SIAM J. Control Optim.*, 24(1):152–156, 1986.

Nicaise and Pignotti,[NP06], examined a system of wave equation with a linear boundary damping term with a delay:

$$\begin{cases} u_{tt} - \Delta u = 0, & x \in \Omega, \ t > 0\,, \\ u(x,t) = 0, & x \in \Gamma_0, \ t > 0\,, \\ \dfrac{\partial u}{\partial \nu}(x,t) = \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau) & x \in \Gamma_1, \ t > 0\,, \\ u(x,0) = u_0(x), & x \in \Omega\,, \\ u_t(x,0) = u_1(x) & x \in \Omega\,, \\ u_t(x, t-\tau) = g_0(x, t-\tau) & x \in \Omega, \tau > 0\,, \end{cases} \tag{3}$$

and proved under the assumption $\mu_2 < \mu_1$ that null stationary state is exponentially stable. They also proved instability if this condition fails.
They also studied [NP08, NVF09], internal feedback, time-varying delay and distributed delay.

[NP06] S. Nicaise and C. Pignotti. Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. *SIAM J. Control Optim.*, 45(5):1561–1585, 2006.

[NP08] S. Nicaise and C. Pignotti. Stabilization of the wave equation with boundary or internal distributed delay. *Diff. Int. Equs.*, 21(9-10):935–958, 2008.

[NVF09] S. Nicaise, J. Valein, and E. Fridman. Stabilization of the heat and the wave equations with boundary time-varying delays. *DCDS-S.*, S2(3):559–581, 2009.

Suppose that :

Coefficient condition

- case 1: $\mu_1 \geq \mu_2$ or
- case 2: $\mu_1 < \mu_2$ and $\alpha > \dfrac{(\mu_1^2 - \mu_2^2)}{2\mu_1} \dfrac{1}{\beta^\star}$ $\beta^\star < 0$ defined later

then Problem (1) has a

- unique global solution,
- this solution decays exponentially to the null solution.

Remark 1

If $\mu_1 \geq \mu_2$, as in the works of Nicaise and Pignotti, we can choose $\alpha = 0$ so that no strong damping is necessary.

Main results

Suppose that :

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Setup and notations

First we reformulate the boundary delay problem, then by a semigroup approach and using the Lumer-Phillips' theorem we will prove the global existence.

Notations

- $H^1_{\Gamma_0}(\Omega) = \left\{u \in H^1(\Omega) /\ u_{\Gamma_0} = 0\right\}$
  γ_1 the trace operator from $H^1_{\Gamma_0}(\Omega)$ on $L^2(\Gamma_1)$
  $H^{1/2}(\Gamma_1) = \gamma_1\left(H^1_{\Gamma_0}(\Omega)\right)$.
- $E(\Delta, L^2(\Omega)) = \left\{u \in H^1(\Omega) \text{ such that } \Delta u \in L^2(\Omega)\right\}$
  For $u \in E(\Delta, L^2(\Omega))$, $\dfrac{\partial u}{\partial \nu} \in H^{-1/2}(\Gamma_1)$ and we have Green's formula:

$$\int_\Omega \nabla u(x) \nabla v(x) dx = \int_\Omega -\Delta u(x) v(x) dx + \left\langle \frac{\partial u}{\partial \nu}; v \right\rangle_{\Gamma_1} \forall v \in H^1_{\Gamma_0}(\Omega),$$

where $\langle .; . \rangle_{\Gamma_1}$ means the duality pairing $H^{-1/2}(\Gamma_1)$ and $H^{1/2}(\Gamma_1)$.

As in [NP06], we add the new variable:

$$z(x,\rho,t) = u_t(x, t - \tau\rho), \; x \in \Gamma_1, \; \rho \in (0,1), \; t > 0. \tag{4}$$

Then, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \; \text{in } \Gamma_1 \times (0,1) \times (0,+\infty). \tag{5}$$

Therefore, problem (1) is equivalent to:

$$\begin{cases}
u_{tt} - \Delta u - \alpha \Delta u_t = 0, & x \in \Omega, \; t > 0, \\
\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & x \in \Gamma_1, \rho \in (0,1), \; t > 0 \\
u(x,t) = 0, & x \in \Gamma_0, \; t > 0, \\
u_{tt}(x,t) = -\left(\dfrac{\partial u}{\partial \nu}(x,t) + \alpha \dfrac{\partial u_t}{\partial \nu}(x,t) + \mu_1 u_t(x,t) + \mu_2 z(x,1,t) \right) & x \in \Gamma_1, \; t > 0, \\
z(x,0,t) = u_t(x,t) & x \in \Gamma_1, t > 0, \\
u(x,0) = u_0(x) & x \in \Omega, \\
u_t(x,0) = u_1(x) & x \in \Omega, \\
z(x,\rho,0) = f_0(x, -\tau\rho) & x \in \Gamma_1, \; \rho \in (0,1).
\end{cases} \tag{6}$$

Let $V := (u, u_t, \gamma_1(u_t), z)^T$; then V satisfies the problem:

$$\begin{cases} V'(t) = (u_t, u_{tt}, \gamma_1(u_{tt}), z_t)^T = \mathcal{A}V(t), & t > 0, \\ V(0) = V_0, \end{cases} \tag{7}$$

where $'$ denotes the derivative with respect to time t, $V_0 := (u_0, u_1, \gamma_1(u_1), f_0(., -.\tau))^T$ and the operator $\mathcal{A}$ is defined by:

$$\mathcal{A}\begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} = \begin{pmatrix} v \\ \Delta u + \alpha \Delta v \\ -\dfrac{\partial u}{\partial \nu} - \alpha \dfrac{\partial v}{\partial \nu} - \mu_1 v - \mu_2 z(., 1) \\ -\dfrac{1}{\tau} z_\rho \end{pmatrix}$$

Energy space:

$$\mathcal{H} = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1) \times L^2(\Gamma_1) \times L^2(0,1),$$

$\mathcal{H}$ is a Hilbert space with respect to the inner product

$$\left\langle V, \tilde{V} \right\rangle_{\mathcal{H}} = \int_\Omega \nabla u . \nabla \tilde{u} dx + \int_\Omega v \tilde{v} dx + \int_{\Gamma_1} w \tilde{w} d\sigma + \xi \int_{\Gamma_1} \int_0^1 z \tilde{z} d\rho d\sigma$$

for $V = (u, v, w, z)^T$, $\tilde{V} = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{z})^T$ and ξ **defined later**.
The domain of $\mathcal{A}$ is the set of $V = (u, v, w, z)^T$ such that:

$$(u, v, w, z)^T \in H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega) \times L^2(\Gamma_1) \times L^2\left(\Gamma_1; H^1(0,1)\right) \quad (8)$$

$$u + \alpha v \in E(\Delta, L^2(\Omega)), \ \frac{\partial(u + \alpha v)}{\partial \nu} \in L^2(\Gamma_1) \quad (9)$$

$$w = \gamma_1(v) = z(.,0) \text{ on } \Gamma_1 \quad (10)$$

For $\beta \in \mathbb{R}$, define :

$$C(\beta) = \inf_{u \in H^1_{\Gamma_0}(\Omega)} \frac{\|\nabla u\|_2^2 + \beta \|u\|^2_{2,\Gamma_1}}{\|u\|_2^2} \tag{11}$$

$C(\beta)$ is the first eigenvalue of the operator $-\Delta$ under the Dirichlet-Robin boundary conditions :

$$\begin{cases} u(x) = 0, & x \in \Gamma_0 \\ \beta u(x) + \dfrac{\partial u}{\partial \nu}(x) = 0 & x \in \Gamma_1 \end{cases}$$

From Kato's perturbation theory, $C(\beta)$ is a continuous decreasing function and as $C(0) > 0$, it exists $\beta^\star < 0$ such that

$$C(\beta^\star) = 0.$$

Definition of $\beta^\star$

it exists $\beta^\star < 0$ such that $\forall \beta > \beta^\star$, $C(\beta) > 0$

Existence result

Suppose that :

Coefficient condition

- case 1: $\mu_1 \geq \mu_2$ or
- case 2: $\mu_1 < \mu_2$ and $\alpha > \dfrac{(\mu_1^2 - \mu_2^2)}{2\mu_1} \dfrac{1}{\beta^\star}$ $\beta^\star < 0$

Theorem 1

Let $V_0 \in \mathcal{H}$, then there exists a unique solution $V \in C(\mathbb{R}_+; \mathcal{H})$ of problem (7). Moreover, if $V_0 \in \mathcal{D}(\mathcal{A})$, then

$$V \in C(\mathbb{R}_+; \mathcal{D}(\mathcal{A})) \cap C^1(\mathbb{R}_+; \mathcal{H}).$$

Let $V = (u, v, w, z)^T \in \mathcal{D}(\mathcal{A})$, we have:

$$\langle \mathcal{A}V, V \rangle_{\mathcal{H}} = \int_\Omega \nabla u . \nabla v dx + \int_\Omega v \left(\Delta u + \alpha \Delta v\right) dx + \int_{\Gamma_1} w \left(-\frac{\partial u}{\partial \nu} - \alpha \frac{\partial v}{\partial \nu} - \mu_1 v - \mu_2 z(\sigma, 1) \right) d\sigma - \frac{\xi}{\tau} \int_{\Gamma_1} \int_0^1 z z_\rho d\rho d\sigma.$$

Since $u + \alpha v \in E(\Delta, L^2(\Omega))$ and $\dfrac{\partial(u + \alpha v)}{\partial \nu} \in L^2(\Gamma_1)$, using Green's formula and the compatibility condition (10) gives:

$$\langle \mathcal{A}V, V \rangle_{\mathcal{H}} = -\mu_1 \int_{\Gamma_1} v^2 d\sigma - \mu_2 \int_{\Gamma_1} z(\sigma, 1) v d\sigma - \alpha \int_\Omega |\nabla v|^2 dx - \frac{\xi}{\tau} \int_{\Gamma_1} \int_0^1 z_\rho z d\rho d\sigma.$$

But from the compatibility condition (10), we get:

$$-\frac{\xi}{\tau} \int_{\Gamma_1} \int_0^1 z_\rho z \, d\rho \, d\sigma = \frac{\xi}{2\tau} \int_{\Gamma_1} \left(v^2 - z^2(\sigma, 1, t)\right) d\sigma \, .$$

$$\langle \mathcal{A}V, V\rangle_{\mathcal{H}} = -\alpha \int_{\Omega} |\nabla v|^2\, dx - \left(\mu_1 - \frac{\xi}{2\tau}\right) \int_{\Gamma_1} v^2 d\sigma - \frac{\xi}{2\tau} \int_{\Gamma_1} \int_0^1 z^2(\sigma, 1, t) d\sigma$$
$$-\mu_2 \int_{\Gamma_1} v(\sigma, t) z(\sigma, 1)\, d\sigma$$

Fix $\delta > 0$, Young's inequality gives :

$$-\int_{\Gamma_1} v(\sigma, t) z(\sigma, 1)\, d\sigma \leq \frac{\delta}{2} \int_{\Gamma_1} z^2(\sigma, 1)\, d\sigma + \frac{1}{2\delta} \int_{\Gamma_1} v^2(\sigma, t) d\sigma$$

Finally

$$\langle \mathcal{A}V, V\rangle_{\mathcal{H}} + \alpha \int_{\Omega} |\nabla v|^2\, dx + \left(\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2\delta}\right) \int_{\Gamma_1} v^2 d\sigma +$$
$$\left(\frac{\xi}{2\tau} - \frac{\delta \mu_2}{2}\right) \int_{\Gamma_1} z^2(\sigma, 1, t) d\sigma \leq 0$$

Fix δ and ξ

Choose $\delta = \dfrac{\mu_1}{\mu_2}$ and $\xi = \dfrac{\mu_2^2 \tau}{\mu_1}$

$$\langle \mathcal{A}V, V\rangle_{\mathcal{H}} + \alpha \int_\Omega |\nabla v|^2 \, dx + \frac{\mu_1^2 - \mu_2^2}{2\mu_1} \int_{\Gamma_1} v^2 d\sigma \leq 0$$

- case 1: $\mu_1 \geq \mu_2$. For all $\alpha \geq 0$

$$\forall V \in \mathcal{H} \ \langle \mathcal{A}V, V\rangle_{\mathcal{H}} \leq 0.$$

- case 2: $\mu_1 < \mu_2$, $\alpha > 0$. Set $\beta = \dfrac{\mu_1^2 - \mu_2^2}{2\alpha\mu_1}$.

$$\langle \mathcal{A}V, V\rangle_{\mathcal{H}} + C(\beta)\|u\|_2^2 \leq 0$$

Suppose : $\alpha > \dfrac{(\mu_1^2 - \mu_2^2)}{2\mu_1} \dfrac{1}{\beta^\star}$. Thus $C(\beta) > 0$ and we get :

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$$\forall V \in \mathcal{H} \ \langle \mathcal{A}V, V\rangle_{\mathcal{H}} \leq 0.$$

$\lambda I - \mathcal{A}$ is surjective for all $\lambda > 0$. Let $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$. We seek $V = (u, v, w, z)^T \in \mathcal{D}(\mathcal{A})$ solution of

$$(\lambda I - \mathcal{A})\, V = F,$$

which writes:

$$\lambda u - v = f_1 \quad (12)$$

$$\lambda v - \Delta(u + \alpha v) = f_2 \quad (13)$$

$$\lambda w + \frac{\partial(u + \alpha v)}{\partial \nu} + \mu_1 v + \mu_2 z(., 1) = f_3 \quad (14)$$

$$\lambda z + \frac{1}{\tau} z_\rho = f_4 \quad (15)$$

To find $V = (u, v, w, z)^T \in \mathcal{D}(\mathcal{A})$ solution of the system (12), (13), (14) and (15), we proceed as in [NP06], with two major changes:

1. the dynamic condition on Γ_1 which adds an unknown and an equation,
2. the presence of $v = u_t$ in this dynamic boundary condition.

Suppose u is determined with the appropriate regularity. Then from (12), we get:

$$v = \lambda u - f_1 \ . \tag{16}$$

Therefore, from the compatibility condition on Γ_1, (10), we determine $z(.,0)$ by:

$$z(x,0) = v(x) = \lambda u(x) - f_1(x), \quad \text{for } \ x \in \Gamma_1 \tag{17}$$

Thus, from (15), z is solution of the linear Cauchy problem:

$$\begin{cases} z_\rho = \boldsymbol{\tau}\Big(f_4(x) - \lambda z(x,\rho)\Big) & \text{for } x \in \Gamma_1 \, , \ \rho \in (0,1) \\ z(x,0) = \lambda u(x) - f_1(x) & \end{cases} \tag{18}$$

The solution of the Cauchy problem (18) is given by:

$$z(x,\rho) = \lambda u(x) e^{-\lambda\rho\boldsymbol{\tau}} - f_1 e^{-\lambda\rho\boldsymbol{\tau}} + \boldsymbol{\tau} e^{-\lambda\rho\boldsymbol{\tau}} \int_0^\rho f_4(x,\sigma) e^{\lambda\sigma\boldsymbol{\tau}} \, d\sigma \quad \text{for } x \in \Gamma_1 \, , \ \rho \in (0,1). \tag{19}$$

So we have at the point $\rho = 1$, for $x \in \Gamma_1$,

$$z(x,1) = \lambda u(x) e^{-\lambda\boldsymbol{\tau}} + z_1(x) \, , \ z_1(x) = -f_1 e^{-\lambda\boldsymbol{\tau}} + \boldsymbol{\tau} e^{-\lambda\boldsymbol{\tau}} \int_0^1 f_4(x,\sigma) e^{\lambda\sigma\boldsymbol{\tau}} \, d\sigma \tag{20}$$

Since $f_1 \in H^1_{\Gamma_0}(\Omega)$ and $f_4 \in L^2(\Gamma_1) \times L^2(0,1)$, $z_1 \in L^2(\Gamma_1)$.

Knowing u, we may deduce v by (16), z by (19) and using (20), we deduce $w = \gamma_1(v)$ by (14).

Set $\overline{u} = u + \alpha v$. From equations (13) and (14), $\overline{u}$ must satisfy:

$$\begin{cases} \dfrac{\lambda^2}{1+\lambda\alpha}\overline{u} - \Delta \overline{u} = f_2 + \dfrac{\lambda}{1+\lambda\alpha} f_1 & \text{in } \Omega \\ \overline{u} = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial \overline{u}}{\partial \nu} = -\dfrac{\lambda\Big(\mu_2 e^{-\lambda\tau} + (\lambda+\mu_1)\Big)}{1+\lambda\alpha}\overline{u} + f(x) & \text{for } x \in \Gamma_1 \end{cases} \tag{21}$$

with $f_1 \in L^2(\Omega)$, $f_2 \in L^2(\Omega)$, $f \in L^2(\Gamma_1)$.
The variational formulation of problem (21) is to find $\overline{u} \in H^1_{\Gamma_0}(\Omega)$ such that:

$$\int_\Omega \frac{\lambda^2}{1+\lambda\alpha}\overline{u}\omega + \nabla\overline{u}\nabla\omega dx \quad + \quad \int_{\Gamma_1} \frac{\lambda\Big(\mu_2 e^{-\lambda\tau} + (\lambda+\mu_1)\Big)}{1+\lambda\alpha}\overline{u}(\sigma)\omega(\sigma)d\sigma = \tag{22}$$

$$\int_\Omega \left(f_2 + \frac{\lambda}{1+\lambda\alpha} f_1\right)\omega dx + \int_{\Gamma_1} f(\sigma)\omega(\sigma)d\sigma \ \forall \omega \in H^1_{\Gamma_0}(\Omega)$$

Since $\lambda > 0$, $\mu_1 > 0$, $\mu_2 > 0$, the left hand side of (22) defines a coercive bilinear form on $H^1_{\Gamma_0}(\Omega)$.

Thus by applying the Lax-Milgram lemma, there exists a unique $\overline{u} \in H^1_{\Gamma_0}(\Omega)$ solution of (22).

Now, choosing $\omega \in \mathcal{C}_c^\infty$, by Green's formula $\overline{u} \in E(\Delta, L^2(\Omega))$.

We recover u , v , z and finally setting $w = \gamma_1(v)$, we have found

$$V = (u, v, w, z)^T \in \mathcal{D}(\mathcal{A}) \text{ solution of } (\lambda Id - \mathcal{A})\, V = F\ .$$

The well-posedness result, Theroem 1, follows from the Lummer-Phillips' theorem.

Outline

Let $\xi > 0$, we define the functional energy of the solution of problem (6) as

$$\begin{aligned} E(t) = E(t, z, u) &= \frac{1}{2}\left[\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 + \|u_t(t)\|_{2,\Gamma_1}^2\right] \\ &+ \frac{\xi}{2}\int_{\Gamma_1}\int_0^1 z^2(\sigma, \rho, t)\, d\rho\, d\sigma. \end{aligned} \quad (23)$$

E is greater than the usual one : $E_1(t) = \dfrac{1}{2}\left[\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 + \|u_t(t)\|_{2,\Gamma_1}^2\right]$.

Set $\beta = \dfrac{\mu_1^2 - \mu_2^2}{2\alpha\mu_1}$.

Lemma 2

For u solution of (6), and for any $t \geq 0$, we have: $\dfrac{dE\,(t)}{dt} \leq -\alpha C(\beta)\|u_t\|_2^2$

Corollary 1

Suppose the damping coefficient condition is fulfilled (that is $\beta > \beta^\star$, $C(\beta) > 0$). Then the energy E is decreasing along the trajectories.

We multiply the first equation in (6) by u_t and perform integration by parts to get:

$$\frac{1}{2}\frac{d}{dt}\left[\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 + \|u_t(t)\|_{2,\Gamma_1}^2\right] + \alpha\,\|\nabla u_t(t)\|_2^2 + \mu_1\,\|u_t(t)\|_{2,\Gamma_1}^2 + \mu_2 \int_{\Gamma_1} u_t(\sigma,t)u_t(\sigma,t-\tau)d\sigma = 0\,. \tag{24}$$

By defintion of z, we have: $\int_{\Gamma_1} u_t(\sigma,t)u_t(\sigma,t-\tau)d\sigma = \int_{\Gamma_1} u_t(\sigma,t)z(\sigma,1,t)d\sigma$

Fix $\delta > 0$, Young's inequality gives :

$$\left|\int_{\Gamma_1} u_t(\sigma,t)z(\sigma,1,t)d\sigma\right| \le \frac{\delta}{2}\int_{\Gamma_1} z^2(\sigma,1)\,d\sigma + \frac{1}{2\delta}\int_{\Gamma_1} u_t^2(\sigma,t)d\sigma$$

Differentiating the term $\int_{\Gamma_1}\int_0^1 z^2(\sigma,\rho,t)\,d\rho\,d\sigma$ with respect to t and using the fact that $z_t = -\frac{z_\rho}{\tau}$, we get

Finally

$$\frac{d\ E}{dt} \le -\alpha\|\nabla u_t\|^2 - \left(\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2\delta}\right)\|u_t\|_{2,\Gamma_1}^2 - \left(\frac{\xi}{2\tau} - \frac{\delta\mu_2}{2}\right)\int_{\Gamma_1} z^2(\sigma,1,t)d\sigma$$

Asymptotic behavior

Fix δ and ξ

Choose $\delta = \frac{\mu_1}{\mu_2}$ and $\xi = \frac{\mu_2^2 \tau}{\mu_1}$, set $\beta = \frac{\mu_1^2 - \mu_2^2}{2\alpha\mu_1}$.

$$\frac{dE(t)}{dt} \leq -\alpha C(\beta) \|u_t\|_2^2$$

The asymptotic stability result reads as follows:

Theorem 3

Assume the damping coefficient relation is fulfiled. Then there exist two positive constants C and γ independent of t such that for u solution of problem (6), we have:

$$E(t) \leq C e^{-\gamma t}, \ \forall \, t \geq 0 \,. \tag{25}$$

For $\varepsilon > 0$, to be chosen later, we define the Lyapunov function:

$$\begin{aligned} L(t) = E(t) \quad &+ \quad \varepsilon \int_\Omega u(x,t) u_t(x,t)\, dx + \varepsilon \int_{\Gamma_1} u(\sigma,t) u_t(\sigma,t)\, d\sigma \\ &+ \quad \frac{\varepsilon\alpha}{2} \int_\Omega |\nabla u(x,t)|^2 \, dx \\ &+ \quad \frac{\varepsilon\xi}{2} \int_{\Gamma_1} \int_0^1 e^{-2\tau\rho} z^2(\sigma,\rho,t)\, d\rho\, d\sigma. \end{aligned} \tag{26}$$

There exist two positive constants β_1 and $\beta_2 > 0$ depending on ε such that for all $t \geq 0$

$$\beta_1 E(t) \leq L(t) \leq \beta_2 E(t) \,. \tag{27}$$

By taking the time derivative of the function L defined by (26), using problem (6), performing several integration by parts, and using the previous inequality on the derivative of E and the same Young's inequality with $\delta = \dfrac{\mu_1}{\mu_2}$ and $\xi = \dfrac{\mu_2^2 \tau}{\mu_1}$, we choose $\varepsilon > 0$ such that there exist two positive constants C_* and γ independent of t:

$$L(t) \leq C_* e^{-\gamma t} \,, \ \forall t \geq 0 \,.$$

Consequently, by using (27) once again, we conclude that it exists $C > 0$ such that:

$$E(t) \leq C e^{-\gamma t} \,, \ \forall t \geq 0 \,.$$

Outline

1. Since the energy associated to (1) is less than the one associated to (6), it is obvious that the exponential stability of the solution associate to problem (6) implies the exponential stability of the one associated to (1).
2. The presence of the strong damping term $-\alpha\Delta u_t$ in equation (1) plays an essential role in the behavior of the system. The condition $\mu_1 < \mu_2$ is a necessary condition in the case $\alpha = 0$, since Nicaise and Pignotti [NP06] showed an instability result if this condition fails.
3. Adapting the same method to the system with internal feedback:

$$
\begin{cases}
u_{tt} - \Delta u - \alpha\Delta u_t + b(x)\Big(\mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau)\Big) = 0, & x \in \Omega,\ t > 0 \\
u(x,t) = 0, & x \in \Gamma_0,\ t > 0 \\
u_{tt}(x,t) = -\left[\frac{\partial u}{\partial \nu}(x,t) + \frac{\alpha \partial u_t}{\partial \nu}(x,t)\right] & x \in \Gamma_1,\ t > 0 \\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & x \in \Omega\ , \\
u(x, t-\tau) = f_0(x, t-\tau) & x \in \Omega \times (0,\tau)
\end{cases}
\tag{28}
$$

with $b \in L^\infty(\Omega)$ is a function which satisfies

$$b(x) \geq 0, \text{ a.e. in } \Omega \text{ and } \quad b(x) > b_0 > 0 \text{ a.e. in } \omega$$

where $\omega \subset \Omega$ is an open neighborhood of Γ_1, **the results are still valid.**

Instability result?

Can we show that if

$$\mu_1 < \mu_2 \text{ and } \alpha \leq \frac{(\mu_1^2 - \mu_2^2)}{2\mu_1} \frac{1}{\beta^\star}$$

we can find solution with constant energy or energy that goes to infinity?

Hint: Try to find a solution of the form:

$$u(t,x) = e^{\lambda t}\phi(x) \text{ with } \lambda \in \mathbb{C}, \Re(\lambda) > 0.$$

Thank you for your attention