# Towards a Theory of Programming Languages

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## 1 Introduction

A large field of research in computer science deals with programming languages and their semantics. In that field, it is common to study a given language to prove programs written in it cannot "go wrong", that is, for example, that they will not crash, or will only crash in a predictable and recoverable way... It is also common to study the semantics of a programming language to show, for example, that it is preserved by compilation. However, the techniques used to prove these properties vary with languages, making the study of each new language a potentially complex one involving new techniques. Rather than techniques, it would be useful to have a theory, i.e. a general framework and general tools to study languages, which would allow the study of languages without having to find new techniques or adapt existing ones to each new language. For example, we would like to have a general theorem that says that a morphism from a language $L$ to a language $L'$ that respects some given properties will preserve and reflect a particular observational equivalence.

To solve this problem, a sufficiently broad approach of the semantics of programming languages is needed. Category theory is a mathematical tool that has proved its worth in being general. It is general enough to describe properties in many fields of mathematics, such as algebraic topology, foundations of mathematics, logics, and semantics of programs, among others. Moreover, it has already produced some results in this field with game semantics and presheaf semantics. Therefore, it looks like an approach based on category theory could yield some interesting results.

Several possible approaches have been suggested to develop general tools to study the semantics of programming languages. One of them, bialgebraic semantics, developed by Marcelo Fiore, Gordon Plotkin, and Daniele Turi, is an algebraic version of operational semantics. Another approach, the Tile Model, developed by Ugo Montanari et al., is based on double categories. Tom Hirschowitz has also developed a 2-categorical approach to higher-order rewriting. However, none of the approaches mentioned above are used in practice, probably because they are too demanding on a mathematical level compared to the results they have produced, and, for most of them, there is no notion of morphism.

Tom Hirschowitz has also started developing a theory of programming languages based on an algebraic notion called *playgrounds* to study programming languages and their semantics. Playgrounds are an algebraic gadget that describe the "rule of the game" of some operational semantics and produce process terms and strategies corresponding to it. Currently, the only known example of playground is CCS [**?**]. In a work in progress, the $\pi$-calculus is shown to also be an instance of playground. The initial goal of the internship was to show that the join-calculus is another instance of playground. Though that goal wasn't achieved during the internship, we have constructed the double category of plays (or candidate playground), which is a crucial step in the construction of a playground, and we have shown that it satisfies most of the playground axioms. Moreover, several playground axioms have been proved not only for the join-calculus, but for a general class of languages, which greatly generalizes the approach.

In a first part, we briefly discuss the categorical notions that are needed to work on playgrounds.

In the next two sections, we discuss two examples of categories of plays. The first example is that of MLL interaction nets, and is more of a toy example that is meant to be simple enough (yet interesting) for the reader to understand how the formal definitions indeed describe plays. The second one is a more involved example, that of the join-calculus.

In a last part, we will discuss one of the most important achievements of this internship, which is a correctness criterion that had already been formulated for the $\pi$-calculus and that is now general enough to encompass a large range of calculi.

# 2 Categorical Prerequisites

We will assume that the reader is familiar with basic notions of category theory (roughly up to Chapter 5 of [**?**]) such as functors, natural transformations, Yoneda embedding, limits and colimits, etc.

## 2.1 Limits and Colimits of Presheaves

We will use presheaves to define positions and plays, so we state here a well-known theorem that we will use a lot.

**Definition 1** (Presheaf)**.** *A (set-valued)* presheaf *on a category $\mathcal{C}$ is a functor $U : \mathcal{C}^{op} \to$ **Set**.*

We will denote $\widehat{\mathcal{C}} = [\mathcal{C}^{op}, \mathbf{Set}]$ the cateogry of presheaves over $\mathcal{C}$.

**Theorem 1.** *In a presheaf category, limits and colimits are computed pointwise.*

This theorem states that limits and colimits of presheaves "behave well", that is, have the same behavior as limits and colimits in **Set**. It comes from the following classical result [**?**] that expresses the fact that limits in functor categories are computed pointwise, provided the pointwise limits exist (and its dual for colimits):

**Theorem 2.** *If $S : J \to X^P$ (where $X^P$ is the category of functors from $P$ to $X$) is such that for every object $p \in P$ the composite $E_pS : J \to X$ (where $E_p : X^P \to X$ is the functor "evaluate at $p$": $E_pH = H_p$ for $H$ functor from $P$ to $X$, $E_p\sigma = \sigma_p : H_p \to H_{p'}$ for $\sigma : H \dashrightarrow H'$ natural transformation between $H$ and $H'$) has a limit $L_p$ with a limiting cone $\tau_p : L_p \dashrightarrow E_pS$, then there is a unique functor $L : P \to X$ with object function $p \mapsto L_p$ such that $p \mapsto \tau_p$ is a natural transformation $\tau : \Delta L \dashrightarrow S$ (where $\Delta : X \to X^J$); moreover, this $\tau$ is a limiting cone from the vertex $L \in X^P$ to the base $S : J \to X^P$.*

## 2.2 Finitely Presentable Presheaves

More precisely, we are interested in "finite" presheaves (because we are especially interested in finite plays on finite positions). A general notion of finiteness in category theory is that of finitely presentable object, and we state and prove here a characterization of finitely presentable presheaves in the case we are interested in.

**Definition 2** (Filtered Category)**.** *A category $\mathcal{I}$ is* filtered *if it is non-empty and:*

- *for any $x, y \in \mathcal{I}$, there exists $z \in \mathcal{I}$ and arrows $f : x \to z$ and $g : y \to z$;*
- *for any pair of parallel arrows $f, g : x \rightrightarrows y$ in $\mathcal{I}$, there exists an object $z$ and a morphism $h : y \to z$ such that $hf = hg$.*

A *filtered diagram* in a category $\mathcal{C}$ is a diagram $F : \mathcal{I} \to \mathcal{C}$ from a small filtered category; a colimit of such a diagram, when it exists, is called a *filtered colimit.*

**Definition 3** (Finitely Presentable Object)**.** *An object $X$ of a category $\mathcal{C}$ is* finitely presentable *if its hom-functor $\mathcal{C}(X, -) : \mathcal{C} \to$ **Set** preserves filtered colimits.*

This exactly means that, if we are given a filtered diagram $F : \mathcal{I} \to \mathcal{C}$ together with a colimiting cone $(f_i : F(i) \to Z)_{i \in \mathcal{I}}$, then:

- every morphism $k : X \to Z$ into the colimit $Z$ factors through one of the injections:

$$X \xrightarrow{k} Z = X \xrightarrow{h} F(i) \xrightarrow{f_i} Z \quad \text{for some } i \in \mathcal{I}$$

- this factorization is essentially unique, that is, if we also have:

$$X \xrightarrow{k} Z = X \xrightarrow{h'} F(i) \xrightarrow{f_i} Z$$

  then there is an object $j \in \mathcal{I}$ and an arrow $g : i \to j$ such that:

$$X \xrightarrow{h} F(i) \xrightarrow{F(g)} F(j) = X \xrightarrow{h'} F(i) \xrightarrow{F(g)} F(j)$$

The intuition behind finitely presentable objects is that they represent "finite" objects, e.g.:

- in **Set**, finitely presentable objects are exactly the finite sets;

- in **Grp** (respectively **Rng** and $R-$**Mod**), finitely presentable objects are exactly the groups (respectively rings and modules over $R$) that can be presented using only finitely many generators and finitely many relations between them.

In our case, we will only work with finitely presentable presheaves on graded categories (though some of the results still hold for arbitrary presheaves on graded categories).

**Definition 4** (Graded Category)**.** *A category* $\mathcal{C}$ *is* graded *if it comes equipped with a functor* $F : \mathcal{C} \to \omega$ *(where* $\omega$ *is the category with objects* $n \in \mathbb{N}$ *and arrows generated by the order on* $\mathbb{N}$*) that reflects identities. The* dimension *of an object* $c \in \mathcal{C}$ *is* $F(c)$*.*

The notion of dimension can be extended to presheaves on $\mathcal{C}$: the empty presheaf has dimension 0, and a non-empty presheaf $X$ has dimension $d$ where $d$ is the greatest integer such that there is an object $c \in \mathcal{C}$ with $X(c) \neq \emptyset$.

**Proposition 1.** *Finitely presentable presheaves over graded categories have finite dimension.*

*Proof.* Assume that $U$ is finitely presentable, and let $U_i$ be the subpresheaf of $U$ defined by:

$$U_i(c) = \begin{cases} U(c) & \text{if } c \text{ has dimension at most } i \\ \emptyset & \text{otherwise} \end{cases}$$

($U_i$ is indeed a presheaf because $\mathcal{C}$ is graded).

We define the following filtered diagram:

$$U_0 \overset{\subseteq}{\to} U_1 \overset{\subseteq}{\to} \ldots \overset{\subseteq}{\to} U_n \overset{\subseteq}{\to} \ldots$$

whose colimit is $U$. Since $U$ is finitely presentable, this means that $\mathrm{id}_U : U \to U$ must factor through one of the $U_i$s. Therefore, $U$ has finite dimension. □

**Definition 5** (Category of elements)**.** *The* category of elements $\mathrm{el}(F)$ *of a presheaf* $F : \mathcal{C}^{op} \to$ ***Set*** *has:*

- *objects: pairs* $(c, x)$ *where* $c$ *is an object of* $\mathcal{C}$ *and* $x \in F(c)$
- *morphisms:* $u : (c, x) \to (c', x')$ *if* $u : c \to c'$ *is a morphism of* $\mathcal{C}$ *such that* $F(u)(x') = x$*.*

Oftentimes, we omit the $c$ in $(c, x)$, as it is obvious from the context or non-important.

In all the cases we have in mind, the category $\mathcal{C}$ is even simpler: not only is it graded, but also, for any object $c \in \mathcal{C}$, $\mathcal{C}(-, c)$ is finite. In this case, finitely presentable presheaves are exactly the presheaves whose categories of elements are finite.

**Proposition 2.** *If* $\mathcal{C}$ *is graded and for any* $c \in \mathcal{C}$*,* $\mathcal{C}(-, c)$ *is finite, then the finitely presentable presheaves are exactly the presheaves whose categories of elements are finite.*

*Proof.* To prove this, we will use an equivalent characterization of finitely presentable objects (this characterization is proven equivalent in [**?**]), which is that its hom-functor preserves directed colimits (the characterization is the same as for filtered limits, namely that there is a factorization that is essentially unique, but with directed diagrams).

Let us prove both implications of the proposition:

- assume that $X$ is finitely presentable: first, we want to show that $X$ is colimit of its subpresheaves $X_i$ with finite categories of elements. Since $\mathcal{C}$ is graded and for any $c \in \mathcal{C}$, $\mathcal{C}(-, c)$ is finite, each representable presheaf has a finite category of elements, therefore, if there is a cocone from $(X_i)_i$ to some $Y$, there is a morphism from $X$ to $Y$ that commutes with this cocone (the cocone itself, because it contains the representabale presheaves, gives the image of each element of $X$).

Moreover, this is a directed diagram, so $X$ is a directed colimit of this diagram. Therefore, $\mathrm{id}_X : X \to X$ must factor through one of the $X_i$s, so $X$ has a finite category of elements.

- assume that $X$ has a finite category of elements: let $(D_i \stackrel{f_i}{\to} C)_{i\in\mathcal{I}}$ be a directed colimit and $k : X \to C$; we want $k$ to factor through one of the $D_i$s. Take an element $x_n \in X(c_n)$, we know that there is a $j_n$ such that $k_{c_n}(x_n)$ lies in the image of $f_{j_n}$. Since $X$ has a finite category of elements $x_1, ..., x_N$ and the diagram is filtered, we can find an $m \in \mathcal{I}$ such that each $D_{j_n} \stackrel{g_n}{\to} D_m$, therefore $k$ factors through $D_m$.

  To show that this factorization is essentially unique, it is sufficient to show that for any $d, d' \in D_i(c)$, if $(f_i)_c(d) = (f_i)_c(d')$, then there is $j \in \mathcal{I}$ and $g : D_i \to D_j$ such that $g_c(d) = g_c(d')$, which is a property of directed colimits in **Set** [?], and since colimits are computed pointwise, it is also true for presheaves.

□

### 2.3 Factorization Systems

Though factorization systems are not used to prove anything written in this report, they are a key elements to proving one of the playground axioms (namely that the vertical codomain functor of the double category of plays is a fibration). Though it is not proved yet for the join-calculus, we contributed a little bit by correcting a proof by Joyal.

**Definition 6** (Factorization System)**.** *A pair $(\mathcal{L}, \mathcal{R})$ of classes of maps in a category $\mathcal{C}$ is a* factorisation system *if the following conditions are satisfied:*

- *every morphism $f : A \to B$ admits a factorisation $f = pu : A \to C \to B$ with $u \in \mathcal{L}$ and $p \in \mathcal{R}$ and this factorisation is unique up to unique isomorphism;*
- *the classes $\mathcal{L}$ and $\mathcal{R}$ contain the isomorphisms and are closed under composition.*

$\mathcal{L}$ is called the *left class* and $\mathcal{R}$ the *right class*. The uniqueness condition in the definition means that for any pair of $(\mathcal{L}, \mathcal{R})$-factorization of $f$, $f = pu : A \to C \to B$ and $f = qv : A \to D \to B$, there is a unique isomorphism $i : C \to D$ such that the diagram in Figure 1 commutes.

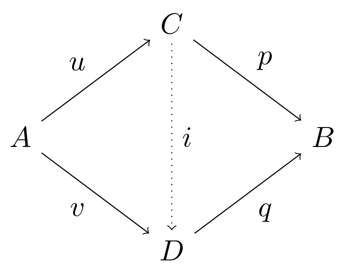



Figure 1: Illustration of the uniqueness condition

**Definition 7** (Left and Right Cancellation Properties)**.** *A class $\mathcal{M}$ of maps has the* left cancellation property *if for all $u : A \to B$, $v : B \to C$, if $vu \in \mathcal{M}$ and $v \in \mathcal{M}$, then $u \in \mathcal{M}$.*

*Dually, a class $\mathcal{M}$ of maps has the* right cancellation property *if for all $u : A \to B$, $v : B \to C$, if $vu \in \mathcal{M}$ and $u \in \mathcal{M}$, then $v \in \mathcal{M}$.*

**Proposition 3.** *The intersection of the classes of a factorization system $(\mathcal{L}, \mathcal{R})$ in a category $\mathcal{C}$ is the class of isomorphisms. Moreover, $\mathcal{L}$ has the right cancellation property and $\mathcal{R}$ has the left cancellation property.*

For the proof, see Joyal's CatLab [**?**].

**Definition 8** (Orthogonality)**.** *A map $u : A \to B$ in a category $\mathcal{C}$ is left orthogonal to a map $f : X \to Y$ (or $f$ is right orthogonal to $u$) if every commutative square such as the one in Figure 2 has a unique diagonal filler $d : B \to X$ ($du = x$ and $fd = y$). We denote this relation by $u \perp f$.*

If $\mathcal{M}$ is a class of maps of $\mathcal{C}$, we denote by $\mathcal{M}^\perp$ (respectively ${}^\perp\mathcal{M}$) its right (respectively left) orthogonal complement, i.e. the class of maps right (respectively left) orthogonal to all maps in $\mathcal{M}$.

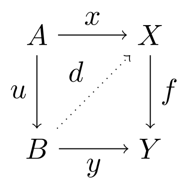


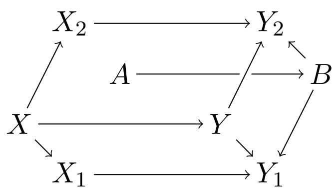
Figure 2: A commutative square and its diagonal filler

**Proposition 4.** *Let $\mathcal{M}$ be a class of maps in $\mathcal{C}$. Then $\mathcal{M}^\perp$ is closed under limits, composition, base changes and it has the left cancellation property. Dually, ${}^\perp\mathcal{M}$ is closed under colimits, composition, cobase changes and it has the right cancellation property.*

*Proof.* Most of the proof can be found in Joyal's CatLab [**?**]. However, the proof that $\mathcal{M}^\perp$ is closed under limits is wrong, as it relies on the fact that the functor $\mathrm{Sq}(u, -) : \mathcal{C}^I \to [I \times I, \mathbf{Set}]$ preserves limits, where $I$ is the category $0 \to 1$ and, for $u : A \to B$ and $f : X \to Y$, $\mathrm{Sq}(u, f)$ is the following square:

$$\begin{array}{ccc} \mathcal{C}(B,X) & \xrightarrow{-\circ u} & \mathcal{C}(A,X) \\ {\scriptstyle f\circ -}\downarrow & & \downarrow{\scriptstyle f\circ -} \\ \mathcal{C}(B,Y) & \xrightarrow[-\circ u]{} & \mathcal{C}(A,Y) \end{array}$$

However, this assumption is wrong, as the following example shows. Let us work in the partial order category generated by the following graph:



Its category of morphisms has for objects pairs $(x, y)$ with $x \leq y$ and there is a morphism $(x, y) \to (x', y')$ when $x \leq x'$ and $y \leq y'$. In this category, we verify that $(X, Y)$ is a product of $(X_1, Y_1)$ and $(X_2, Y_2)$, but $\mathrm{Sq}((A, B), (X, Y))$ is not a product of $\mathrm{Sq}((A, B), (X_1, Y_1))$ and $\mathrm{Sq}((A, B), (X_2, Y_2))$ because:

$$\mathrm{Sq}((A,B),(X,Y)) = \begin{array}{ccc} \emptyset & \to & \emptyset \\ \downarrow & & \downarrow \\ \emptyset & \to & \emptyset \end{array} \quad \mathrm{Sq}((A,B),(X_1,Y_1)) = \mathrm{Sq}((A,B),(X_2,Y_2)) = \begin{array}{ccc} \emptyset & \to & \emptyset \\ \downarrow & & \downarrow \\ 1 & \to & 1 \end{array}$$

and limits are computed pointwise in functor categories.

Here's an alternative proof of the fact that $\mathcal{M}^\perp$ is closed under limits.

We redefine $\mathrm{Sq}(u, -) : \mathcal{C}^I \to [I, \mathbf{Set}]$ (note that its type has changed). For $u : A \to B$ and $f : X \to Y$:

$$\mathrm{Sq}(u,f) = \left\{ \begin{array}{ccc} \mathcal{C}(B,X) & \to & [I,\mathcal{C}](u,f) \\ d & \mapsto & (du, fd) \end{array} \right.$$

and this new functor $\mathrm{Sq}(u, -)$ preserves limits: since limits are computed pointwise in presheaf categories, we only have to verify it for each point:

- for 0: the functor is equal to $[I, \mathcal{C}] \xrightarrow{\mathrm{dom}} \mathcal{C} \xrightarrow{y'_B} \mathbf{Set}$ (where $y'_B$ is the covariant Yoneda functor at $B$), which preserves limits as a composite of functors that preserve limits;

- for 1: the functor is equal to $[I, \mathcal{C}] \xrightarrow{y'_u} \mathbf{Set}$, which preserves limits.

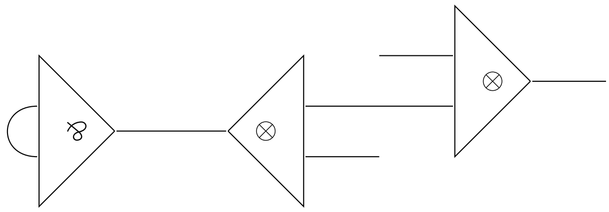


Figure 3: The graphical representation of an interaction net

The rest of the proof is similar to that of Joyal, since $u \perp f$ if and only if $\mathrm{Sq}(u, f)$ is an isomorphism and isomorphisms of $\mathcal{C}$ form a full reflexive subcategory of $[I, \mathbf{Set}]$. □

The following theorem by Bousfield [**?**] gives a way to build factorization systems:

**Theorem 3.** *In a locally presentable category, given a class $\mathcal{T}$ of morphisms, $({}^{\perp}(\mathcal{T}^{\perp}), \mathcal{T}^{\perp})$ forms a factorization system.*

# 3 Plays in MLL

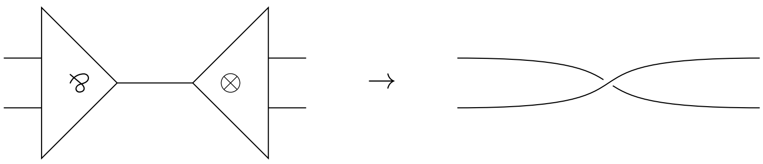
## 3.1 MLL Interaction Nets

**Definition 9.** *An* MLL interaction net *is given by:*

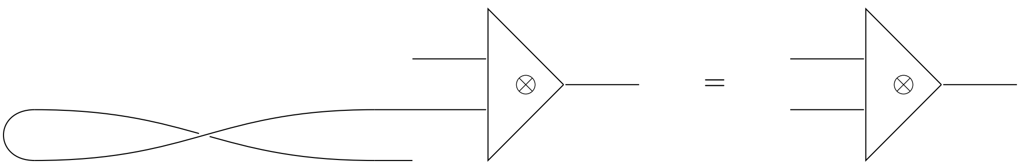
- *a finite set of* tensor gates *and a finite set of* par gates*, each gate has three* ports *numbered 1, 2, and 3;*
- *a finite set of* wires *that form a pseudo partition (a partition in which empty sets are authorized) of the ports, with each wire containing 0, 1 or 2 ports (wires that are empty sets of ports are called loops).*

There is a nice graphical representation of MLL interaction nets. For example, the one shown in Figure 3 represents an interaction net with a single par gate, two tensor gates, and six wires partitioning the ports.

Graphically, the reduction rule of MLL interaction nets is the following (there is a formal definition, but it is not intuitive and can easily be found after having seen the graphical definition):



It may be applied in any "context", that is, as long as there is a wire regrouping the 1st port of a par gate and the 1st port of a tensor gate, the rule can be applied, e.g., if the rule is applied to the interaction net in Figure 3, we get:



The next subsection aims at giving a categorical formalism to describe MLL interaction nets and their reductions.

### 3.2 The Base Double Category

Here, we define the base pseudo double category $\mathcal{D}$ for MLL. It is a sub-double category of the pseudo double category of cospans over some presheaf category.

We start by defining the base category $\mathcal{C}$ on which we will take presheaves.

Consider the graph $G_{\mathcal{C}}$ with

- four vertices: $*$, $\parr$, $\otimes$ and cut;
- edges $s_1, s_2, s_3 : * \to \parr$, $s_1, s_2, s_3 : * \to \otimes$, $p : \parr \to \text{cut}$ and $t : \otimes \to \text{cut}$.

Now, we define $\mathcal{C}$ to be the free category on $G_{\mathcal{C}}$, modulo the relations:

$$p \circ s_i = t \circ s_i \quad i \in \{1, 2, 3\}$$

The intuition behind this construction is the following.

The object $*$ represents wires in MLL interaction nets.

The object $\parr$ represents par gates. The morphisms $s_1, s_2, s_3$ are the three ports of each gate, so each associates a wire to each gate (in a presheaf $U$ over this category, the arrows are functions $U(s_i) : U(\parr) \to U(*)$, so they do associate wires to gates). The object $\otimes$ represents tensor gates, and the morphisms $s_i$ the three ports of the gate.

The object cut represents a cut-elimination step in the MLL interaction net. The morphisms $p$ and $t$ show that both a par gate and a tensor gate are part of the cut-elimination step. The equality $p \circ s_1 = t \circ s_1$ means that, for a cut-elimination step to occur, the par gate and the tensor gate have to face each other. The other two equations are more complicated to understand, but their utility will be demonstrated later.

We equip the objects of $\mathcal{C}$ with a dimension: $*$ has dimension 0, $\parr$ and $\otimes$ have dimension 1 and cut has dimension 2. A *channel* is an object of dimension 0, a *player* is an object of dimension 1, and a *move* is an object of dimension greater than 1.

A *position* is a presheaf of dimension at most 1 such that at most two gates point to a given wire, an *interface* is a presheaf of dimension 0.

Note that, for each object $c$ of $\mathcal{C}$, $\mathcal{C}(-, c)$ is finite and that, with the definition of dimension above, $\mathcal{C}$ is a graded category.

We give concrete examples of positions and plays in the next subsection.

Now, consider the pseudo double category [?] $\mathcal{D}^0$ with:

- positions as objects;
- natural transformations $h : X \to Y$ that are injective above dimension 0 (the components of $h$ are injective, except possibly $h_*$) as horizontal morphisms;
- cospans $Y \to U \leftarrow X$ in $\widehat{\mathcal{C}}$ as vertical morphisms;
- the commutative diagrams shown in Figure 4 as double cells, where $h$, $k$ and $l$ injective above dimension 0.

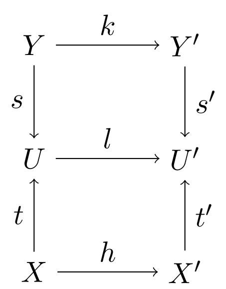
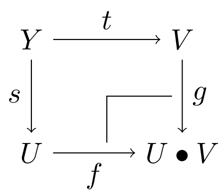


Figure 4: A double cell

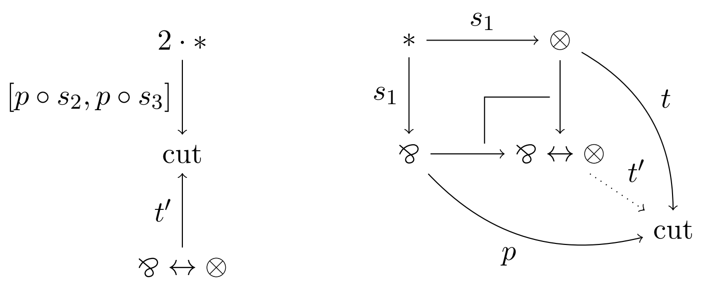
This forms a pseudo double category with the following compositions:

- the horizontal composition is obvious (it is the composition of natural transformations);
- the vertical composition is the composition of cospans in categories with pushouts, i.e. the composition of $Z \xrightarrow{s'} V \xleftarrow{t'} Y$ and $Y \xrightarrow{s} U \xleftarrow{t} X$ is $Z \xrightarrow{gs'} U \bullet V \xleftarrow{ft} X$, where $U \bullet V$ is defined by the pushout:




  This composition is associative only up to isomorphism, which is why $\mathcal{D}^0$ is a pseudo double category;
- the horizontal composition of double cells is given by the composition of natural transformations;
- the vertical composition of double cells is given by the composition of cospans and the universal property of the pushout.

Let us call the following cospan (on the left) a *seed*, where the position $\parr \leftrightarrow \otimes$ is obtained as the pushout on the right and the morphism $t'$ is obtained by universal property of the pushout:




Note that the seed hints the fact that we should have $p \circ s_2 = t \circ s_2$ and $p \circ s_3 = t \circ s_3$, so that the morphism $2 \cdot * \to \text{cut}$ we chose is not arbitrary.

The intuition behind the seed is that it is a move that goes from an initial position (the bottom position of the cospan) to a final position (the top position of the cospan), and that the bottom position is the minimal position necessary so that the move can be played. We show in the next subsection what these positions concretely correspond to in MLL interaction nets and how cut does indeed model cut-elimination.

From here on, the construction of the pseudo double category of plays $\mathcal{D}$ is automated.

The *canonical interface* of a seed $Y \to U \leftarrow X$ is the pullback of $t$ along $s$ in dimension 0. It obviously comes with the commuting diagram shown in Figure 5.

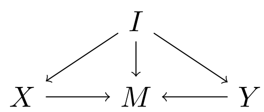



Figure 5: Canonical interface

A move can be played in an arbitrary position, as long as there is a copy of the seed corresponding to the move in it. The intuition behind the canonical interface is that it represents the channels that may link the seed to the rest of the position. Again, we will give concrete examples of how this works in the next section.

In the case of cut, we get that $I = 4 \cdot *$, i.e. that there are four wires that may link the initial position of the seed to the rest of the position when a cut-elimination step occurs, which is what we were expecting (there are four wires that can link the initial position to the rest of an interaction net). Had we not forced that $p \circ s_2 = t \circ s_2$ and $p \circ s_3 = t \circ s_3$, we would have found that $I = 2 \cdot *$, which is not what we were expecting.

A move is a cospan obtained by pushout of a seed (in the case of MLL, there is only one seed) along some $h : I \to Z$ in $\mathcal{D}^0_h$ (such that $X$ is a position) as shown in Figure 6, where the dotted arrows are obtained by universal property of pushout.

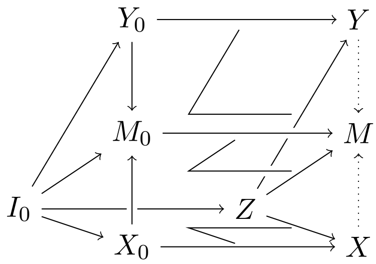


Figure 6: Construction of a move

**Proposition 5.** *In the diagram above, $Y$ is a position.*

**Definition 10** (Play)**.** *A play is any cospan that is isomorphic to a (possibly empty) composite of moves. We note $\mathcal{D}$ the sub-double category of $\mathcal{D}^0$ obtained by restricting vertical morphisms to plays.*

### 3.3 An Example

To give an idea of why what we call *positions* indeed represent MLL interaction nets, let us give an example of a net, show what the corresponding presheaf is, and compute its category of elements.

Let us take the position shown in Figure 3.

This position corresponds to the following presheaf $X$ on $\mathcal{C}$:

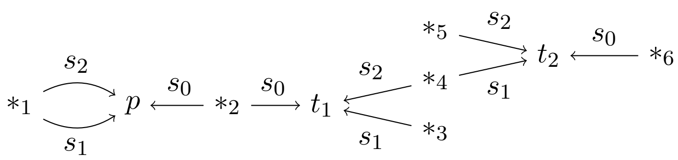
$X(*) = \{*_1, *_2, *_3, *_4, *_5, *_6\}, X(⅋) = \{p\}, X(\otimes) = \{t_1, t_2\}, X(\text{cut}) = \emptyset$

$p \cdot s_0 = t_1 \cdot s_0 = *_2, p \cdot s_1 = p \cdot s_2 = *_0, t_1 \cdot s_1 = *_3, t_1 \cdot s_2 = t_2 \cdot s_1 = *_4, t_2 \cdot s_2 = *_5, t_2 \cdot s_0 = *_6$

where $x \cdot s$ is a notation for $X(s)(x)$.

$X$ corresponds exactly to the interaction net above, as it exactly encodes the fact that there are 6 wires, 1 tensor gate and 2 par gates, and how the gates are connected to the wires (we assume that $s_0$ is the "front" port, $s_1$ the "bottom" port and $s_2$ the "top" port).

But there is a visual way to represent a presheaf, that is to compute its category of elements. If we do this for $X$, we find the following category:

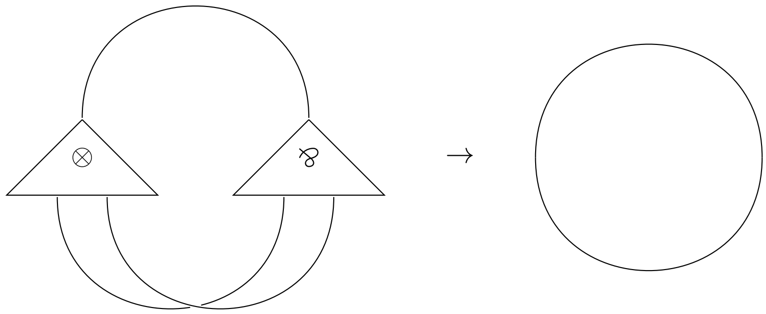



which is exactly another representation of the MLL interaction net in Figure 3.

Now, let us show what $⅋ \leftrightarrow \otimes$ is. By construction of pushout in presheaf categories, it consists of a par gate and tensor gate, whose ports $s_0$ point to the same wire. Therefore, it is exactly the initial position for a cut-elimination step. $2 \cdot *$ is just a position with two wires, exactly the final position in a cut-elimination step.

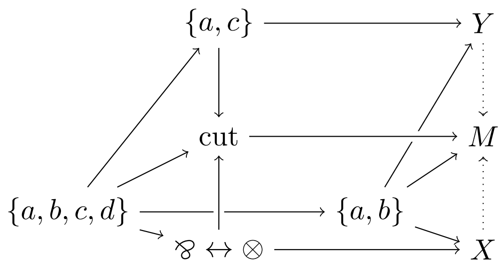
Remember that the canonical interface for cut is $4 \cdot *$, which we will abusively denote as a set $\{a, b, c, d\}$. We can also write $X(*) = \{a, b, c, d, e\}$ ($t \cdot s_1 = a$, $t \cdot s_2 = b$, $p \cdot s_1 = c$, $p \cdot s_2 = d$, and $t \cdot s_0 = p \cdot s_0 = e$) and $Y(*) = \{a, c\}$. The morphisms from the canonical interface to $X$ and $Y$ that commute with the morphisms in the seed are $l : I \to X$, $l(x) = x$, and $k : I \to Y$, $k(a) = k(d) = a$, $k(b) = k(c) = c$.

Now, we give a simple example of play to illustrate how the category of plays $\mathcal{D}$ indeed corresponds to reductions in MLL interaction nets.




The base position is described by the presheaf $X$: $X(*) = \{a, b, e\}$, $X(⅋) = \{p\}$, $X(\otimes) = \{t\}$, $p \cdot s_0 = t \cdot s_0 = a$, $p \cdot s_1 = t \cdot s_1 = b$, $p \cdot s_2 = t \cdot s_2 = e$.

The canonical interface for the cut move is $\{a, b, c, d\}$, with the morphisms. If we pushout the seed along the morphism $f : \{a, b, c, d\} \to \{a, b\}$, $f(a) = f(c) = a$, $f(b) = f(d) = b$, we have the diagram:




By construction of pushouts in presheaf categories, the bottom-right presheaf in the diagram above is indeed $X$ (because it is $⅋ \leftrightarrow \otimes$ in which $c$ is identified with $a$ and $d$ with $b$). And, by construction of pushouts in presheaf categories, the top position $Y$ is $\{a, b, c, d\}$ where $c$ is identified with $a$ and $d$ with $b$ (by $\{a, b, c, d\} \to \{a, b\}$), but $a$ is also identified with $d$ (by $\{a, b, c, d\} \to \{a, c\}$), so $Y = \{a\}$, i.e. a single wire, which is what we were expecting.

# 4 Plays in Join-Calculus

## 4.1 The Join-Calculus

The join-calculus is a process calculus that gives a model of concurrency. It has the same expressive power as the asynchronous $\pi$-calculus, up to their weak output-only barbed congruences [**?**].

The syntax presented here is that of the core recursive join-calculus (which has the same expressive power as the join-calculus, up to observational congruence [**?**]):

$$\begin{array}{rclll} P & = & x\langle u\rangle \quad | \quad P \,|\, P \quad | \quad \text{def } D \text{ in } P & \text{processes} \\ D & = & x\langle u\rangle \,|\, y\langle v\rangle \rhd P & \text{definitions} \end{array}$$

where $x$, $y$, $u$ and $v$ are variable names taken from an infinite set of names.

In the term def $x\langle u\rangle \,|\, y\langle v\rangle \rhd P$ in $Q$, the scope of $x$ and $y$ is the whole term, while the scope of $u$ and $v$ is $P$.

The defined variables of a definition $D = x\langle u\rangle \mid y\langle v\rangle \triangleright P$ are $\mathrm{dv}(D) = \{x, y\}$.

The free variables of a process $P$ are recursively defined in the following way:

$$\begin{array}{rcl} \mathrm{fv}(x\langle u\rangle) & = & \{x, u\} \\ \mathrm{fv}(P \mid Q) & = & \mathrm{fv}(P) \cup \mathrm{fv}(Q) \\ \mathrm{fv}(\mathrm{def}\ x\langle u\rangle \mid y\langle v\rangle \triangleright P \ \mathrm{in}\ Q) & = & (\mathrm{fv}(Q) \cup (\mathrm{fv}(P)\backslash\{u, v\}))\backslash\{x, y\} \end{array}$$

The structural congruence $\equiv$ on these terms is the smallest relation such that for all processes $P$, $Q$, $R$, $S$ and definitions $D$ and $D'$ such that $\mathrm{dv}(D)$ and $\mathrm{dv}(D')$ only contain fresh variables:

$$\begin{array}{rcl} P \mid Q & \equiv & Q \mid P \\ (P \mid Q) \mid R & \equiv & P \mid (Q \mid R) \\ P \mid \mathrm{def}\ D \ \mathrm{in}\ Q & \equiv & \mathrm{def}\ D \ \mathrm{in}\ P \mid Q \\ \mathrm{def}\ D \ \mathrm{in}\ \mathrm{def}\ D' \ \mathrm{in}\ P & \equiv & \mathrm{def}\ D' \ \mathrm{in}\ \mathrm{def}\ D \ \mathrm{in}\ P \\ P \equiv_\alpha Q & \Longrightarrow & P \equiv Q \\ P \equiv Q & \Longrightarrow & P \mid R \equiv Q \mid R \\ P \equiv Q, R \equiv S & \Longrightarrow & \mathrm{def}\ J \triangleright R \ \mathrm{in}\ P \equiv \mathrm{def}\ J \triangleright S \ \mathrm{in}\ Q \end{array}$$

where $\equiv_\alpha$ is $\alpha$-conversion (i.e., renaming of bound variables) and $J$ is a *join pattern* $x\langle u\rangle \mid y\langle v\rangle$.

Now, we define the transitions of this system as the smallest relation such that for every definition $D = x\langle u\rangle \mid y\langle v\rangle \triangleright R$:

$$x\langle s\rangle \mid y\langle t\rangle \xrightarrow{D} R\{u \leftarrow s, v \leftarrow t\}$$

and, for every transition $P \xrightarrow{\delta} P'$ (where $\delta$ is either a definition $D$ or $\tau$):

$$\begin{array}{rcll} P \mid Q & \xrightarrow{\delta} & P' \mid Q & \\ \mathrm{def}\ D \ \mathrm{in}\ P & \xrightarrow{\delta} & \mathrm{def}\ D \ \mathrm{in}\ P' & \text{if } \mathrm{fv}(D) \cap \mathrm{dv}(\delta) = \emptyset \\ \mathrm{def}\ D \ \mathrm{in}\ P & \xrightarrow{\tau} & \mathrm{def}\ D \ \mathrm{in}\ P' & \text{if } \delta = D \\ Q & \xrightarrow{\delta} & Q' & \text{if } P \equiv Q \text{ and } P' \equiv Q' \end{array}$$

The reduction relation is defined as the transitions of this labelled translation system that are labelled by $\tau$.

Here are some examples of processes written in core recursive join-calculus:

$$\mathrm{def}\ x\langle v\rangle \mid y\langle \kappa\rangle \triangleright \kappa\langle v\rangle \ \mathrm{in}\ P$$
$$\mathrm{def}\ \mathit{once}\langle a\rangle \mid y\langle v\rangle \triangleright x\langle v\rangle \ \mathrm{in}\ y\langle 1\rangle \mid y\langle 2\rangle \mid y\langle 3\rangle \mid \mathit{once}\langle a\rangle$$

The first one implements $\pi$-calculus-like channels, as values are sent on $x$ and requests for values are sent on $y$, to be matched using the definition. The second one implements some kind of non-determinism, as only one of $x\langle 1\rangle$, $x\langle 2\rangle$, and $x\langle 3\rangle$ can be spawned, using $\mathit{once}\langle a\rangle$ as a lock.

## 4.2 The Candidate Playground

Here, we define the base pseudo double category $\mathcal{D}$ that is the candidate playground for the join-calculus. It is a sub-double category of the pseudo double category of cospans over some presheaf category.

We start by defining the base category $\mathcal{C}$ on which we will take presheaves.

Consider the graph $G_{\mathcal{C}}$ with vertices:

- a vertex $*$

- for every $n \in \mathbb{N}$, a vertex $[n]$
- for every $n \geq 2$, a vertex $\langle n \rangle$
- for every $n \in \mathbb{N}$, vertices $\pi_n^l$, $\pi_n^r$, $\pi_n$, $\delta_n^l$, $\delta_n^r$, $\delta_n$, $\text{reac}_n$, and $\text{freac}_n$
- for every $n \in \mathbb{N}$ and every $i, j \in \{1, ..., n\}$, a vertex $\sigma_{n,j\langle i \rangle}$
- for every $n \in \mathbb{N}$, every $p \in \mathbb{N}$, every $i, j \in \{1, ..., p\}$, every $q \in \mathbb{N}$ and every $k, m \in \{1, ..., q\}$, a vertex $\tau_{n,p,j\langle i \rangle,q,m\langle k \rangle}$

and edges:

- for every vertex $[n]$, edges $s_1, ..., s_n : * \to [n]$
- for every vertex $\langle n \rangle$, edges $s_1, .., s_n : * \to \langle n \rangle$
- for every vertex $\pi_n^l$, edges $s, t : [n] \to \pi_n^l$ (idem for $\pi_n^r$)
- for every vertex $\pi_n$, an edge $l : \pi_n^l \to \pi_n$ and an edge $r : \pi_n^r \to \pi_n$
- for every vertex $\delta_n^l$, an edge $t : [n] \to \delta_n^l$ and an edge $s : \langle n+2 \rangle \to \delta_n^l$
- for every vertex $\delta_n^r$, an edge $t : [n] \to \delta_n^r$ and an edge $s : [n+2] \to \delta_n^r$
- for every vertex $\delta_n$, an edge $l : \delta_n^l \to \delta_n$ and an edge $r : \delta_n^r \to \delta_n$
- for every vertex $\text{reac}_n$, an edge $t : \langle n \rangle \to \text{reac}_n$ and an edge $s : [n+2] \to \text{reac}_n$
- for every vertex $\text{freac}_n$, an edge $f : \text{reac}_n \to \text{freac}_n$
- for every vertex $\sigma_{n,j\langle i \rangle}$, an edge $t : [n] \to \sigma_{n,j\langle i \rangle}$
- for every vertex $\tau_{n,p,j\langle i \rangle,q,m\langle k \rangle}$, an edge $d : \text{freac}_n \to \tau_{n,p,j\langle i \rangle,q,m\langle k \rangle}$, an edge $l : \sigma_{p,j\langle i \rangle} \to \tau_{n,p,j\langle i \rangle,q,m\langle k \rangle}$, and an edge $r : \sigma_{q,m\langle k \rangle} \to \tau_{n,p,j\langle i \rangle,q,m\langle k \rangle}$

Now, we define $\mathcal{C}$ to be the free category on $G_{\mathcal{C}}$, modulo the relations:

$$\begin{array}{ll}
t \circ s_i = s \circ s_i & \text{for any } \pi_n^l, \pi_n^r, \delta_n^l, \delta_n^r, \text{reac}_n,\ i \in n \\
l \circ s \circ s_i = r \circ s \circ s_i & \text{for any } \delta_n,\ i \in \{n+1, n+2\} \\
l \circ t = r \circ t & \text{for any } \pi_n, \delta_n \\
f \circ d \circ t \circ s_{n-1} = l \circ t \circ s_j \quad f \circ d \circ t \circ s_n = r \circ t \circ s_m & \multirow{2}{*}{for any $\tau_{n,p,j\langle i \rangle,q,m\langle k \rangle}$} \\
f \circ d \circ s \circ s_{n+1} = l \circ t \circ s_i \quad f \circ d \circ s \circ s_{n+2} = r \circ t \circ s_k &
\end{array}$$

The reasoning behind the construction of this category is the following.

The object $*$ represents channels on which the processes communicate.

The objects $[n]$ are processes that can communicate on $n$ channels, the $s_i : * \to [n]$ represent the channels on which the process can communicate (since positions will be modelled as presheaves, the $s_i$'s will go from a process to a channel).

The objects $\langle n \rangle$ are definitions that know $n$ channels and listen on channels $n-1$ and $n$, the $s_i$'s represent the channels it knows.

The object $\sigma_{n,j\langle i \rangle}$ represents a process that knows $n$ channels and sends the name $i$ on channel $j$, i.e. that a process can be of the form $x\langle u \rangle$.

The object $\pi_n$ represents the move of a process that knows $n$ channels and forks into two different processes, i.e. that a process can be of the form $P_1 \,|\, P_2$ ($\pi_n^l$ and $\pi_n^r$ are the left and right "half-fork" moves, with $l$ and $r$ being morphisms that link a fork move to its left and right half-fork moves). The morphisms $t$ (target) and $s$ (source) represent respectively the player before and after the move (note that "source" and "target" may sound counter-intuitive). We require that $t \circ s_i = s \circ s_i$ because the avatars of the forking player know exactly the same channels as this player, and the fact that $l \circ t = r \circ t$ is because $\pi_n$ is a forking move, so there is only one player before the move is made.

The object $\delta_n$ represents the move of a process that knows $n$ channels and forks into a definition that listens on two new channels $n+1$ and $n+2$ and another process (the two channels it creates are shared between the definition and the process), i.e. that a process can be of the form def $D$ in $P$ ($\delta^l_n$ and $\delta^r_n$ are the left and right half-fork moves). The morphism equations required express exactly the same constraints as for $\pi_n$, plus that the two new channels are indeed shared between the definition and the process.

The object $\tau_{n,p,j\langle i\rangle,q,m\langle k\rangle}$ represents the synchronization move between a definition that listens on channels $n-1$ and $n$ and two processes that send the channel names $i$ and $k$ on channels $j$ and $m$ respectively ($\text{reac}_n$ is the part of the synchronization that creates a new process from a definition and $\text{freac}_n$ is here for technical purposes). The morphism equations required express the fact that channels $n-1$ and $n$ (on which the definition listens) are indeed channels $j$ and $m$ (on which the names are sent) and that channels $n+1$ and $n+2$ of the newly created process are channels $i$ and $k$ (that were sent to the definition).

We equip the objects of $\mathcal{C}$ with a dimension: $*$ has dimension 0, all $[n]$ and $\langle n\rangle$ have dimension 1, all $\pi^l_n$, $\pi^r_n$, $\delta^l_n$, $\delta^r_n$, $\text{reac}_n$ and $\sigma_{n,j\langle i\rangle}$ have dimension 2, all $\pi_n$ and $\delta_n$ have dimension 3, and all $\tau_{n,p,j\langle i\rangle,q,m\langle k\rangle}$ have dimension 4. A *channel* is an object of dimension 0, a *player* is an object of dimension 1, and a *move* is an object of dimension greater than 1.

Note that for any $c \in \mathcal{C}$, $\mathcal{C}(-,c)$ is finite, and that, with the definition of dimension given above, $\mathcal{C}$ is a graded category. Therefore, finitely presentable presheaves are exactly the presheaves with finite categories of elements. Plays represented by finitely presentable presheaves may be understood as finite plays on finite positions.

We define *positions* as presheaves of dimension at most 1 and *interfaces* as presheaves of dimension 0.

We define the pseudo double category $\mathcal{D}^0$ in the same way as in MLL. The construction of $\mathcal{D}$ will also be the same as in MLL, the only thing that changes is what cospans the seeds are.

We define seeds to be the cospans shown in Figure 7, where the positions $n \mid n$, $n+2 \rhd n+2$, $p \overset{j\langle i\rangle}{\rightsquigarrow} n \overset{m\langle k\rangle}{\leftsquigarrow} q$, $n \trianglerighteq n+2$, and $n \trianglerighteq n+2, p, q$ are defined as shown in Figure 8, and the corresponding morphisms $t'$ and $s'$ are defined by the universal properties of these colimits.

Just like in the case of MLL, the canonical interface of a seed $Y \to U \leftarrow X$ is defined as the pullback of $t$ along $s$ in dimension 0, and it has the same commuting diagram.

A move is a cospan obtained by pushout of a seed along some $h : I \to Z$ in $\mathcal{D}^0_h$ in the same way as in MLL.

A *basic move* is a move isomorphic to one of the seeds $\text{reac}_n$, $\pi^l_n$, $\pi^r_n$, $\delta^l_n$, and $\delta^r_n$.

**Definition 11** (Play)**.** *A play is any cospan that is isomorphic to a (possibly empty) composite of moves. We note $\mathcal{D}$ the sub-double category of $\mathcal{D}^0$ obtained by restricting vertical morphisms to plays.*

**Proposition 6** (Views)**.** *For any move $Y \to M \leftarrow X$ and player $x : p \to Y$ (where $p = [n]$ or $p = \langle n\rangle$), there exists a play $p \to V^{x,M} \leftarrow p^{x,M}$ and double cell $\alpha^{x,M} : V^{x,M} \to M$ with $\text{dom}(\alpha^{x,M}) = x$, which either is a basic move or has length 0, and which is unique up to canonical isomorphism.*

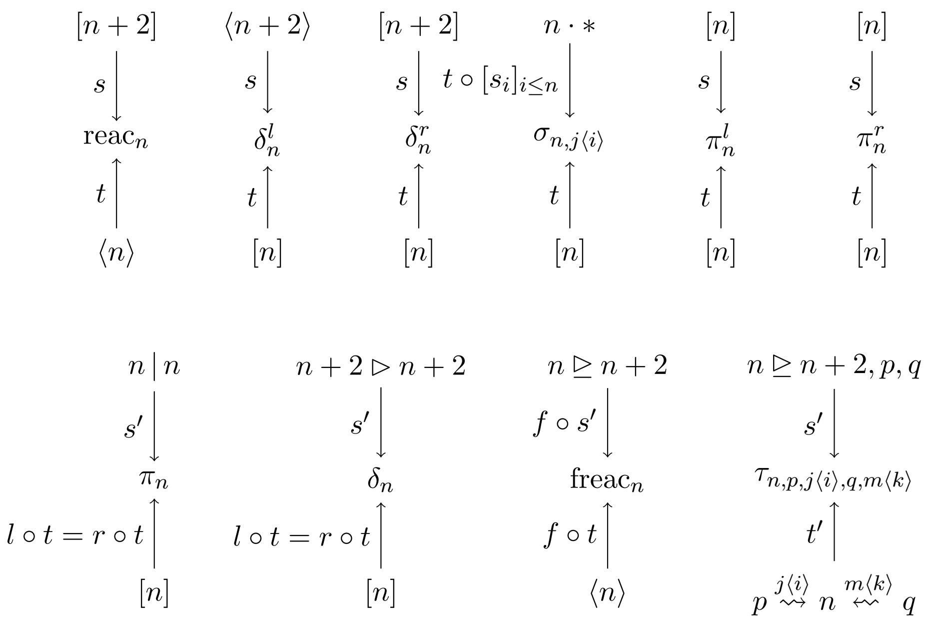
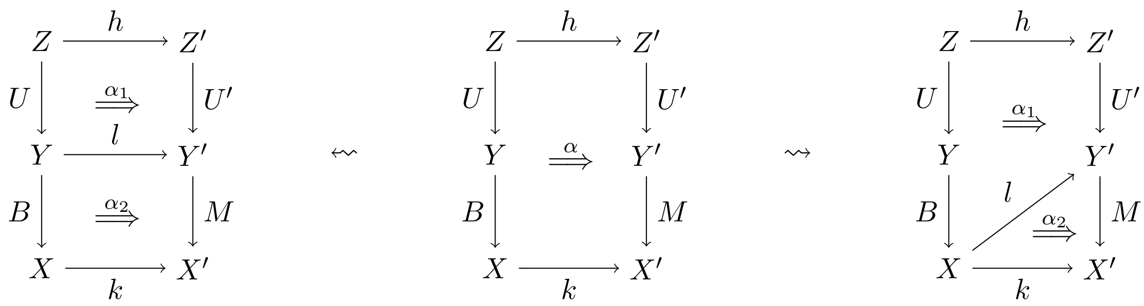


Figure 7: Seeds in the join-calculus

*Proof.* A straightforward case analysis. □

This is one of the playground axioms, known as the axiom of views, that was proved for the join-calculus.

**Proposition 7** (Right decomposition)**.** *Any double cell (in the middle), where $B$ is a basic move and $M$ is a move, decomposes into exactly one of the forms (on the left and on the right):*



This is also one of the playground axioms, which has been proved in a general case with some hypotheses on the shape of the seeds.

## 5 A General Correctness Criterion

In order to know when a cospan $Y \to U \leftarrow X$ is a play, it is useful to have a combinatorial criterion. This criterion may be formulated and proved correct for a large class of games. In this section, we present the definitions necessary to formulate the criterion and prove it. The proof of the criterion is in appendix A.

We will assume that the base category $\mathcal{C}$ is graded, and "presheaf" will mean "finitely presentable presheaf".

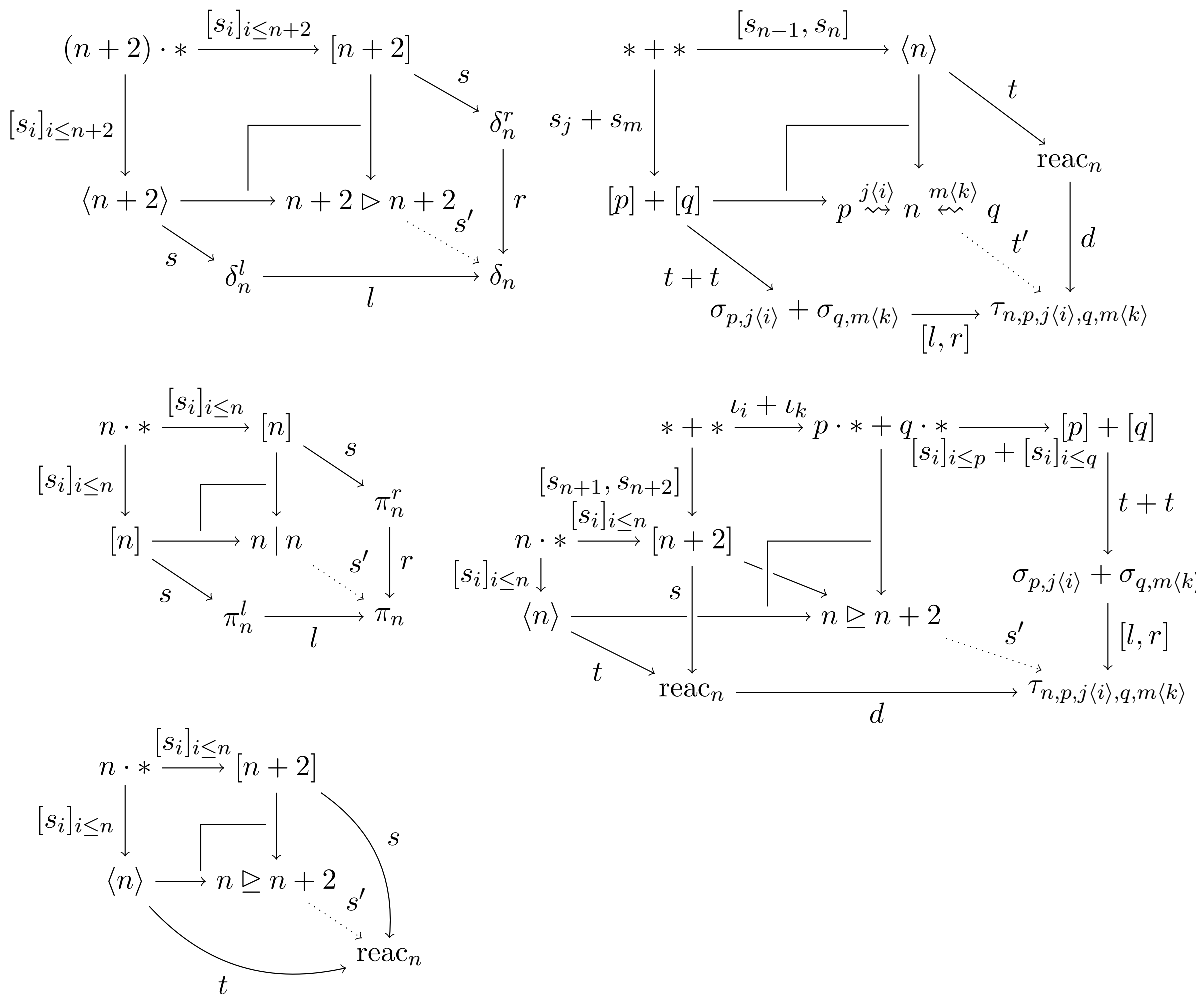



Figure 8: Auxiliary definitions for seeds

**Definition 12** (Dimension)**.** *The* dimension *of an object $c$ of $\mathcal{C}$ is $F(c)$. Based on this notion of dimension, the objects of $\mathcal{C}$ are partitioned into three classes:*

- channels *are objects $*$ of dimension* $0$*;*
- players *are objects $d$ of dimension* $1$*;*
- moves *are objects $m$ of dimension $\geq 2$.*

Remember that the base category we have given for the join-calculus is graded and that the definitions of channels, players, and moves coincide with those we have given for the join-calculus. The base category for MLL is graded as well.

**Definition 13** (Seed)**.** *To each object $m$ of $\mathcal{C}$ we associate a* seed*, that is a cospan of presheaves $Y \overset{s}{\hookrightarrow} M \overset{t}{\hookleftarrow} X$, such that $M$ is the representable presheaf $y(m)$, where $y$ is the Yoneda embedding $y : \mathcal{C} \to [\mathcal{C}^{op}, \boldsymbol{Set}]$, $y(c) = \mathcal{C}(-, c)$. Moreover, we ask that, for all seeds, $(X + Y)(p) \xrightarrow{[t_p, s_p]} M(p)$ is surjective for all players and all channels $p$.*

Note that $t$ and $s$ have to be injective, which is true for all seeds in the base category for the join-calculus, but not for that of MLL (because there are 5 wires in the initial position and only 3 wires in cut).

**Definition 14** (Core)**.** *Given a presheaf $U$, we say that $\mu$ is a* core *of $U$ if $\mu$ is a move and for any element $x$ in $\mathrm{el}(U)$, if $f : \mu \to x$ in $\mathrm{el}(U)$, then $x = \mu$ and $f = \mathrm{id}_\mu$.*

**Definition 15** (Consumed, Created, and Surviving Players)**.** *For every seed $Y \overset{s}{\hookrightarrow} M \overset{t}{\hookleftarrow} X$, we partition the set of players $M(d)$ in the following way:*

- consumed players*:* $\mathrm{Co}(M)(p) = t_p(X(p)) - s_p(Y(p))$
- created players*:* $\mathrm{Cr}(M)(p) = s_p(Y(p)) - t_p(X(p))$
- surviving players*:* $\mathrm{Sr}(M)(p) = t_p(X(p)) \cap s_p(Y(p))$

*These notions are extended to cores in the obvious way.*

**Definition 16** (Initial and Final Players)**.** *We define the initial and final players and channels of a presheaf $U$ in the following way:*

- *a player (or a channel) is* initial *if it is not created by any core of $U$:* $\mathrm{Init}(U)(p) = \{x \in U(p) \mid \nexists \mu \in \mathrm{el}(U), x \in \mathrm{Cr}(\mu)(p)\}$*;*
- *a player (or a channel) is* final *if it is not consumed by any core of $U$:* $\mathrm{Fin}(U)(p) = \{x \in U(p) \mid \nexists \mu \in \mathrm{el}(U), x \in \mathrm{Co}(\mu)(p)\}$*.*

**Definition 17** (Local 1-Injectivity)**.** *A presheaf $U$ on $\mathcal{C}$ is* locally 1-injective *if for every seed $Y \overset{s}{\hookrightarrow} M \overset{t}{\hookleftarrow} X$ with canonical interface $u : I \to M$ and every corresponding core $\mu \in \mathrm{el}(U)$ (seen as a morphism $\mu : M \to U$), if $x \neq y \in M$ are such that $\mu(x) = \mu(y)$, then $x, y$ are in the image of $u$.*

**Definition 18** (Core Separation)**.** *A presheaf $U$ on $\mathcal{C}$ is* core-separating *if for all cores $\mu \neq \mu' \in \mathrm{el}(U)$, the pullback of $\mu$ along $\mu'$ is a position.*

**Definition 19** (Past)**.** *The* past *of a seed $Y \overset{s}{\hookrightarrow} M \overset{t}{\hookleftarrow} X$ is the set* $\mathrm{past}(M) = \Sigma_{o \in \mathcal{C}} M(o) - s_o(Y(o))$*. This notion is extended to cores: the past of a core $\mu \in \mathrm{el}(U)$ is the set $\mu(\mathrm{past}(M))$.*

**Definition 20** (Causal Graph)**.** *Given a presheaf $U$ on $\mathcal{C}$, we define its* causal graph *$G_U$ in the following way:*

- *its vertices are all the players, channels, and cores of $U$*
- *there is an edge:*
  - *$d \to \mu$ for each core $\mu$ and player or channel $d$ that is in* $\mathrm{Cr}(\mu)$
  - *$\mu \to d$ for each core $\mu$ and player or channel $d$ that is in* $\mathrm{Co}(\mu) \cup \mathrm{Sr}(\mu)$
  - *$\mu \to \mu'$ for all cores $\mu \neq \mu'$ such that there is a player or a channel that is in* $\mathrm{Co}(\mu) \cap (\mathrm{Co}(\mu') \cup \mathrm{Sr}(\mu'))$

**Definition 21** (Source Linearity)**.** *A causal graph $G_U$ is* source-linear *if for all players $d$ and cores $\mu, \mu'$, if $d \to \mu$ and $d \to \mu'$ in $G_U$, then $\mu = \mu'$.*

**Theorem 4** (Correctness Criterion)**.** *A cospan $Y \overset{s}{\hookrightarrow} M \overset{t}{\hookleftarrow} X$ is a play if and only if:*

- *$U$ is locally 1-injective and core-separating*
- *$X$ contains exactly the initial players and channels*
- *$Y$ contains exactly the final players and channels*
- *$G_U$ is source-linear and acyclic*

# 6 Conclusion

Though the initial goal of the internship, which was to show that the join-calculus is an instance of playgrounds, has not been achieved, most playground axioms have been proven for the join-calculus. Moreover, significant steps have been taken to generalize the proofs of the playground axioms to a large class of operational semantics.

An immediate direction for future research would be to indeed show that the join-calculus is an instance of playground, and study the semantics given by that playground. Once this is done, there would be three known instances of playgrounds (CCS, $\pi$, and join), so it would be interesting to study the notion of morphisms between these playgrounds. Another direction for future research would be to continue proving the playground axioms in the general case for a large class of operational semantics. A last direction for research would be to find a way to automatically build the seeds corresponding to an operational semantics (because their construction is currently done by hand), so that the construction of a playground may become completely automated.

# A Proof of the Correctness Criterion

The following lemmas are used to prove lemma 3, which is used in the proof of the correctness criterion.

**Lemma 1.** *Let $U$ be a finitely presentable presheaf on $\mathcal{C}$ and $m$ be a move in* $\mathrm{el}(U)$*. Then there is a core $\mu$ such that $m \to \mu$ in* $\mathrm{el}(U)$*. If $U$ is core-separating, then $\mu$ is unique.*

*Proof.* We have proved in Proposition 1 that $U$ has finite dimension. Let us show the lemma: if there is no $m_1 \neq m$ such that $m \to m_1$, then we can take $\mu = m$, which is indeed a core. Otherwise, we have found $f_1 : m \to m_1$. We can repeat this process so that either $m_1$ is a core or we find $f_2 : m_1 \to m_2$, and so on. This process will ultimately stop for some $i \in \mathbb{N}$ since $U$ has finite dimension and a new $m_j$ has dimension greater than that of all the previous ones (because $\mathcal{C}$ is graded). Therefore, we have a core $\mu = m_i$ and $m \xrightarrow{(f_j)} \mu$.

If $U$ is core-separating, then assume that there are cores $\mu \neq \mu'$ such that $m \to \mu$ and $m \to \mu'$, then $m$ is in the pullback of $\mu$ along $\mu'$, which contradicts core separation. □

**Lemma 2.** *If $U \in \widehat{\mathcal{C}}^f$, $\mu$ is a core of $U$ and $x$ a player in $U(p)$ (or a channel in $U(*)$), there is a morphism $f : x \to \mu$ in* $\mathrm{el}(U)$ *if and only if there is either an edge $\mu \to x$ or an edge $x \to \mu$ in $G_U$.*

*Proof.* Here is the proof of both directions if $x$ is a player (the proof is exactly the same if $x$ is a channel):

The reverse direction is obvious by construction of the causal graph and because the seeds are representable.

Let us show $\Rightarrow$: through the bijection between seeds and moves, $f$ is in the image of a seed $Y \overset{s}{\hookrightarrow} M \overset{t}{\hookleftarrow} X$. Therefore, by surjectivity of $(X+Y)(p) \xrightarrow{[t_p, s_p]} M(p)$, $x$ is in the image of $X$ or in that of $Y$, so, by construction of the causal graph, there is either an edge $\mu \to x$ or an edge $x \to \mu$ in $G_U$. □

**Lemma 3.** *Let $U$ be a presheaf on $\mathcal{C}$ and $\mu$ be maximal in $G_U$ (i.e., there is no path from $\mu$ to any other core). Assume the final players and channels of $U$ form a presheaf $Y$ and that $U$ is core-separating. Then for all $c \in \mathrm{el}(U)$, $U(c) - \mathrm{past}(\mu) = (U \setminus\!\!\setminus \mu)(c)$.*

*Proof.* Let us show both inclusions:

- the direction $(U \setminus\!\!\setminus \mu)(c) \subseteq U(c) - \mathrm{past}(\mu)$ is obvious;
- let us show that $U(c) - \mathrm{past}(\mu) \subseteq (U \setminus\!\!\setminus \mu)(c)$:

  we only need to show that $U - \mathrm{past}(\mu)$ forms a subpresheaf of $U$. Let us take $x \in U(c') - \mathrm{past}(\mu)$ and $f : c \to c'$, and show that $x' = x \cdot f \notin \mathrm{past}(\mu)$: let us assume that $x' \in \mathrm{past}(\mu)$, we want to derive a contradiction.

  Since $\mathcal{C}$ is graded and $f$ cannot be the identity, $x$ and $x'$ are one of the following:

  - $x$ is a move: by lemma 1, there is a core $\widetilde{x}$ of $U$ with a morphism $g : x \to \widetilde{x}$, now there are two possible cases for $x'$:
    * $x'$ is also a move: in this case, $x'$ is in the pullback of $\widetilde{x}$ along $\mu$, and we know that $\widetilde{x} \neq \mu$ because $x$ is not in $\mathrm{past}(\mu)$, which contradicts core separation of $U$;
    * $x'$ is a player or a channel: in this case, by lemma 2, we know that there is either an arrow $x' \to \widetilde{x}$ or an arrow $\widetilde{x} \to x'$ in $G_U$:

- $\cdot$ if there is an arrow $x' \to \widetilde{x}$ in $G_U$: we have $\mu \to x' \to \widetilde{x}$ in $G_U$, which contradicts maximality of $\mu$ in $G_U$;
  - $\cdot$ if there is an arrow $\widetilde{x} \to x'$ in $G_U$: we have $\mu \to x'$ and $\widetilde{x} \to x'$ in $G_U$, and since $x'$ is in past$(\mu)$, there is an arrow $\mu \to \widetilde{x}$, which contradicts maximality of $\mu$ in $G_U$;
- $x$ is a player and $x'$ is a channel: since $x' \notin Y$, we know that $x \notin Y$, therefore there is a core $\mu'$ such that $x \in \text{past}(\mu')$. $\mu' \neq \mu$ (otherwise, $x \in \text{past}(\mu)$), and since $x \in \text{past}(\mu') \subseteq \mu'(t'(X'))$, we know that $x' = x \cdot f \in \mu'(t'(X'))$ as well, so there is an edge $\mu' \to x'$ in $G_U$. Since $x' \in \text{past}(\mu)$, there is an edge $\mu \to \mu'$ in $G_U$, which contradicts maximality of $\mu$ in $G_U$

□

The following lemmas prove that if $Y \overset{s}{\hookrightarrow} U \overset{t}{\hookleftarrow} X$ is a play, then it verifies the correctness criterion.

**Lemma 4.** *Let $Y \overset{s}{\hookrightarrow} U \overset{t}{\hookleftarrow} X$ be a play. Then $U$ is locally 1-injective.*

*Proof.* Decompose $U$ into moves. Each core $\mu$ corresponds to exactly one such move, say $M'$, obtained by pushout of a seed $M$ along some $I \to Z$. By construction of pushouts in presheaf categories, $M'$ is obtained from $M$ with identifications only occurring in the image of $I$. □

**Lemma 5.** *Let $Y \overset{s}{\hookrightarrow} U \overset{t}{\hookleftarrow} X$ be a play. Then $U$ is core-separating.*

*Proof.* Decompose $U$ into moves. Each core corresponds to exactly such a move. Therefore, $\mu \neq \mu'$ correspond to different moves. By construction of pushouts in presheaf categories, the play is obtained by identifying only positions between the different moves, so the pullback of two cores is a position. □

*Proof.* Let us prove this by induction on the play $Y \overset{s}{\hookrightarrow} U \overset{t}{\hookleftarrow} X$:

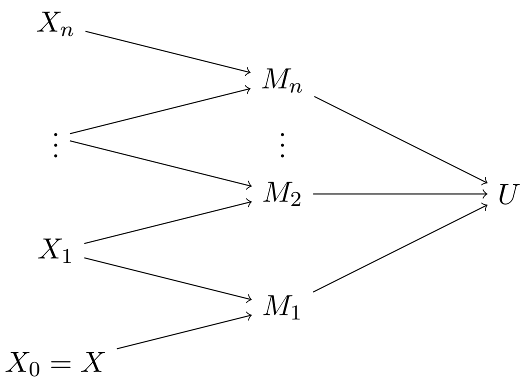
- if the play is empty: there is no core, so the property is obviously true
- if the play is $Z \overset{s'}{\hookrightarrow} U' \overset{t'}{\hookleftarrow} Y \overset{s}{\hookrightarrow} M \overset{t}{\hookleftarrow} X$: let us take $c \in U(*)$:
  - either $c \in U'(*)$: if $c = x \cdot s$:
    - $*$ either $x \in U'(p)$: then the property holds by induction hypothesis
    - $*$ or $x \in M(p) - U'(p)$: then $x \in \text{past}(\mu_0)$, so the property holds
  - or $c \in M(*) - U'(*)$: if $c = x \cdot s$, then $x \notin U'(p)$, so $x \in M(p) - U'(p)$, so $x \in \text{past}(\mu_0)$, so the property holds

□

**Lemma 6.** *Let $Y \overset{s}{\hookrightarrow} U \overset{t}{\hookleftarrow} X$ be a play. Then $X$ contains exactly the initial channels and players and $Y$ exactly the final channels and players.*

*Proof.* Let us show that $X$ contains exactly the initial players and channels.

Choose a decomposition of $U$ into moves:




If $x$ is in $X$, then it is in $X_0$, ..., $X_i$ for some $i$ (because identifications can only occur between $X_j$ and $X_{j+1}$). Therefore, it is not in $\mathrm{Cr}(\mu)$ for any core $\mu$ of $U$, so it is initial.

If $x$ is initial, it is not in $\mathrm{Cr}(\mu)$ for any core $\mu$ of $U$. Therefore, if it is in $X_i$, it is in $X_{i-1}$ and therefore in $X_0 = X$. Since $x$ is at least in one $X_i$, it is in $X$.

A similar reasoning shows that $Y$ only contains final players and channels. □

**Lemma 7.** *Let* $Y \overset{s}{\hookrightarrow} U \overset{t}{\hookleftarrow} X$ *be a play. Then* $G_U$ *is source-linear.*

*Proof.* Using the same kind of arguments as in lemma 6, if a player $x$ is in $X_i$, then it is exactly in $X_j$, ..., $X_k$ for some $j \leq i \leq k$. Therefore, it may only be created by at most a single move, which happens exactly when $j \geq 1$, in which case it is created by $M_j$. This shows that $G_U$ is source-linear. □

**Lemma 8.** *Let* $Y \overset{s}{\hookrightarrow} U \overset{t}{\hookleftarrow} X$ *be a play. Then* $G_U$ *is acyclic.*

*Proof.* Choose a decomposition of $U$ into moves. We can build the graph inductively in the following way:

- add all the players and channels in the initial position and the edges between them (this does not form a cycle since the only edges are from players to channels)
- add the core corresponding to $M_1$ and all the edges from $M_1$ to the vertices already in the graph (notice that all the edges are indeed *from* $M_1$, so it cannot create cycles)
- add all the players and channels in $X_1$ and all the edges between them and the vertices already in the graph (again, notice that, since we only added channels and players that were created by the core corresponding to $M_1$, all the edges are *from* them, so this cannot create cycles)
- add the core corresponding to $M_2$...

This construction shows that there is no cycle in $G_U$. □

**Theorem 5** (Correctness Criterion)**.** *A cospan* $Y \overset{s}{\hookrightarrow} U \overset{t}{\hookleftarrow} X$ *is a play if and only if:*

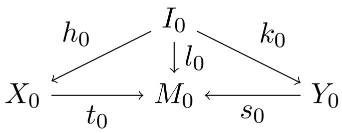
- $U$ *is locally 1-injective and core-separating*
- $X$ *contains exactly the initial players and channels*
- $Y$ *contains exactly the final players and channels*
- $G_U$ *is source-linear and acyclic*

*Proof.* Let us prove both directions. If $Y \overset{s}{\hookrightarrow} U \overset{t}{\hookleftarrow} X$ is a play, then it verifies the properties by lemmas 4, 5, 6, 7 and 8.

Now assume $Y \overset{s}{\hookrightarrow} U \overset{t}{\hookleftarrow} X$ verifies the properties, let us show it is a play. We proceed by induction on the number of moves in $U$.

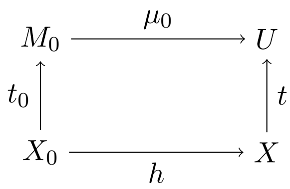
- if there is no move in $U$, then $U$ is a position, all players and channels are both initial and final, so $s$ and $t$ are isomorphisms, so $Y \overset{s}{\hookrightarrow} U \overset{t}{\hookleftarrow} X$ is a play.
- if there is at least a move in $U$, we want to decompose $Y \overset{s}{\hookrightarrow} U \overset{t}{\hookleftarrow} X$ into $Y \overset{s''}{\hookrightarrow} U' \overset{t''}{\hookleftarrow} Z \overset{s'}{\hookrightarrow} M \overset{t'}{\hookleftarrow} X$ with $Z \overset{s'}{\hookrightarrow} M \overset{t'}{\hookleftarrow} X$ a move and $Y \overset{s''}{\hookrightarrow} U' \overset{t''}{\hookleftarrow} Z$ that verifies the hypotheses and such that $U'$ contains strictly fewer moves than $U$.

  Take $\mu_0$ maximal in $G_U$ (possible since $G_U$ is acyclic) and let

  be the corresponding seed with its canonical interface.

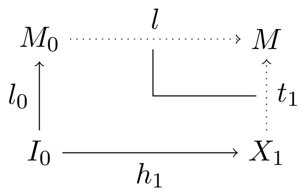
  We define $h : X_0 \to X$ to be the unique morphism such that

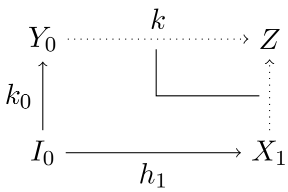
  
  

  commutes. It exists indeed because the image of $\mu_0 \circ t_0 : X_0 \to U$ are initial players and channels: be $x_0 \in X_0$ and let $u_0 = \mu_0(t_0(x_0))$, if $u_0$ is not initial, then there is a $\mu' \in \mathrm{el}(U)$ such that $u_0 \to \mu'$ in $G_U$, but we know that $\mu_0 \to^* u_0$ in $G_U$ by construction, which contradicts the fact that $\mu_0$ is maximal in $G_U$. It is unique because $t$ is monic.

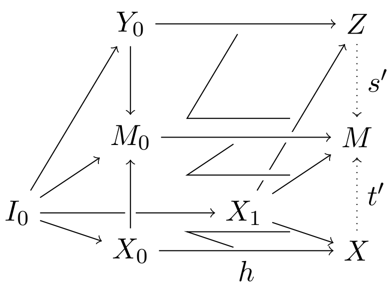
  Now, we define $X_1$ to be $X$ to which the players in the image of $X_0$ have been removed. $X_1$ is still a presheaf because it contains all the channels. Since colimits are computed pointwise in presheaf categories, it is easy to see that $X$ is the pushout of $h_0 : I_0 \to X_0$ and $h_1 = I_0 \xrightarrow{h_0} X_0 \xrightarrow{h} X \xrightarrow{\subseteq^{-1}} X_1$ ($\subseteq^{-1}$ is well-defined on the image of $I_0$).

  Now, we define $M$ and $Z$ to be the following pushouts:

  We define $t' : X \to M$ and $s' : Z \to M$ by universal property of the pushout. What we have constructed up to this point is a move $Z \overset{s'}{\hookrightarrow} M \overset{t'}{\hookleftarrow} X$, as the following diagram shows:




By the pushout lemma, the following diagrams are also pushouts:

$$\begin{array}{ccc} M_0 & \xrightarrow{\;\;l\;\;} & M \\ {\scriptstyle t_0}\big\uparrow & & \big\uparrow{\scriptstyle t'} \\ X_0 & \xrightarrow[\;\;h\;\;]{} & X \end{array} \qquad \begin{array}{ccc} Y_0 & \xrightarrow{\;\;k\;\;} & Z \\ {\scriptstyle s_0}\big\downarrow & & \big\downarrow{\scriptstyle s'} \\ M_0 & \xrightarrow[\;\;l\;\;]{} & M \end{array}$$

Since we know that, by construction,

$$\begin{array}{ccc} M_0 & \xrightarrow{\;\;\mu_0\;\;} & U \\ {\scriptstyle t_0}\big\uparrow & & \big\uparrow{\scriptstyle t} \\ X_0 & \xrightarrow[\;\;h\;\;]{} & X \end{array}$$

commutes, by property of the pushout, we define $\mu : M \to U$. Now, we define $U' = U - \text{past}(\mu_0)$, which is a presheaf by lemma 3. We have the inclusion $U' \to U$.

We define $t'' : Z \to U'$ as $Z \xrightarrow{s'} M \xrightarrow{\mu} U \xrightarrow{\subseteq^{-1}} U'$ ($\subseteq^{-1}$ is well-defined on the image of $Z$ because the only players and channels in $\text{past}(\mu_0)$ are in the image of $X_0 \backslash Y_0$ and this does not intersect the image of $Z$) and $s'' : Y \to U'$ as $Y \xrightarrow{s} U \xrightarrow{\subseteq^{-1}} U'$ ($\subseteq^{-1}$ is well-defined on the image of $Y$ because $Y$ only contains final players and channels and the players and channels in $\text{past}(\mu_0)$ cannot be final).

Now, we want to show that

$$\begin{array}{ccc} Z & \xrightarrow{\;\;t''\;\;} & U' \\ {\scriptstyle s'}\big\downarrow & & \big\downarrow{\scriptstyle \subseteq} \\ M & \xrightarrow[\;\;\mu\;\;]{} & U \end{array}$$

is a pushout. By the pushout lemma, it is sufficient to show that

$$\begin{array}{ccc} Y_0 & \xrightarrow{\;\;f = \mu_0 \circ s_0\;\;} & U' \\ {\scriptstyle s_0}\big\downarrow & & \big\downarrow{\scriptstyle \subseteq} \\ M_0 & \xrightarrow[\;\;\mu_0\;\;]{} & U \end{array}$$

is a pushout. It is sufficient to show that:

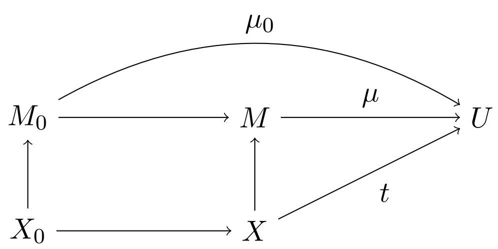
- $[\mu_0, \subseteq]$ is surjective
- if $\mu_0(m) = \subseteq (u)$, then there exists $y$ such that $s_0(y) = m$ and $f(y) = u$
- if $\mu_0(m) = \mu_0(m') \notin \Im(\subseteq)$, then $m = m'$

The first point is obvious (each $u$ in $U$ is either in the past of $\mu_0$ and therefore has an antecedent by $\mu_0$ or not and is therefore in $U'$), the second one is a simple case analysis on whether $x$ is in $Y_0$ or not and the third one is true because elements that are not in $\Im(\subseteq)$ are not in $I_0$ and therefore cannot be identified. Therefore, it is indeed a pushout.

Hence, $Y \to U \leftarrow X$ is indeed the composition of the cospans $Y \to U' \leftarrow Z$ and $Z \to M \leftarrow X$. Since it is obvious that $U'$ contains strictly fewer moves than $U$, the only thing left to show is that $Y \stackrel{s''}{\to} U' \stackrel{t''}{\leftarrow} Z$ verifies the properties.

First, we need to show that $s''$ and $t''$ are injective. $s''$ obviously is because $s$ is.

To show that $t''$ is injective, let us first show that $\mu$ is. Let us focus on the following diagram:




Let us choose $x$ and $y$ in $M$ such that $\mu(x) = \mu(y)$ and show that necessarily $x = y$. Since $M$ is the pushout of $X$ and $M_0$, there are three possible cases:

- either $x$ and $y$ are both in $X$, in which case $x = y$ by injectivity of $t$
- or $x$ and $y$ are both in $M_0$, in which case, since $U$ is locally 1-injective, $x$ and $y$ are both in $X_0$, so both in $X$, so $x = y$ by injectivity of $t$
- $x$ is in $X \backslash X_0$ and $y$ is in $M_0 \backslash Y_0$, in which case $\mu(x)$ is initial, but $\mu(y)$ is not, which is absurd.

From the fact that $U'$ is a subpresheaf of $U$, it is obvious that $U$ is locally 1-injective and core-separating and that $G_U$ is source-linear and acyclic. We need to show that $Z$ contains exactly the initial players and channels of $U'$ and $Y$ the final ones.

Let us show that a player (or a channel) is in $Z$ exactly when it is initial in $U'$:

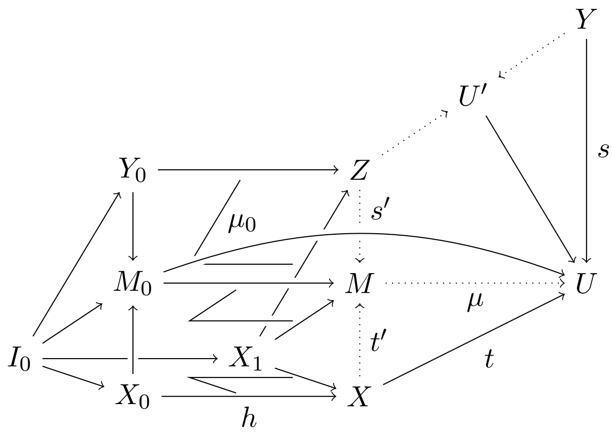
- take $x \in Z$, then either $x$ is in $X_1 \subseteq X$ so it is initial in $U$ and therefore initial in $U'$, or $x$ is in $Y_0$, in which case:
  * either $x$ is in $Y_0 \cap X_0$, so $x$ is initial in $U$, hence initial in $U'$
  * or $x$ is in $Y_0 \backslash X_0$, so there is an edge $\mu(x) \to \mu_0$ in $G_U$, so, since $G_U$ is source-linear, there is no edge $\mu(x) \to \mu'$ in $G_U$ for $\mu' \neq \mu_0$. Hence, there is no arrow $\mu(x) \to \mu'$ in $G'_U$.
- take $x$ initial in $U'$, there is no edge $\mu(x) \to \mu'$ to a core $\mu'$ in $G_{U'}$, so:
  * if $\mu(x) \not\to \mu_0$ in $G_U$, then $x$ is initial in $U$, so in $X$, so there are three possible cases:

    · $x$ is in $X_1$: then $x$ is indeed in $Z$
    · $x$ is in $X_0 \cap Y_0$: then $x$ is indeed in $Z$
    · $x$ is in $X_0 \backslash Y_0$: then $x$ is not in $U'$, which is absurd.
  * if $\mu(x) \to \mu_0$ in $G_U$, then $x$ is in $Y_0 \backslash X_0$, so indeed in $Z$.

Let us show that a player (or a channel) is in $Y$ exactly when it is final in $U'$, that is to say that final players (and channels) in $U'$ are exactly the same as in $U$:

- if $x$ is final in $U$, then it is obviously final in $U'$
- if $x$ is final in $U'$, then there is no core $\mu'$ of $U'$ such that $\mu(x) \in \mathrm{Co}(\mu')$, moreover, $\mu(x)$ cannot be in $\mathrm{Co}(\mu_0)$ (otherwise it would be in $\mathrm{past}(\mu_0)$ and couldn't be in $U'$), so it is also final in $U$.

Here is the diagram resuming the whole construction:




□