A kinetic formulation for a model coupling free surface and pressurised flows in closed pipes

ICCAM 2006, July 10 - July 14, KU Leuven, Belgium

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Non stationnary mixed flows can occurs in closed pipes:

- Free surface flow:

  the pipe is not filled by the water which is considered as incompressible

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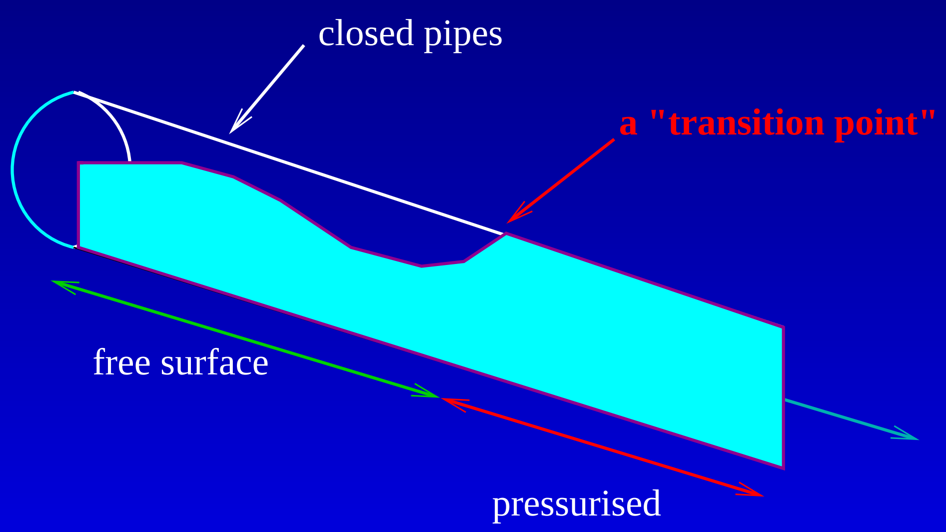
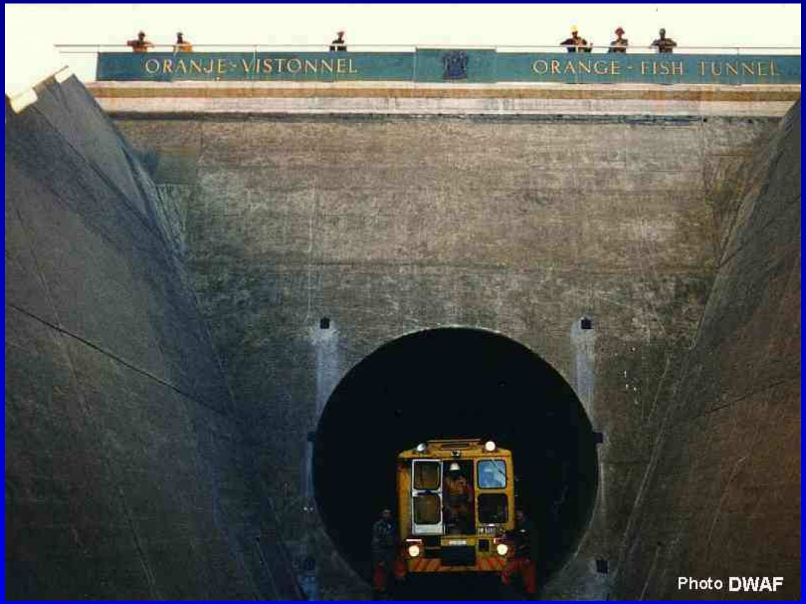


Exemples of closed pipes

a forced pipe

a sewer in Paris

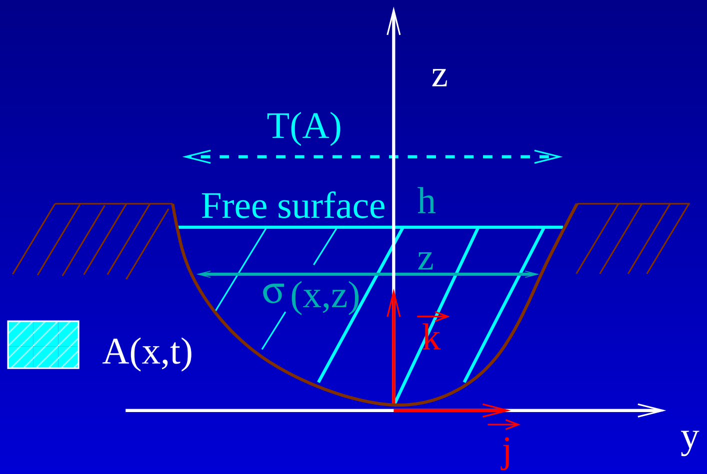


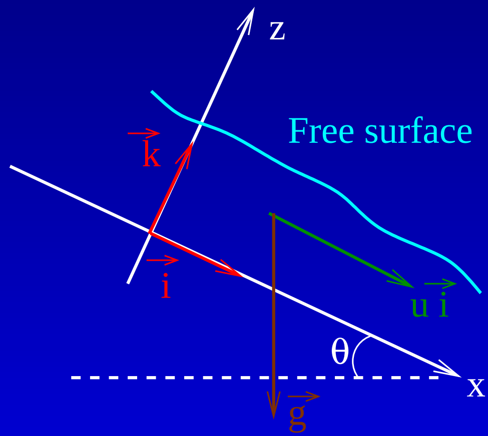
The Orange-Fish Tunnel (in Canada)

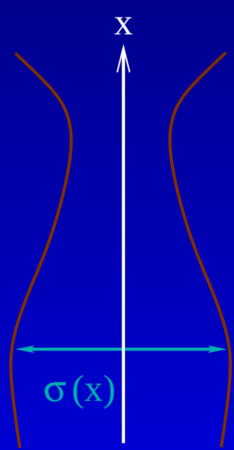
Saint Venant system for free surface flows

A the wetted area, Q the discharge.

$$\begin{aligned} \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} &= 0 \\ \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A} + g I_1\right) &= -gA\frac{\partial Z}{\partial x} \end{aligned}$$







Under the hydrostatic pressure:

$$I_1 = \int_0^y (y - z)\,\sigma(x, z)\,dz$$

Properties of Saint Venant system for free surface flows

- The Saint Venant system is strictly hyperbolic for $A(x,t) > 0$.
  It admits a mathematical entropy

  $$E(A,Q,Z) = \frac{Au^2}{2} + gAZ + gAh(A) - gI_1(A)$$

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$$Q = Au = C_1 \quad , \frac{u^2}{2} + g\,h(A) + g\,Z = C_2$$

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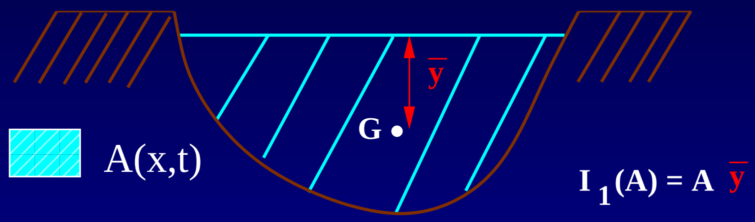

$$\mathcal{M}_{FS}(t,x,\xi) = \frac{A(x,t)}{\sqrt{g\overline{\overline{y}}}}\chi\left(\frac{\xi - u(t,x)}{\sqrt{g\overline{\overline{y}}}}\right)$$

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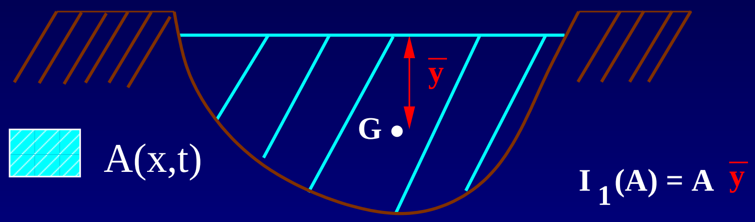

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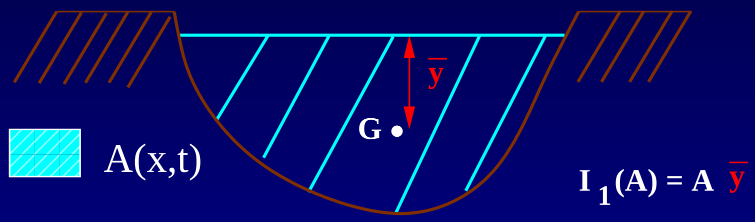

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(A, Q) is a solution of Saint-Venant system if and only if

$\mathcal{M}$ is a solution of the kinetic equation :

$$\frac{\partial}{\partial t}\mathcal{M} + \xi \cdot \frac{\partial}{\partial x}\mathcal{M} - g\frac{\partial}{\partial x}Z \cdot \frac{\partial}{\partial \xi}\mathcal{M} = K(t,x,\xi)$$

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  - ▷ **Not convex for the circular case**.

Saint-Venant like system for pressurised flows

3D system of compressible Euler equations + integration over sections orthogonal to the flow axis + Boussinesq law for lightly compressible fluid: $P = P_a + \frac{1}{\beta}\left(\frac{\rho}{\rho_0} - 1\right),\ c^2 = \frac{1}{\beta\,\rho_0}$ and conservative variables: $M = \rho\, A\,,\ D = \rho\, Q$:

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"FS-equivalent wet area" A_{eq}: $M = \rho\,A_{max} = \rho_0\,A_{eq}$

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The dual model and its kinetic formulation

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The pressure law term writes:

$$\left\{ \begin{array}{lcl} p(x, A, E) & = & g\, I_1(A) \text{ if } A \leq A_{max} \text{ and } E = FS, \\ \\ p(x, A, E) & = & g\, I_1(A_{max}) + c^2\,(A - A_{max}) \text{ if } E = PF. \end{array} \right.$$

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At each point x of the pipe, and at each time t, if we know the state of the flow (free surface of pressurised), define the Gibbs equilibrium by:

$$\mathcal{M}(t, x, E, \xi) = \left\{ \begin{array}{lll} \dfrac{A}{c(A)} \chi \left(\dfrac{\xi - u(t,x)}{c(A)} \right) & \text{with } c(A) = \sqrt{g \overline{y}(A)} & \text{if } E = FS, \\ \dfrac{A}{c} \chi \left(\dfrac{\xi - u(t,x)}{c} \right) & \text{with } c = \sqrt{\dfrac{1}{\beta\, \rho_0}} & \text{if } E = PF \end{array} \right.$$

Wiggert test

Horizontal rectangular pipe.
Initial condition: free surface still water steady state

Downstream water height increases : a pressurised flows comes from downstream
Downstream water height decreases after

Upstream water level constant until the wave comes ...