AN INTENSIONALLY FULLY-ABSTRACT SHEAF MODEL FOR π
(EXPANDED VERSION)

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Abstract. Following previous work on CCS, we propose a compositional model for the pi-calculus in which processes are interpreted as sheaves on certain simple sites. We define an analogue of fair testing equivalence in the model and show that our interpretation is intensionally fully abstract for it. That is, the interpretation preserves and reflects fair testing equivalence; and furthermore, any strategy is fair testing equivalent to the interpretation of some process. The central part of our work is the construction of our sites, whose heart is a combinatorial presentation of pi-calculus traces in the spirit of string diagrams. As in previous work, the sheaf condition is analogous to innocence in Hyland-Ong/Nickau games.

Contents

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1. Introduction

1.1. **Causal models and beyond.** Operational semantics of programming languages standardly model the execution of programs as paths in a certain labelled transition system (LTS). Under this interpretation, different possible interleavings of parallel actions yield different paths. Verification on LTSs thus incurs a well-known state explosion problem. Similarly, causality between various actions, visible in the syntax, is lost in the LTS, thus making, e.g., error diagnostics difficult [27].

Causal models, originally designed for Petri nets [59] and Milner's CCS [67], intend to remedy both problems, but have yet to be applied to full-scale programming languages. They have recently been extended in two different directions: (1) by Crafa et al. [15] to Milner's π-calculus, and (2) by Melliès [53] to Girard's linear logic. The former extension accounts for the subtle interaction of channel creation with synchronisation in π, a significant technical achievement, 30 years after the first causal semantics for CCS. The latter is the first causal model for functional languages (inspired by Hyland-Ong's and Nickau's *games* models for PCF [58, 37]). An important challenge is now the search for a causal model of full-fledged languages with both concurrent and functional features. Winskel and collaborators are currently working in this direction, using extensions of Melliès's approach [63, 68, 12].

In previous work [35, 33, 34], we have proposed a causal model for CCS based on a different approach. We here push this approach further by applying it to the π-calculus.

1.2. **Traces and naive concurrent strategies.** In standard causal models, execution traces essentially consist of partially ordered sets of atomic 'events'. Our approach relies on a new notion of trace, which we briefly sketch. There is first a (straightforward) notion of *configuration*, which is essentially a finite hypergraph whose nodes are thought of as *agents*, and whose hyperedges between nodes $x_1, \ldots, x_n$ are thought of as communication channels shared by $x_1, \ldots, x_n$. There is then a notion of *atomic action* from one configuration to another, thought of as a 'rule of the game'. Examples of atomic actions are: an agent creates a new, private communication channel; an agent forks into two new agents connected to the same channels; an agent sends some channel a over some channel b to some other agent. We finally have a notion of trace which allows several atomic actions to occur, in a way that only retains some minimal causality information between them. We here mean, e.g., information such as: 'such agent outputs on such channel only after having created such other channel'.

The main purpose of our notion of trace is to interpret π-calculus processes as some kind of strategies over them. Most naively, a strategy on some configuration X is a prefix-closed set of 'accepted' traces from X. But what should prefix mean in our setting? Well, we may view traces with initial configuration X and final configuration Y as morphisms $Y \mathrel{-\!\!\bullet\!\!\to} X$. Sequential composition of traces, denoted by $\bullet$, yields an analogue of prefix ordering, defined by $t \leq t \bullet w$. This however fails to suit our needs on three counts.

We start by examining the first two problems. The first, easy one is that there is an obvious notion of isomorphism between traces, under which strategies should be closed. The second problem is more serious: until now, these too naive strategies are not concurrent enough to adequately model CCS or the π-calculus.

Example 1.1 (Milner's coffee machines)**.** Consider the CCS processes $P = (a.b + a.c)$ and $Q = a.(b + c)$. The process P has two ways of inputting on a and then, depending on the chosen way, inputs either on b or on c. The process Q inputs on a and then has both possibilities of inputting on b or c. Both processes, however, accept exactly the same traces (in the standard sense), namely $\{\epsilon, a, ab, ac\}$, where ϵ denotes the empty trace.

Thus, taking strategies to be prefix-closed sets of traces would prevent us from directly modelling any reasonably fine behavioural equivalence on processes. Inspired by *presheaf models* [40], we remedy both problems at once by passing from prefix-closed sets of traces to presheaves (of finite sets) on traces. Indeed, in the simple case where traces on X form a mere poset $\mathbb{T}(X)$ by prefix ordering, a prefix-closed set of traces is nothing but a contravariant functor from $\mathbb{T}(X)$ to the ordinal 2, viewed as a category. The latter has two objects 0 and 1 and just one non-trivial morphism $0 \to 1$. The idea is that a functor $S: \mathbb{T}(X)^{op} \to 2$ maps any trace to 1 when it is accepted, and to 0 otherwise. Furthermore, if $t \leq t'$, i.e., t is a prefix of t', then we have a morphism $t \to t'$ which should be mapped by S to some morphism $S(t') \to S(t)$. If t' is accepted then $S(t') = 1$, so this has to be a morphism $1 \to S(t)$. Because there are no morphisms $1 \to 0$, this entails $S(t) = 1$, hence prefix-closedness of the corresponding strategy.

Now our traces naturally form a proper category $\mathbb{T}(X)$, encompassing both prefix ordering and isomorphisms between traces, so we are led to considering functors $\mathbb{T}(X)^{op} \to 2$. This retains prefix-closedness and solves our first problem: for any $t \cong t'$, functoriality imposes $S(t) \cong S(t')$. Our second problem is then solved by replacing such functors with *presheaves*, i.e., functors $\mathbb{T}(X)^{op} \to \mathsf{Set}$.

Example 1.2. In Example 1.1, the two ways that P has to accept inputting on a may be reflected by mapping the trace a to some two-element set. More precisely, P may be modelled by the presheaf S defined on the left and pictured on the right:

- $S(\epsilon) = \{\star\}$,
- $S(a) = \{x, x'\}$,
- $S(ab) = \{y\}$,
- $S(ac) = \{y'\}$,
- S empty otherwise,
- $S(\epsilon \hookrightarrow a) = \{x \mapsto \star, x' \mapsto \star\}$,
- $S(a \hookrightarrow ab) = \{y \mapsto x\}$,
- $S(a \hookrightarrow ac) = \{y' \mapsto x'\}$,

$$\begin{tikzcd} & \star \arrow[dl, "a"'] \arrow[dr, "a"] & \\ x & & x' \\ y \arrow[u, "b"] & & y' . \arrow[u, "c"'] \end{tikzcd}$$

Presheaves thus may 'accept a trace in several ways': the trace a is here accepted in two ways, x and x'. The process Q is of course modelled by identifying x and x'.

As it turns out, we actually only need *finitely many* ways of accepting each trace. Thus, we arrive at a first sensible notion of strategy given by presheaves of finite sets, i.e., functors $\mathbb{T}(X)^{op} \to \mathsf{set}$, where set denotes the category with as objects all finite subsets of $\mathbb{N}$, with all maps between them. We call them *(naive) strategies* in the sequel.

Notation 1.3. For any $\mathcal{C}$, let $\widehat{\mathcal{C}}$ denote the category of presheaves of finite sets over $\mathcal{C}$.

1.3. **Innocence as a sheaf condition.** The third problem evoked above is that functors $\mathbb{T}(X)^{op} \to \mathsf{set}$ allow some undesirable behaviours. Intuitively, in π just as in CCS, agents should not have any control over the routing of messages.

Example 1.4. Consider a configuration X with three agents x, y, and z sharing a communication channel a, and a strategy S accepting (1) the trace where x outputs on a, (2) the trace where y inputs on a, and (3) the trace where z inputs on a. Then, both synchronisations should be accepted by S. However, one easily constructs a naive strategy in which one is refused (see Example 4.2).

In order to rectify this deficiency, we enrich strategies with 'local' information. The idea is that a strategy should not only accept or refuse traces on the whole configuration X, but also traces on all possible subconfigurations of X. Furthermore, this local information should fit together coherently.

Example 1.5. Consider the configuration X of Example 1.4. Any strategy on X should now in particular include independent strategies for each of the three agents x, y, and z. Coherence means that in order for a trace to be accepted, it should be enough for it to be 'locally accepted', i.e., at every stage in the trace, each agent should approve what she sees of the next action. E.g., if the next action is a synchronisation $x \to y$ with x outputting and y inputting on some channel a, then all that's required for the synchronisation to be accepted is that x accepts to output and y accepts to input. Consequently, if some other agent z also accepts to input on a at this stage, then the synchronisation $x \to z$ is also accepted.

We call this putative coherence condition *innocence* by analogy with Hyland and Ong's notion [37]. In order to formalise it, we first extend our category of traces $\mathbb{T}(X)$ on X with new objects representing traces on subconfigurations of X. We also add new morphisms, which are about 'locality':

Example 1.6. Consider the configuration X with two unary agents x_1 and x_2. There is a trace t on X in which both agents fork. Consider now the subconfiguration Y of X consisting solely of x_1 and the trace t' on Y in which x_1 merely forks. There is a morphism $t' \to t$ in our new category.

This extended category, $\mathbb{T}_X$, yields an intermediate notion of strategy, given by functors $\mathbb{T}_X^{op} \to \mathsf{set}$. Among the new objects, we have in particular traces on just one agent of X, which are obtained by sequentially composing atomic actions whose final configuration again consists of one agent. We call this particular kind of trace a *view*. Views are the most 'local' kind of objects in $\mathbb{T}_X$. They form a subcategory $\mathbb{V}_X$ of $\mathbb{T}_X$.

Example 1.7. If X merely consists of an agent x linked to n communication channels, consider the atomic action given by x forking into two new agents, say x_1 and x_2. This action, viewed as an object of $\mathbb{T}_X$ has three subobjects which are views: (1) the 'identity' view, in which nothing happens, (2) π^l_n, which represents the left-hand branch (to x_1), (3) and π^r_n, which represents the right-hand branch (to x_2).

The inclusion $\mathbb{V}_X \hookrightarrow \mathbb{T}_X$ induces a simple Grothendieck topology [51] on $\mathbb{T}_X$, which amounts to decreeing that any trace is covered by its views. We finally call any $S\colon \mathbb{T}_X^{op} \to \mathsf{set}$ *innocent* precisely when it is a *sheaf* for this Grothendieck topology. In particular, giving an innocent presheaf on $\mathbb{T}_X$ is equivalent (up to isomorphism) to separately giving an innocent presheaf for each agent of X, which rules out the undesirable behaviour described in Example 1.4.

Sheaves on $\mathbb{T}_X$ form a category S_X, which is small thanks to our use of set instead of Set. They furthermore map back to naive strategies, i.e., presheaves on $\mathbb{T}(X)$, by forgetting the local information. (This forgetful functor has a left adjoint.) Finally, because the considered topology is particularly simple, sheaves are equivalent to presheaves on views, i.e., $\mathsf{S}_X \simeq \widehat{\mathbb{V}_X}$ (recalling Notation 1.3). In summary, we have three categories of strategies: *naive* strategies are presheaves on traces $\widehat{\mathbb{T}(X)}$, *innocent* strategies S_X are sheaves on the extended category of traces $\mathbb{T}_X$, and so-called *behaviours* B_X are presheaves on the category of views $\mathbb{V}_X$. The last two are equivalent, and we furthermore have an adjunction $\widehat{\mathbb{T}(X)} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{S}_X$.

We use both sides of the equivalence: behaviours directly lead to our compositional interpretation $[\![-]\!]\colon Pi \to \mathsf{S}$ of π-calculus processes, and innocent strategies are used below as the basis for our semantic definition of fair testing equivalence.

1.4. **Main result.** What should we do in order to demonstrate adequacy of our model? By definition, causal models expose some intensional information. Hence, equality is generally much finer than any reasonable behavioural equivalence, so we should not base our main result on it. On the other hand, causal models are supposed to be 'compositional', i.e., to come equipped with an interpretation of syntactic operations in the model. The natural thing to do is thus to choose some behavioural equivalence from the operational side, use compositionality to transpose it to the model, and prove that the two coincide. More precisely, the considered equivalence induces by quotienting two 'extensional collapses', one syntactic and the other semantic, and we want to prove that the translation $[\![-]\!]$ induces a bijection between both extensional collapses. Following [1], we call this *intensional full abstraction* for the considered equivalence.

We here focus on so-called *testing equivalences* [17, 57, 7, 62], which are defined in two stages. First, one chooses a 'mode of interaction'. That is, one defines what the relevant *tests* are for a given process and specifies how the two should interact. Typically, tests for P are other processes T with the same free communication channels as P, and interaction is just parallel composition $P \mid T$. The second stage amounts to choosing when $P \mid T$ is *successful*. E.g., in may testing equivalence $P \mid T$ is successful just when there exists a transition $(P \mid T) \overset{\heartsuit}{\Longrightarrow} P'$ (that is, a $\heartsuit$ transition, possibly surrounded by silent transitions), where $\heartsuit$ is some action fixed in advance. In must testing equivalence, success is when all maximal (possibly infinite) transition sequences contain at least one $\heartsuit$ transition. In fair testing equivalence (see [11] for some motivation and an adaptation to π), one requires that all silent sequences $(P \mid T) \Longrightarrow P'$ extend to some sequence $P' \Longrightarrow P'' \xrightarrow{\heartsuit} P'''$ ending with a ♡ transition. In this paper, we focus on the latter, i.e., we prove (Corollary 5.44) that our model is intensionally fully-abstract for fair testing equivalence. We finally show (Section 5.4) that the result generalises to a wide range of testing equivalences, obtained by varying the notion of success.

1.5. **Contributions.** Since this paper follows the same approach as previous work on CCS [35, 33, 34], we should explain in which sense extending the approach to π is more than an easy application.

A first contribution comes from the fact that, in order to even define composition in our category of traces, we need to show that traces form the total category of a *fibration* [38] over configurations. In previous work, this was done in an *ad hoc* way. We here introduce a more satisfactory approach based on *factorisation systems* [49, 24].

A second significant contribution is prompted by the interplay between synchronisation and private channels in π, which is notoriously subtle to handle. And indeed, our proof method for CCS fails miserably on π. One reason for this, we think, is that our notion of trace for π, though simple and natural, is not 'modular' enough, in the sense that a trace contains strictly more information than the collection of all 'local' information accessible to agents (i.e., of all of its views, in the above sense). Thus, adapting our proof technique from CCS would have required us to define a much more complex but more modular notion of trace. Instead, we here take a somewhat rougher route.

Finally, as mentioned above, our proof now applies not only to fair testing equivalence, but also to a whole class of testing equivalences.

1.6. **Related work.** Beyond the obviously closely related, already mentioned work of Winskel et al., we should mention other causal models for π [9, 55, 21, 5, 18, 13, 8, 15, 10], as well as interleaving models [23, 22, 65, 13, 56, 31] and the early approach [39] based on Girard's Geometry of Interaction. All of these models are based on some LTS for π. Instead, ours is rather based on *reduction rules*. The subtleties usually showing up in LTSs, related to mixing synchronisation and private channels, do resurface in our proof of intensional full abstraction, but not in the definition of our model. Indeed, it merely goes by describing the 'rule of the game' in π, and applying the general framework of *playgrounds* [34].

Another general framework relating operational and denotational descriptions of programs is Kleene coalgebra [4], which is mainly designed for automata theory. Playgrounds may be viewed as adapting ideas from Kleene coalgebra to the process algebraic setting.

We should also mention Laird's games model of (a fragment of) π [42], which accounts for *trace* (a.k.a. *may testing*) equivalence. Standard game models view strategies as *sets* of traces (with well-formedness conditions), so, as we have seen, lend themselves better to modelling trace equivalence. In a non-deterministic, yet not concurrent setting, Harmer and McCusker [30] resort to an explicit action for divergence, which allows them to recover a finer behavioural equivalence. We feel that the presheaf-based approach is more general.

Furthermore, recent work by Tsukada and Ong [66] adapts and extends some ideas of [35, 33] to nondeterministic, simply-typed λ-calculus.

Let us moreover mention less closely related work: Girard's *ludics* [26], Levy's morphisms between plays [46], Melliès's game semantics in string diagrams [54], Harmer et al.'s categorical combinatorics of innocence [30], McCusker et al.'s graphical foundation for schedules [52]. Finally, Hildebrandt's work [32] also uses sheaves, though as a tool to correctly handle infinite behaviour, as opposed to their use here to force reactions of agents to depend only on their views.

1.7. **Plan.** In previous work [34], we have defined an algebraic notion called *playground*, which encompasses what's needed for our approach to apply. Namely, it organises configurations, atomic actions, and traces into a *pseudo double category* [19, 20, 28, 29, 44, 25] with additional structure. Any playground $\mathbb{D}$ automatically gives rise, among other things, to

- categories of innocent strategies S_X, on each configuration X, organised into a pseudo double functor from $\mathbb{D}^{op}$ to small categories;
- a simple, yet complete syntax for innocent strategies, together with an LTS $\mathcal{S}_{\mathbb{D}}$ for them over an alphabet built from atomic actions.

After introducing some notation, the considered variant of the π-calculus, and fair testing equivalence, in Section 2, we construct such a playground $\mathbb{D}$ for the π-calculus in Section 3. We continue in Section 4 by applying results from [34] to define our sheaf model and semantic fair testing equivalence. In Section 5, we prove that our model is equivalent to a more standard presentation in terms of an LTS $\mathcal{S}$ for strategies (derived from $\mathcal{S}_{\mathbb{D}}$). We then define a further, more syntactic LTS $\mathcal{M}$ which we prove equivalent to $\mathcal{S}$. We finally give our translation $[\![-]\!]$ of π, whose intensional full abstraction is reduced by the previous results to that of intensional full abstraction of an induced translation to $\mathcal{M}$, which lives in the realm of LTSs. We prove the latter, which leads to our main result (Corollary 5.44).

2. Notation and preliminaries

First of all, we adopt the notation of [34, Section 2], with the slight modification that set now denotes the category with finite subsets of $\mathbb{N}$ as objects, and all maps as morphisms. (This category is equivalent to what we used in [34], but slightly easier to work with for our purposes.) We denote by $\widehat{\mathbb{C}}$ the category of presheaves on $\mathbb{C}$, and by $\widehat{\mathbb{C}}$ the category of presheaves of finite sets, i.e., of contravariant functors to set. For any category $\mathcal{C}$, let $\mathcal{C}_f$ denote the full subcategory of *finitely presentable* objects [2], or *finite* objects for short. In the only case where we'll use this, $\mathcal{C}$ will be a presheaf category $[\mathbb{C}^{op}, \mathsf{set}]$ and furthermore due to the special form of $\mathbb{C}$, $F \in \widehat{\mathbb{C}}_f$ will be equivalent to the category of elements of F being finite, i.e., $\sum_{c \in \mathrm{ob}(C)} F(c)$ is finite.

To recall some bare minimum: we often confuse objects C of a category $\mathbb{C}$ with the corresponding representable presheaves $\mathsf{y}_C \in \widehat{\mathbb{C}}$. Gph denotes the category of reflexive graphs, and all our graphs are reflexive so we often omit mentioning it. We think of morphisms $p: G \to A$ as LTSs over the alphabet A, except that for reasons specific to playgrounds our convention is that a transition from x to y is represented as an edge $x \leftarrow y$.

For any graph G, $G^\star$ denotes the graph with the same objects and all paths between them; on the other hand, $\mathsf{fc}(G)$ denotes the free category on G, i.e., the category with the same objects and identity-free paths between them. Both $(-)^\star$ and fc extend to functors, i.e., act on morphisms. We often silently coerce $\mathsf{fc}(G)$ into a reflexive graph, and denote by $\widetilde{-}$ the obvious morphism $G^\star \to \mathsf{fc}(G)$.

For any graph $p: G \to A$ over A, $x, y \in \mathrm{ob}(G)$, and edge $a: p(y) \to p(x)$ in A, we denote by $x \xleftarrow{a} y$ the existence of an edge $e: y \to x$ in G such that $p(e) = a$. We denote strong bisimilarity over A by $\sim_A$.

For any graph $p: G \to A$ over A, $x, y \in \mathrm{ob}(G)$, and path $\rho: p(y) \to p(x)$ in $A^\star$, we denote by $x \overset{\rho}{\Longleftarrow} y$ the existence of a path $r: y \to x$ in $G^\star$ such that $\widetilde{p^\star(r)} = \widetilde{\rho}$. When ρ is the empty path we just write $x \Longleftarrow y$. We denote weak bisimilarity over A by $\approx_A$.

2.1. **Fair testing equivalence.** In [34], a non-trivial abstract framework was defined for studying fair testing equivalence and its relationship with weak bisimilarity. We won't use this here, and instead work with the following simpler definition of fair testing equivalence.

We first define the reflexive graph Σ as follows:

$$\begin{tikzcd} \bullet \arrow[loop left, "\heartsuit"] \arrow[loop right, "\tau"] \end{tikzcd},$$

where τ is the chosen identity edge.

The free category $\mathsf{fc}(\Sigma)$ has τ as identity, and powers of $\heartsuit$ as other morphisms.

Definition 2.1. A *graph with testing* is a graph G together with a morphism $p: G \to \mathsf{fc}(\Sigma)$ and a relation $R: (\mathrm{ob}(G))^2 \mathrel{\longrightarrow\!\!\!\!\!\!\!+} \mathrm{ob}(G)$ whose domain is an equivalence relation and which is partially functional up to strong bisimilarity over Σ.

The domain being an equivalence relation more precisely means that the binary relation $(x, y) \mapsto \exists z.(x, y)\ R\ z$ is an equivalence relation.

Partial functionality up to strong bisimilarity means that if $(x, y)\ R\ z$ and $(x, y)\ R\ z'$, then $z \sim_{\mathsf{fc}(\Sigma)} z'$.

Notation 2.2. The relation is called the *testing* relation, and we denote it by $|_G$, i.e., $(x, y)\ R\ z$ is denoted by $z \in (x \,|_G\, y)$. Furthermore, its domain is denoted by $\supset_G$. We use $|$ and $\supset$ when there is no ambiguity. Since $|$ is partially functional up to strong bisimilarity, for any $(x, y)\ R\ z$, as long as what we say about z is invariant under strong bisimilarity, then it also holds for any other z' such that $(x, y)\ R\ z'$. In such cases, we implicitly make some global choice of z and consider $|$ as partially functional.

Definition 2.3. For any graph with testing, let $\bot^G$ denote the set of objects x such that for all $x \leftarrow y$, there exists $y \xleftarrow{\heartsuit} z$.

Any two objects x and y are *fair testing equivalent* iff $x \supset_G y$ and for all $z \supset_G x$, $(x \,|_G\, z) \in \bot^G$ iff $(y \,|_G\, z) \in \bot^G$.

Notation 2.4. We denote fair testing equivalence in G by $\sim_f^G$. Given x, any z such that $x \supset z$ is called a *test* for x, and x *passes* the test iff $(x \,|_G\, z) \in \bot^G$.

Lemma 2.5. *For any morphism $p^G: G \to \Sigma$ and relation $R: (\mathrm{ob}(G))^2 \mathrel{\longrightarrow\!\!\!\!\!\!\!+} \mathrm{ob}(G)$ whose domain is an equivalence relation, R equips $\mathsf{fc}(p^G)$ with testing structure iff it is partially functional up to weak bisimilarity over Σ.*

Proof. R is a strong bisimulation for $\mathsf{fc}(G)$ iff it is a weak bisimulation for G. $\square$

Definition 2.6. A graph with testing is *free* iff it is of the form $\mathsf{fc}(p^G)$.

Definition 2.7. A relation $R{:}\,\mathrm{ob}(G) \twoheadrightarrow \mathrm{ob}(H)$ between the vertex sets of two graphs with testing $p^G{:}\,G \to \mathsf{fc}(\Sigma)$ and $p^H{:}\,H \to \mathsf{fc}(\Sigma)$ is *fair* iff

- $x \mathrel{R} y$ and $x' \mathrel{R} y'$ implies $(x \frown_G x') \Leftrightarrow (y \frown_H y')$;
- R is total and surjective, i.e.,
  - for all $y \in H$, there exists $x \in G$ such that $x \mathrel{R} y$, and
  - for all $x \in G$, there exists $y \in H$ such that $x \mathrel{R} y$;
- $x \mathrel{R} y$ implies $x \sim_{\mathsf{fc}(\Sigma)} y$;
- if $x \mathrel{R} y$, $x' \mathrel{R} y'$, and $x \frown_G x'$, then $(x \mid_G x') \mathrel{R} (y \mid_H y')$.

Lemma 2.8. *A relation $R{:}\,\mathrm{ob}(G) \twoheadrightarrow \mathrm{ob}(H)$ between the vertex sets of two graphs with testing $p^G{:}\,G \to \mathsf{fc}(\Sigma)$ and $p^H{:}\,H \to \mathsf{fc}(\Sigma)$ is* fair *iff its converse $R^\dagger$, defined by $(y \mathrel{R^\dagger} x) \Leftrightarrow (x \mathrel{R} y)$, is.*

Proof. Easy. □

Lemma 2.9. *For any fair relation $R{:}\,\mathrm{ob}(G) \twoheadrightarrow \mathrm{ob}(H)$, if $x \mathrel{R} y$ and $x' \mathrel{R} y'$, then $(x \sim_f^G x') \Leftrightarrow (y \sim_f^H y')$.*

For proving this lemma, we need:

Lemma 2.10. *For any graphs with testing $p^G{:}\,G \to \mathsf{fc}(\Sigma)$ and $p^H{:}\,H \to \mathsf{fc}(\Sigma)$, $x \in G$ and $y \in H$, if $x \sim_{\mathsf{fc}(\Sigma)} y$, then $(x \in \bot^G) \Leftrightarrow (y \in \bot^H)$.*

Proof. Assume $x \in \bot^G$ and consider any transition $y \leftarrow y'$ in H. By bisimilarity, $x \leftarrow x' \sim_{\mathsf{fc}(\Sigma)} y'$. By hypothesis, we find $x' \xleftarrow{\heartsuit} x''$, so by bisimilarity, $y' \xleftarrow{\heartsuit} y''$. Thus, $y \in \bot^H$, which entails the result by symmetry. □

Proof of Lemma 2.9. Consider any such x, y, x', and y'. By Lemma 2.8, it suffices to show one direction of the desired equivalence. So let us assume that $x \sim_f^G x'$. Then $x \frown_G x'$, hence also $y \frown_H y'$ by fairness of R. By symmetry, it again suffices to check one direction of the desired implication. Consider thus any $v \frown_H y$ such that $(y \mid v) \in \bot^H$. By surjectivity of R, we find $u \in G$ such that $u \mathrel{R} v$. By fairness again, we have $(x \mid u) \mathrel{R} (y \mid v)$, so $(x \mid u) \sim_{\mathsf{fc}(\Sigma)} (y \mid v)$, and hence by the previous lemma $(x \mid u) \in \bot^G$. Because $x \sim_f^G x'$, this entails $(x' \mid u) \in \bot^G$, hence by fairness again $(y' \mid v) \in \bot^H$, as desired. □

Definition 2.11. A relation $R{:}\,\mathrm{ob}(G) \twoheadrightarrow \mathrm{ob}(H)$ between the vertex sets of two free graphs with testing respectively generated by $p^G{:}\,G \to \Sigma$ and $p^H{:}\,H \to \Sigma$ is *weakly fair* iff it satisfies the conditions of Definition 2.7, except for the third one, which is replaced by: $x \mathrel{R} y$ implies $x \approx_\Sigma y$.

Corollary 2.12. *For any weakly fair relation $R{:}\,\mathrm{ob}(G) \twoheadrightarrow \mathrm{ob}(H)$, if $x \mathrel{R} y$ and $x' \mathrel{R} y'$, then $(x \sim_f^{\mathsf{fc}(G)} x') \Leftrightarrow (y \sim_f^{\mathsf{fc}(H)} y')$.*

Proof. Because weakly bisimilar over Σ is the same as being strongly bisimilar over $\mathsf{fc}(\Sigma)$, being weakly fair is the same as being fair for the generated graphs with testing. □

2.2. **A π-calculus.** We now make precise the variant of π with which we'll work. We use a chemical abstract machine presentation, based on infinite terms, thus sparing us the need for recursion or replication constructs. Also, we keep track of the channels known to the considered process, i.e., we work with a 'natural deduction' presentation of terms.

Processes are infinite terms coinductively generated by the grammar

$$\frac{\gamma \vdash_{\mathsf{g}} P_1 \quad \dots \quad \gamma \vdash_{\mathsf{g}} P_n}{\gamma \vdash \sum_{i \in n} P_i} \qquad \frac{\gamma \vdash P \quad \gamma \vdash Q}{\gamma \vdash P \mid Q}$$

$$\frac{\gamma, a \vdash P}{\gamma \vdash_{\mathsf{g}} \nu a.P} \qquad \frac{\gamma \vdash P}{\gamma \vdash_{\mathsf{g}} \heartsuit.P} \qquad \frac{\gamma \vdash P}{\gamma \vdash_{\mathsf{g}} \tau.P} \qquad \frac{a \in \gamma \quad \gamma, b \vdash P}{\gamma \vdash_{\mathsf{g}} a(b).P} \qquad \frac{a, b \in \gamma \quad \gamma \vdash P}{\gamma \vdash_{\mathsf{g}} \bar{a}\langle b \rangle.P},$$

where

- we have two judgements, $\vdash$ for *processes* and $\vdash_{\mathsf{g}}$ for *guarded processes*;
- γ ranges over finite sets of natural numbers, and
- γ, a is defined iff $a \notin \gamma$ and then denotes $\gamma \uplus \{a\}$.

Notation 2.13. Let Pi be the set of all such (non-guarded) processes. Let Pi_γ denote the set of processes $\gamma \vdash P$.

In the following, processes are considered equivalent up to renaming of bound channels. Capture-avoiding substitution extends the assignment $\gamma \mapsto Pi_\gamma$ to a functor $\mathsf{set} \to \mathsf{Set}$ mapping $\sigma : \gamma \to \gamma'$ to $P \mapsto P[\sigma]$.

Notation 2.14. For any $\gamma \vdash_{\mathsf{g}} P$, $\gamma \vdash Q$ of the form $\sum_{i \in n} Q_i$, and injection $h : n \hookrightarrow n+1$, we denote by $P +_h Q$ the sum $\sum_{j \in n+1} P_j$, where $P_{h(i)} = Q_i$ for all $i \in n$ and $P_k = P$, for k the unique element of $(n+1) \setminus \mathrm{Im}(h)$.

Definition 2.15. Let $-^{\odot}$ denote the finite multiset monad on sets.

Definition 2.16. A *configuration* is an element of $\mathrm{Conf} = \sum_{\gamma \in \mathcal{P}_f(\mathbb{N})} Pi_\gamma^{\odot}$.

Notation 2.17. Configurations (γ, S) will be denoted by $\langle \gamma \| S \rangle$, and we will use list syntax $[P_1, \dots, P_n]$ for multisets, sometimes dropping brackets, e.g., as in $\langle \gamma \| P_1, \dots, P_n \rangle$. We sometimes resort to a hopefully clear 'multiset comprehension' notation $[P \mid \varphi(P)]$. We use $\cup$ for multiset union and $x :: M = [x] \cup M$.

Just as Pi, Conf extends to a functor $\mathsf{set} \to \mathsf{Set}$ by capture-avoiding substitution.

In Figure 1, we extend Conf to a graph over Σ (omitting identity edges). There, we let R and R' range over processes of the form $\sum_{i \in n} P_i$. The last rule makes sense because each transition as in the premise implicitly comes with an inclusion $h : \gamma_1 \hookrightarrow \gamma_2$, and the second occurrence of S is implicitly $S[h]$.

We now make $\mathsf{p}^{\mathrm{Conf}} : \mathrm{Conf} \to \Sigma$ into a free graph with testing.

Definition 2.18. For any $\langle \gamma \| S \rangle, \langle \gamma' \| S' \rangle \in \mathrm{Conf}$, let $\langle \gamma \| S \rangle @ \langle \gamma' \| S' \rangle$ denote $\langle \gamma \| S \cup S' \rangle$ if $\gamma = \gamma'$ and be undefined otherwise. Let furthermore $\epsilon_\gamma = \langle \gamma \| \rangle$.

Then @, as a partial map, is in particular a relation $\mathrm{ob}(\mathrm{Conf})^2 \twoheadrightarrow \mathrm{ob}(\mathrm{Conf})$, whose domain is an equivalence relation. Being partially functional, it is trivially partially functional up to weak bisimilarity. Lemma 2.5 entails:

Proposition 2.19. *The morphism $\mathsf{fc}(\mathsf{p}^{\mathrm{Conf}}) : \mathsf{fc}(\mathrm{Conf}) \to \mathsf{fc}(\Sigma)$, with @ as testing relation, forms a graph with testing.*

$$\frac{}{\langle\gamma\|P\,|\,Q\rangle \overset{\tau}{\leftarrow} \langle\gamma\|P,Q\rangle} \qquad \frac{}{\langle\gamma\|\tau.P +_h R\rangle \overset{\tau}{\leftarrow} \langle\gamma\|P\rangle} \qquad \frac{}{\langle\gamma\|\heartsuit.P +_h R\rangle \overset{\heartsuit}{\leftarrow} \langle\gamma\|P\rangle}$$

$$\frac{}{\langle\gamma\|\nu a.P +_h R\rangle \overset{\tau}{\leftarrow} \langle\gamma, a\|P\rangle} \qquad \frac{}{\langle\gamma\|a(b).P +_h R, \bar{a}\langle c\rangle.Q +_{h'} R'\rangle \overset{\tau}{\leftarrow} \langle\gamma\|P[b\mapsto c], Q\rangle}$$

$$\frac{\langle\gamma_1\|S_1\rangle \overset{\alpha}{\leftarrow} \langle\gamma_2\|S_2\rangle}{\langle\gamma_1\|S\cup S_1\rangle \overset{\alpha}{\leftarrow} \langle\gamma_2\|S\cup S_2\rangle}\ (\alpha\in\{\tau,\heartsuit\})$$

Figure 1: Reduction rules for Conf

Definition 2.3 thus yields $\sim_f^{\mathrm{Conf}}$ which we take as our definition of fair testing equivalence for π.

2.3. **Playgrounds.** We here recall the axioms for playgrounds [34]. Some constructions and results are developped from these axioms in op. cit. Some of the main ideas are reviewed and reworked in Sections 4 and 5.

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "u"'] \arrow[dr, phantom, "\Nearrow\alpha"] & X' \arrow[r, "k"] \arrow[d, "u'"'] \arrow[dr, phantom, "\Nearrow\alpha'"] & X'' \arrow[d, "u''"] \\
Y \arrow[r, "h'"] \arrow[d, "v"'] \arrow[dr, phantom, "\Nearrow\beta"] & Y' \arrow[r, "k'"] \arrow[d, "v'"'] \arrow[dr, phantom, "\Nearrow\beta'"] & Y'' \arrow[d, "v''"] \\
Z \arrow[r, "h''"'] & Z' \arrow[r, "k''"'] & Z''
\end{tikzcd},$$

Let us start with a brief recap of pseudo double categories. A pseudo double category $\mathbb{D}$ consists of a set $\mathrm{ob}(\mathbb{D})$ of *objects*, shared by a 'horizontal' category $\mathbb{D}_h$ and a 'vertical' bicategory $\mathbb{D}_v$. Since we won't consider strict double categories, we'll often omit the word 'pseudo'. Following Paré [60], $\mathbb{D}_h$, being a mere category, has standard notation (normal arrows, $\circ$ for composition, *id* for identities), while the bicategory $\mathbb{D}_v$ earns fancier notation ($\relbar\!\!\bullet\!\!\rightarrow$ for arrows, $\bullet$ for composition, $id^\bullet$ for identities). $\mathbb{D}$ is furthermore equipped with a set of *double cells* α, which have vertical, resp. horizontal, domain and codomain, denoted by $\mathrm{dom}_v(\alpha)$, $\mathrm{cod}_v(\alpha)$, $\mathrm{dom}_h(\alpha)$, and $\mathrm{cod}_h(\alpha)$. We picture this as, e.g., α above, where $u = \mathrm{dom}_h(\alpha)$, $u' = \mathrm{cod}_h(\alpha)$, $h = \mathrm{dom}_v(\alpha)$, and $h' = \mathrm{cod}_v(\alpha)$. $\mathbb{D}$ is furthermore equipped with operations for composing double cells: $\circ$ composes them along a common vertical morphism, $\bullet$ composes along horizontal morphisms. Both vertical compositions (of morphisms and double cells) may only be associative up to coherent isomorphism. The full axiomatisation is given by Garner [25], and we here only mention the *interchange law*, which says that the two ways of parsing the above diagram coincide: $(\beta'\circ\beta)\bullet(\alpha'\circ\alpha) = (\beta'\bullet\alpha')\circ(\beta\bullet\alpha)$.

For any (pseudo) double category $\mathbb{D}$, we denote by $\mathbb{D}_H$ the category with vertical morphisms as objects and double cells as morphisms, and by $\mathbb{D}_V$ the bicategory with horizontal morphisms as objects and double cells as morphisms. Domain and codomain maps arrange into functors $\mathrm{dom}_v, \mathrm{cod}_v\colon \mathbb{D}_H \to \mathbb{D}_h$ and $\mathrm{dom}_h, \mathrm{cod}_h\colon \mathbb{D}_V \to \mathbb{D}_v$. We will refer to dom_v and cod_v simply as dom and cod, reserving subscripts for dom_h and cod_h.

We then need to recall the notion of fibration (see [38]). Consider any functor $p\colon\mathbb{E}\to\mathbb{B}$. A morphism $r\colon E'\to E$ in $\mathbb{E}$ is *cartesian* when, as on the right above, for all $t\colon E''\to E$ and $k\colon p(E'')\to p(E')$, if $p(r)\circ k = p(t)$ then there exists a unique $s\colon E''\to E'$ such that $p(s) = k$

and $r \circ s = t$, as in

$$\begin{tikzcd}
E'' \arrow[drr, bend left, "t"] \arrow[dr, dashed, "s"'] & & \\
& E' \arrow[r, "r"'] & E \\
p(E'') \arrow[drr, bend left, "p(t)"] \arrow[dr, "k"'] & & \\
& p(E') \arrow[r, "p(r)"'] & p(E)
\end{tikzcd}$$

Definition 2.20. A functor $p: \mathbb{E} \to \mathbb{B}$ is a *fibration* iff for all $E \in \mathbb{E}$, any $h: B' \to p(E)$ has a cartesian lifting, i.e., a cartesian antecedent by p.

We may now state the definition of playgrounds. The following differs slightly from the original definition, but only in presentation and terminology.

Definition 2.21. A *playground* is a pseudo double category $\mathbb{D}$ such that cod is a fibration, equipped with

- a full subcategory $\mathbb{I} \hookrightarrow \mathbb{D}_h$ of objects called *individuals*,
- a replete[1] class $\mathbb{M}$ of vertical morphisms called *actions*, with replete subclasses $\mathbb{B}$ and $\mathbb{F}$, respectively called *basic* and *full* actions,
- a map $|-|: \mathrm{ob}(\mathbb{D}_H) \to \mathbb{N}$ called the *length*,

satisfying the following conditions:

(P1) $\mathbb{I}$ is discrete. Basic actions have no non-trivial automorphisms in $\mathbb{D}_H$. Vertical identities on individuals have no non-trivial endomorphisms.

(P2) (Individuality) Basic actions have individuals as both domain and codomain.

(P3) (Atomicity) For any cell $\alpha: v \to u$, if $|u| = 0$ then also $|v| = 0$. Up to a special isomorphism in $\mathbb{D}_H$, all plays u of length $n > 0$ admit decompositions into n actions. For any $u: X \multimap Y$ of length 0, there is an isomorphism $id^\bullet_X \to u$ as on the right in $\mathbb{D}_H$.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Rightarrow \alpha^u"] & X \arrow[d, "u"] \\
X \arrow[r, "\bar{u}"'] & Y
\end{tikzcd}$$

(P4) (Fibration, continued) Restrictions of actions (resp. full actions) to individuals either are actions (resp. full actions), or have length 0.

(P5) (Views) For any action $M: Y \multimap X$, and $y: d \to Y$ with $d \in \mathbb{I}$, there exists a cell

$$\begin{tikzcd}
d \arrow[r, "y"] \arrow[d, dashed, "v^{y,M}"'] \arrow[dr, phantom, "\Rightarrow \alpha^{y,M}"] & Y \arrow[d, "M"] \\
d^{y,M} \arrow[r, dashed, "y^M"'] & X,
\end{tikzcd}$$

where $v^{y,M}$ either is a basic action or has length 0, which is unique up to canonical isomorphism, i.e., for any $y': d' \to X$, $v': d \multimap d'$, and $\alpha': v' \to M$, we have $y = y^M$ and there exists a unique $\beta: v \to v'$ making the diagram

[1]Replete means stable under isomorphism.

$$
\begin{tikzcd}
 & d \arrow[r, "y"] \arrow[dl, "v^{y,M}"'] \arrow[ddr, "v'"] \arrow[r, phantom, shift right=3, "\Uparrow \alpha^{y,M}"'] & Y \arrow[d, "M"] \arrow[dl, phantom, "\Uparrow \alpha'"] \\
d^{y,M} \arrow[rr, "y^M"] \arrow[dr, equal] \arrow[r, phantom, "\beta \Rightarrow"] & & X \\
 & d' \arrow[ur, "y'"'] &
\end{tikzcd}
$$

commute.

(P6) (Left decomposition) Any double cell

$$
\begin{tikzcd}
A \arrow[r, "h"] \arrow[dd, "u"'] \arrow[ddr, phantom, "\Uparrow \alpha"] & X \arrow[d, "w_1"] \\
 & Y \arrow[d, "w_2"] \\
B \arrow[r, "k"'] & Z
\end{tikzcd}
\qquad \text{decomposes as} \qquad
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "u_1"] \arrow[dd, bend right=60, "u"'] \arrow[dr, phantom, "\Uparrow \alpha_1"] \arrow[d, phantom, shift right=4, "\alpha_3 \Rightarrow"'] & X \arrow[d, "w_1"] \\
C \arrow[r, "l"] \arrow[d, "u_2"] \arrow[dr, phantom, "\Uparrow \alpha_2"] & Y \arrow[d, "w_2"] \\
B \arrow[r, "k"'] & Z
\end{tikzcd}
$$

with α_3 an isomorphism, in an essentially unique way.

(P7) (Right decomposition) Any double cell as in the center below, where b is a basic action and M is an action, decomposes in exactly one of the forms on the left and right:

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\Uparrow \alpha_1"] & X \arrow[d] \\
B \arrow[r] \arrow[d] \arrow[dr, phantom, "\Uparrow \alpha_2"] & Y \arrow[d] \\
C \arrow[r] & Z
\end{tikzcd}
\quad \leftsquigarrow \quad
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "w"'] \arrow[ddr, phantom, "\Uparrow \alpha"] & X \arrow[d, "u"] \\
B \arrow[d, "b"'] & Y \arrow[d, "M"] \\
C \arrow[r, "k"'] & Z
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\Uparrow \alpha_1"] & X \arrow[d] \\
B \arrow[d] & Y \arrow[d] \\
C \arrow[r] \arrow[ur] \arrow[ur, phantom, shift right=3, "\alpha_2 \Rightarrow"'] & Z.
\end{tikzcd}
$$

(P8) (Finiteness) For any position X, the category $\mathbb{I}/X$ is finite.

(P9) (Basic vs. full) For any $d \in \mathbb{I}$ and actions $M : X \multimap d$, $M' : X' \multimap d$, and $b : d' \multimap d$ with M and M' full and b basic, if there exist cells $M \leftarrow b \rightarrow M'$ then $M \cong M'$.

3. A playground for π

3.1. **A pseudo double category of traces.** In this section, we introduce our notion of trace, which is based on certain combinatorial objects, close in spirit to string diagrams. We first define these string diagrams, and then use them to define traces. Configurations are special, hypergraph-like string diagrams whose vertices represent agents and whose hyperedges represent channels. A perhaps surprising point is that actions are not just a binary relation between configurations, because we not only want to say *when* there is an action from one configuration to another, but also *how* this action is performed. This will be implemented by viewing actions from X to Y as *cospans* $Y \to M \leftarrow X$ in a certain category $\widehat{\mathbb{C}}_f$, whose objects we call higher-dimensional string diagrams for lack of a better term. The idea is that X and Y respectively are the initial and final configurations, and that M describes how one goes from X to Y. By combining such actions (by pushout), we get a bicategory $\mathbb{D}_v$ of configurations and traces. Finally, we recast $\mathbb{D}_v$ as the vertical bicategory of a pseudo double category $\mathbb{D}$.

3.1.1. *String diagrams.* The category $\widehat{\mathbb{C}}_f$ will be a category of finite presheaves over a base category, $\mathbb{C}$. Let us motivate the definition of $\mathbb{C}$ by recalling that (directed, multi) graphs may be seen as presheaves over the category with two objects $\star$ and $[1]$, and two non-identity morphisms $s, t \colon \star \to [1]$. Any such presheaf G represents the graph with vertices in $G(\star)$ and edges in $G[1]$, the source and target of any $e \in G[1]$ being respectively $G(s)(e)$ and $G(t)(e)$, or $e \cdot s$ and $e \cdot t$ for short. A way to visualise how such presheaves represent graphs is to compute their *categories of elements* [51]. Recall that the category of elements $\int G$ for a presheaf G over $\mathbb{C}$ has as objects pairs (c, x) with $c \in \mathbb{C}$ and $x \in G(c)$, and as morphisms $(c,x) \to (d,y)$ all morphisms $f \colon c \to d$ in $\mathbb{C}$ such that $y \cdot f = x$. This category admits a canonical functor π_G to $\mathbb{C}$, and G is the colimit of the composite $\int G \xrightarrow{\pi_G} \mathbb{C} \xrightarrow{\mathsf{y}} \widehat{\mathbb{C}}$ with the Yoneda embedding. E.g., the category of elements for $\mathsf{y}[1]$ is the poset $(\star, s) \xrightarrow{s} ([1], id_{[1]}) \xleftarrow{t} (\star, t)$, which could be pictured as , where dots represent vertices, the triangle represents the edge, and links materialise the graph of $G(s)$ and $G(t)$, the convention being that t connects to the apex of the triangle. We thus recover some graphical intuition.

Our string diagrams will also be defined as particular presheaves over some base category $\mathbb{C}$. However, since we'll only be interested in *finite* structures, we restrict ourselves to the category $\widehat{\mathbb{C}}$ of presheaves of finite sets. In the case of graphs, presheaves of finite sets are graphs whose nodes and edges are identified using finitely many natural numbers. Such graphs are thus finite. In our case, the base category $\mathbb{C}$ is infinite, so presheaves of finite sets may represent infinite structures. However, our notion of trace will only involve finite ones.

Let us give the formal definition of $\mathbb{C}$ for reference. We advise to skip it on a first reading, as we then attempt to provide some graphical intuition.

Definition 3.1. Let $G_{\mathbb{C}}$ be the graph with, for all n, m, with $a, b \in n$ and $c, d \in m$:

- vertices $\star$, $[n]$, π_n^l, π_n^r, π_n, ν_n, $\heartsuit_n$, τ_n, $\iota_{n,a}$, $o_{n,a,b}$, and $\tau_{n,a,m,c,d}$;
- edges $s_1, \dots, s_n : \star \to [n]$, plus, $\forall v \in \{\pi_n^l, \pi_n^r, \heartsuit_n, \tau_n, o_{n,a,b}\}$, edges $s, t : [n] \to v$;
- edges $[n] \xrightarrow{t} \nu_n \xleftarrow{s} [n+1]$ and $[n] \xrightarrow{t} \iota_{n,a} \xleftarrow{s} [n+1]$;
- edges $\pi_n^l \xrightarrow{l} \pi_n \xleftarrow{r} \pi_n^r$ and $\iota_{n,a} \xrightarrow{\rho} \tau_{n,a,m,c,d} \xleftarrow{\epsilon} o_{m,c,d}$.

Let $\mathbb{C}$ be the free category on $G_{\mathbb{C}}$, modulo the equations

$$s \circ s_i = t \circ t_i \qquad l \circ t = r \circ t \qquad \rho \circ t \circ s_a = \epsilon \circ t \circ s_c \qquad \rho \circ s \circ s_{n+1} = \epsilon \circ s \circ s_d.$$

The first equation should be understood in $\mathbb{C}(\star, v)$ for all $n \in \mathbb{N}$, $i \in n$, and $v \in \cup_{a,b \in n} \{\pi_n^l, \pi_n^r, \heartsuit_n, \tau_n, \iota_{n,a}, o_{n,a,b}, \nu_n\}$. (This is rather elliptic: if v has the shape $\iota_{n,a}$ or ν_n, $s \circ s_i$ is really $\star \xrightarrow{s_i} [n+1] \xrightarrow{s} v$.) The second equation should be understood in $\mathbb{C}(\star, \pi_n)$ for all n, and the last two in $\mathbb{C}(\star, \tau_{n,a,m,c,d})$, for all n, m, $a \in n$, and $c, d \in m$.

Our category of string diagrams is the category of finite presheaves $\widehat{\mathbb{C}}_f$. To explain the design of $\mathbb{C}$, let us compute a few categories of elements. Let us start with an easy one, that of $[3] \in \mathbb{C}$ (we implicitly identify any $c \in \mathbb{C}$ with $\mathsf{y}c$). An easy computation shows that it is the poset pictured in the top left part of Figure 2. We think of it as a configuration with one agent $([3], id_{[3]})$ connected to three channels, and draw it as in the top right part, where

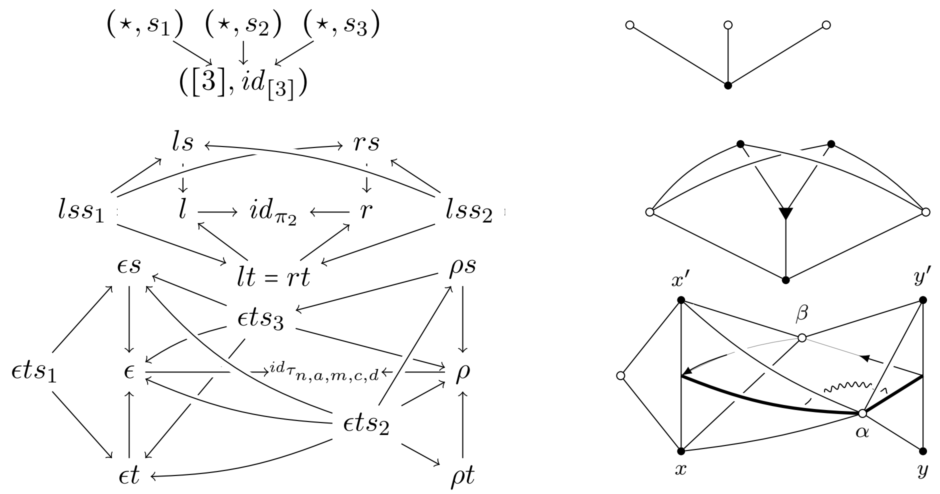


Figure 2: Categories of elements for $[3]$, π_2, and $\tau_{1,1,3,2,3}$, with graphical representation

the bullet represents the agent, and circles represent channels. In the presheaf, elements over $[3]$ represent ternary agents, while elements over $\star$ represent channels.

Definition 3.2. *Configurations* are finite presheaves empty except perhaps on $\star$ and $[n]$'s.

Other objects will represent actions. In fact, we may equip the objects of $\mathbb{C}$ with a *dimension*: $\star$ has dim 0, any $[n]$ has dim 1, all of $\tau_n, \pi^l_n, \pi^r_n, \heartsuit_n, \iota_{n,i}, o_{n,j,k}, \nu_n$ have dim 2, π_n has dim 3, $\tau_{n,i,m,j,k}$ has dim 4.

Definition 3.3. We accordingly define the *dimension* of a presheaf X on $\mathbb{C}$ to be the lowest $d \in \mathbb{N}$ such that for any $c \in \mathbb{C}$ of dimension strictly greater than d, $X(c) = \emptyset$.

A configuration is thus equivalently a finite presheaf in $[\mathbb{C}^{op}, \mathsf{set}]$ of dimension at most 1. An *interface* is one of dimension 0.

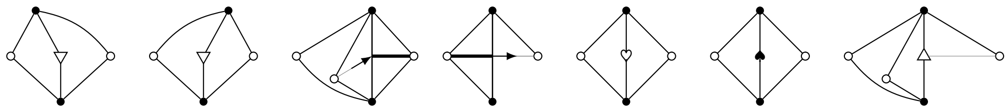
Definition 3.4. A map in $\widehat{\mathbb{C}}$ is *1-injective* iff it is injective in all strictly positive dimensions.

A *morphism of configurations* is a 1-injective morphism in $\widehat{\mathbb{C}}$. The intuition for a morphism $X \to Y$ between configurations is thus that X embeds into Y, possibly identifying some channels.

Definition 3.5. Configurations and morphisms between them form a category $\mathbb{D}_h$.

A more difficult category of elements is that of π_2. It is the poset generated by the left-hand graph in the second row of Figure 2 (omitting base objects for conciseness). We think of it as a binary agent (lt) forking into two agents (ls and rs), and draw it as on the right. The graphical convention is that a black triangle stands for the presence of id_{π_2}, l, and r. Below, we represent just l as a white triangle with only a left-hand branch, and symmetrically for r. Furthermore, in all our pictures, time flows 'upwards'.

Another category of elements, characteristic of the π-calculus, is the one for synchronisation $\tau_{n,a,m,c,d}$. The case $(n,a,m,c,d) = (1,1,3,2,3)$ is the poset generated by the graph on the bottom left of Figure 2, which we will draw as on the right. The left-hand ternary agent x outputs its 3rd channel, here β, on its 2nd channel, here α. The right-hand unary agent y receives the sent channel on its 1st channel, here α. Both agents have two occurrences, one before and one after the action, respectively marked as x/x' and y/y'. Both x



$$\begin{array}{ccccccc}
[p] & [p] & [m] & [n+1] & [p] & [p] & [p+1] \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\pi^l_p & \pi^r_p & o_{m,c,d} & \iota_{n,a} & \heartsuit_p & \tau_p & \nu_p \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
[p] & [p] & [m] & [n] & [p] & [p] & [p]
\end{array}$$

Figure 3: Pictures and corresponding cospans for π^l_p, π^r_p, $o_{m,c,d}$, $\iota_{n,a}$, $\heartsuit_p$, τ_p, and ν_p

and x' are ternary here, while y is unary and y', having gained knowledge of β, is binary. There are actually three actions here, in the sense that there are three higher-dimensional objects. The first is the output action ϵ from x to x', graphically represented as the middle point of (intended to evoke the point where β enters channel α). The second is the input action ρ from y to y', graphically represented as the middle point of (where β exits channel α). The third action is the synchronisation itself, which 'glues' the other two together, as represented by the squiggly line.

We leave the computation of other categories of elements as an exercise to the reader. The remaining string diagrams are depicted in the top row of Figure 3, for $p = 2$ and $(n, a, m, c, d) = (1, 1, 3, 2, 3)$. The first two are *views*, in the game semantical sense, of the fork action π_2 explained above. The next two, $o_{m,c,d}$ (for 'output') and $\iota_{n,a}$ (for 'input'), respectively are views for the sender and receiver in a synchronisation action. The τ_p action is a silent, dummy action as standard in the π-calculus. The $\heartsuit_n$ action is a special 'tick' action used for defining fair testing equivalence. The last one is a channel creation action.

3.1.2. *From string diagrams to actions.* In the previous section, we have defined our category of string diagrams as $\widehat{\mathbb{C}}_f$, and provided some graphical intuition on its objects. The next step is to construct a bicategory whose objects are configurations, and whose morphisms represent traces. We start in this section by defining in which sense higher-dimensional objects of $\mathbb{C}$ represent actions, and continue in the next one by explaining how to compose actions to form traces. Actions are defined in two stages: *seeds*, first, give the local form of actions, which are then defined by embedding seeds into bigger configurations.

To start with, until now, our string diagrams contain no information about the 'flow of time', although we mentioned it informally in the previous section. To add this information, for each string diagram M representing an action, we define its initial and final configurations, say X and Y, and view the whole action as a cospan $Y \xrightarrow{s} M \xleftarrow{t} X$. We have taken care, in drawing our pictures before, of placing initial configurations at the bottom, and final configurations at the top. So, e.g., the initial and final configurations for the synchronisation action are pictured above and they map into (the representable presheaf over) $\tau_{1,1,3,2,3}$ in the obvious ways, yielding the cospan $Y \xrightarrow{s} \tau_{1,1,3,2,3} \xleftarrow{t} X$.

We leave it to the reader to define, based on the above pictures, the expected cospans for forking and synchronisation as on the right, plus the remaining ones

$$\begin{array}{ccc}
[p] \mid\mid [p] & \qquad & [m]\,{}_{c,d}|_{a,n+1}\,[n+1] \\
\downarrow & & \downarrow \\
\pi_p & & \tau_{n,a,m,c,d} \\
\uparrow & & \uparrow \\
[p] & & [m]\,{}_{c}|_{a}\,[n]
\end{array}$$

specified in the bottom row of Figure 3. Initial configurations are at the bottom, and we use:

Notation 3.6. We denote by $[m]_{a_1,\ldots,a_p}|_{c_1,\ldots,c_p}[n]$ the configuration consisting of an m-ary agent x and an n-ary agent y, quotiented by the equations $x \cdot s_{a_k} = y \cdot s_{c_k}$ for all $k \in p$. When both lists are empty, by convention, $m = n$ and the agents share all channels in order.

Definition 3.7. These cospans are called *seeds*.

We now define actions from seeds by embedding the latter into bigger configurations. E.g., we allow a fork action to occur in a configuration with more than one agent.

Definition 3.8. The *interface* I_F of a presheaf $F \in \widehat{\mathbb{C}}$ is $F(\star) \cdot \star$, the $F(\star)$-fold coproduct of $\star$ with itself, or in other words the configuration consisting solely of F's channels. The interface of a seed $Y \xrightarrow{s} M \xleftarrow{t} X$ is I_X.

Since channels occurring in the initial configuration remain in the final one, we have for each seed a cone from I_X to the seed. For any morphism of positions $I_X \to Z$, pushing the cone along $I_X \to Z$ using the universal property of pushout as on the right yields a new cospan, say $Y' \to M' \leftarrow X'$.

$$\begin{tikzcd}
 & Y \arrow[rr] \arrow[d] & & Y' \arrow[d, dashed] \\
 & M \arrow[rr] & & M' \\
I_X \arrow[uur] \arrow[ur] \arrow[rr] \arrow[dr] & & Z \arrow[ur] \arrow[uur] \arrow[dr] & \\
 & X \arrow[uu] \arrow[rr] & & X' \arrow[uu, dashed]
\end{tikzcd} \tag{3.1}$$

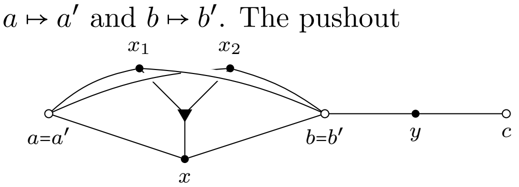
Definition 3.9. Let *actions* be all such pushouts of seeds.

Intuitively, taking pushouts glues string diagrams together. Let us do a few examples.

Example 3.10. The seed $[2]\,|\,[2] \xrightarrow{[ls,rs]} \pi_2 \xleftarrow{lt} [2]$ has as interface the presheaf $I_{[2]} = \star + \star$, consisting of two channels, say a and b. Consider the configuration $[2] + \star$ consisting of an agent y connected to two channels b' and c, plus an additional channel a'. Further consider the map $h\colon I_{[2]} \to [2] + \star$ defined by $a \mapsto a'$ and $b \mapsto b'$. The pushout

$$\begin{tikzcd}
I_{[2]} \arrow[r] \arrow[d] & {[2] + \star} \arrow[d] \\
\pi_2 \arrow[r] & M'
\end{tikzcd}$$

is



.

The meaning of such an action is that x forks while y is passive.

Example 3.11. Because we push along *initial* channels, the interface of a seed may not contain all involved channels. E.g., in an input action (not part of any synchronisation), the received channel cannot be part of the initial configuration.

3.1.3. *From actions to traces.* Having defined actions, we now define their composition to yield our bicategory $\mathbb{D}_v$ of configurations and traces. Consider $\mathsf{Cospan}(\widehat{\mathbb{C}}_f)$, the bicategory which has as objects all finite presheaves on $\mathbb{C}$, as morphisms $X \to Y$ all cospans $X \to U \leftarrow Y$, and obvious 2-cells. Composition is given by pushout, and hence is not strictly associative.

Notation 3.12. By convention, the initial configuration is the *target* of the morphism in $\mathsf{Cospan}(\widehat{\mathbb{C}}_f)$. We denote morphisms in $\mathsf{Cospan}(\widehat{\mathbb{C}}_f)$ with special arrows $Y \mathrel{\rightarrow\!\!\bullet\!\!\rightarrow} X$; composition and identities are denoted with $\bullet$ and $id^{\bullet}$.

Figure 4: Example traces

Definition 3.13. A *trace* is any cospan in $\widehat{\mathbb{C}}_f$ which is isomorphic to some finite, possibly empty composite of actions in $\mathsf{Cospan}(\widehat{\mathbb{C}}_f)$. Let $\mathbb{D}_v$ denote the subbicategory of $\mathsf{Cospan}(\widehat{\mathbb{C}}_f)$ obtained by restricting to configurations, traces, and 1-injective 2-cells.

Thus, arrows $X \to Y$ in $\mathbb{D}_h$ denote embeddings of X into Y (up to identification of channels), whereas arrows $Y \rightarrow\!\!\!\!\bullet\!\!\!\rightarrow X$ in $\mathbb{D}_v$ denote traces with X initial and Y final. Intuitively, composition in $\mathbb{D}_v$ glues string diagrams on top of each other, which yields a truly concurrent notion of trace: the only information retained in a trace about the order of occurrence of actions is their causal dependencies.

Example 3.14. Composing the action of Example 3.10 with a forking action by y yields the first string diagram of Figure 4, which shows that the ordering between remote actions is irrelevant. To illustrate how composition retains causal dependencies between actions, consider the second string diagram. It is unfolded for readability: one should identify both framed nodes, resp. both circled ones. In the initial configuration, there are channels a, b, and c, and three agents $x(a, b)$, $y(b)$, and $z(a, c)$ (channels known to each agent are in parentheses). In a first action, x sends a on b, and y receives it. In a second action, z sends c on a, and the avatar y' of y receives it. The second action is enabled by the first, by which y gains knowledge of a.

3.1.4. *The main double category.* At last, we define the base double category $\mathbb{D}$ of our playground for the π-calculus. It is a sub-double category of a double category of cospans in $\widehat{\mathbb{C}}$.

Consider the double category $\mathbb{D}^0$ with

- configurations as objects,
- horizontal morphisms $X \to Y$ given by all natural transformations $h: X \to Y$,
- vertical morphisms $X \to Y$ given by cospans $X \xrightarrow{s} U \xleftarrow{t} Y$ in $\widehat{\mathbb{C}}$,
- and double cells $U \to V$ given by commuting diagrams

$$
\begin{tikzcd}
X' \arrow[r, "k"] \arrow[d, "s_U"'] & Y' \arrow[d, "s_V"] \\
U \arrow[r, "l"] & V \\
X \arrow[r, "h"'] \arrow[u, "t_U"] & Y. \arrow[u, "t_V"']
\end{tikzcd}
\tag{3.2}
$$

Definition 3.15. Let $\mathbb{D}$ denote the sub-double category of $\mathbb{D}^0$ obtained by restricting

- vertical morphisms to traces,
- horizontal morphisms to 1-injective maps,
- double cells to diagrams (3.2) in which k, l, and h are 1-injective.

Proposition 3.16. *$\mathbb{D}$ indeed forms a sub-double category of $\mathbb{D}^0$, i.e., is stable under all composition operations.*

Proof. The only non-obvious point is that double cells in $\mathbb{D}$ are stable under vertical composition, and in particular that the middle component of the composite is 1-injective. This follows from Lemma 3.18 and Corollary 3.20 below. □

Definition 3.17. Let $\mathcal{V}_0$ denote the set of 't'-legs (i.e., lower legs) of seeds.

Lemma 3.18. *For any morphism* (3.2) *in $\mathbb{D}^0_H$, if U and V are traces, then the upper square is a pullback.*

Proof. For any trace $Y \xrightarrow{s} P \xleftarrow{t} X$, s is mono, and furthermore $s_\star : Y(\star) \cong P(\star)$ is an iso. Finally, for all n, $Y[n]$ consists of all elements of $P[n]$ which are not in the image of (the action of) any map in $\mathcal{V}_0$.

Now, consider any double cell as in (3.2). By construction, the mediating arrow $X' \to U \times_V Y'$ is mono. To show that it is epi, we proceed pointwise. Over $\star$, the result follows from s_U and s_V being isomorphisms. Over $[n]$, if $x \in (U \times_V Y')[n]$ then $x \in U[n]$, and $l(x) \in V[n]$ is not in the image of (the action of) any $t \in \mathcal{V}_0$. But if there existed $y \in U(c)$ such that $y \cdot t = x$, then by naturality we would have $l(y) \cdot t = l(x)$, contradicting the latter. □

Lemma 3.19. *In sets, consider any cube*

$$\begin{tikzcd}
I \arrow[rr] \arrow[dr] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & B \arrow[dr] \arrow[dd] & \\
& A \arrow[rr, crossing over] & & C \arrow[dd, "f"] \\
I' \arrow[rr] \arrow[dr] & & B' \arrow[dr] & \\
& A' \arrow[rr] \arrow[from=uu, crossing over] & & C',
\end{tikzcd}$$

with the marked pushouts and pullback, and with all arrows mono except perhaps f. Then, f is also mono and the front square is also a pullback.

Proof. Any such cube is naturally isomorphic to some cube of the shape

$$\begin{tikzcd}
n \arrow[rr] \arrow[dr] \arrow[dd, "\mathrm{inj}_l"'] & & n+q \arrow[dr] \arrow[dd, "\mathrm{inj}_l + k" near end] & \\
& n+m \arrow[rr, crossing over] & & n+m+q \arrow[dd, "f=n+h+k"] \\
n+n' \arrow[rr] \arrow[dr, "\mathrm{inj}_l"'] & & n+n'+r \arrow[dr] & \\
& n+n'+p \arrow[rr] \arrow[from=uu, crossing over, "n+h"' near start] & & n+n'+p+r,
\end{tikzcd}$$

the only non-trivial point being that the map $n + q \to n + n' + r$ has the given shape. But that's because we know that its pullback along $n + n' \to n + n' + r$ is inj_1, so the image of q has to lie in r. □

Corollary 3.20. *Consider any cube*

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] \arrow[dd] & & B \arrow[dr] \arrow[dd] & \\
& A \arrow[rr] \arrow[dd] & & C \arrow[dd, "f"] \\
X' \arrow[rr] \arrow[dr] & & B' \arrow[dr] & \\
& A' \arrow[rr] & & C',
\end{tikzcd}$$

in $\widehat{\mathbb{C}}$ in which all arrows except perhaps f are 1-injective, and the marked squares are pushouts, resp. pullbacks. Then f is also 1-injective.

Proof. We proceed pointwise. On any object C of dimension > 0, we obtain a diagram in sets for which the lemma applies. □

3.2. **Codomain is a fibration.** In this section, we prove that $\mathbb{D}$ satisfies the primary axiom for playgrounds, namely that the codomain functor $\mathbb{D}_H \to \mathbb{D}_h$ is a fibration. We proceed as follows. We first define a (strong) factorisation system on $\widehat{\mathbb{C}}$, from which we derive an intermediate sub-double category $\mathbb{D} \hookrightarrow \mathbb{D}^1 \hookrightarrow \mathbb{D}^0$ such that $\mathbb{D}^1_v$ contains traces. By the properties of factorisation, the codomain functor $\mathbb{D}^1_H \to \mathbb{D}^1_h$ is a fibration. Finally, $\mathbb{D}_H \to \mathbb{D}_h$ being a fibration follows from the fact that cartesian liftings of any trace along any morphism in $\mathbb{D}_h$ are in $\mathbb{D}_H$. We first check this for seeds. Next, using a covariant *oplifting* construction, we check the desired property for actions. Finally, we extend the result to arbitrary traces.

3.2.1. *A factorisation system.* Our main tool is the following (strong) factorisation system $(\mathcal{V}, \mathcal{H})$ on $\widehat{\mathbb{C}}$ (see the definition right after Proposition 3.21 below). The idea is that all three components of cartesian morphisms in $\mathbb{D}_H$ are in $\mathcal{H}$, while t-legs of vertical morphisms are in $\mathcal{V}$. The cartesian lifting of V in (3.2) along any $h\colon X \to Y$ is then given by factoring $t_V \circ h$ as $l \circ t_U$ with $t_U \in \mathcal{V}$ and $l \in \mathcal{H}$ to obtain

$$\begin{tikzcd}
X' \arrow[r, dashed, "k"] \arrow[d, dashed, "s_U"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, "s_V"] \\
U \arrow[r, dashed, "l"] & V \\
X \arrow[u, dashed, "t_U"] \arrow[r, "h"'] & Y, \arrow[u, "t_V"']
\end{tikzcd} \tag{3.3}$$

where the upper square is a pullback.

We recall from Definition 3.17 that $\mathcal{V}_0$ denote the set of 't'-legs (i.e., lower legs) of seeds. Following Bousfield's [6] construction of 'cofibrantly generated' factorisation systems, we define $\mathcal{H} = \mathcal{V}_0^{\perp}$ to be the class of maps f such that for any $t \in \mathcal{V}_0$ and commuting square $(u, v)\colon t \to f$ in $\widehat{\mathbb{C}}^{\to}$, there exists a unique filler h making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "t"'] & X' \arrow[d, "f"] \\ Y \arrow[r, "v"'] \arrow[ur, dashed, "h"] & Y'. \end{tikzcd}$$

In this situation, one says that f is *right-orthogonal* to t, and t is *left-orthogonal* to f, which is denoted by $t \perp f$.

We define $\mathcal{V} = {}^{\perp}\mathcal{H}$ to consist of all maps which are left-orthogonal to any map in $\mathcal{H}$. The following is then an instance of easy results in Bousfield [6]:

Proposition 3.21. *The pair* $(\mathcal{V}, \mathcal{H})$ *forms a factorisation system.*

What does that mean? Here is a modern definition [24]:

Definition 3.22. The classes of maps $\mathcal{V}$ and $\mathcal{H}$ form a factorisation system iff $\mathcal{V} = {}^{\perp}\mathcal{H}$, $\mathcal{V}^{\perp} = \mathcal{H}$, and any arrow factors as $h \circ v$ with $h \in \mathcal{H}$ and $v \in \mathcal{V}$.

In the case where $\mathcal{H} = \mathcal{V}_0^{\perp}$ and $\mathcal{V} = {}^{\perp}\mathcal{H}$, Bousfield proves that any map in $\widehat{\mathbb{C}}$ admits a factorisation using a transfinite construction (a so-called *small object* argument). But here we will only need factorisations of particular morphisms, which we will actually be able to calculate by hand. Bousfield's results include:

Lemma 3.23. $\mathcal{V}$ *is stable under pushout and composition, contains all isomorphisms, and enjoys the right cancellation property, i.e., if* $v \in \mathcal{V}$ *and* $fv \in \mathcal{V}$*, then* $f \in \mathcal{V}$*.*

$\mathcal{H}$ *is stable under pullback and composition, contains all isomorphisms, and enjoys the left cancellation property, i.e., if* $h \in \mathcal{H}$ *and* $hf \in \mathcal{H}$*, then* $f \in \mathcal{H}$*.*

Remark 3.24. Stability under pushout is ambiguous here: we mean that for any pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d, "v"'] & Y \arrow[d, "v'"] \\ Z \arrow[r] & T, \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

if $v \in \mathcal{V}$, then $v' \in \mathcal{V}$. Stability under pullback is defined dually.

3.2.2. *A first 'fibred' double category.* We now make concrete the idea evoked in the previous section, of using our factorisation system to obtain a codomain fibration. Consider the sub-double category $\mathbb{D}^1$ of $\mathbb{D}^0$ obtained by restricting vertical morphisms to cospans $X \xrightarrow{s} U \xleftarrow{t} Y$ with $t \in \mathcal{V}$. Its vertical morphisms are stable under composition and contain identities by Lemma 3.23, i.e.:

Lemma 3.25. $\mathbb{D}^1$ *forms a sub-double category of* $\mathbb{D}^0$*.*

Lemma 3.26. *Traces are in* $\mathbb{D}^1_v$*, i.e., we have* $\mathbb{D} \subseteq \mathbb{D}^1$*.*

Proof. By Lemma 3.23. □

The main interest of introducing $\mathbb{D}^1$ is:

Lemma 3.27. *The codomain functor* $\mathrm{cod}\colon \mathbb{D}^1_H \to \mathbb{D}^1_h$ *is a fibration. Double cells as in* (3.2) *for which* $l \in \mathcal{H}$ *and whose upper square is a pullback are cartesian.*

We first observe that any morphism in $\mathbb{D}_h$ is automatically in $\mathcal{H}$, as its codomain has dimension ≤ 1.

Proof. We show that the lifting candidate computed in (3.3) is cartesian. Indeed, consider any double cell (3.3), and any morphism from some vertical morphism $Z' \to W \leftarrow Z$ to V whose bottom component factors through $h\colon X \to Y$. By unique lifting in $(\mathcal{V}, \mathcal{H})$, we obtain a unique dashed arrow making

$$\begin{tikzcd}
Z \arrow[r] \arrow[dd] \arrow[dr] & X \arrow[r] \arrow[d] & U \arrow[dd] \\
& Y \arrow[dr] & \\
W \arrow[rr] \arrow[urr, dashed, bend right] & & V
\end{tikzcd}$$

commute. We finally obtain the desired arrow $Z' \to X'$ by universal property of pullback. □

3.2.3. *Codomain is a fibration.* We finally prove that $\mathbb{D}_H \to \mathbb{D}_h$ is a fibration. In fact, we start by constructing covariant liftings, which we here call *opliftings*.

To compute the oplifting of $Y \to U \leftarrow X$ along $h\colon X \to X'$, observe that, h being 1-injective, we can complete the solid part of

$$\begin{tikzcd}
I_X \arrow[r, dashed] \arrow[d, hook, "\subseteq"'] & Z \arrow[d, hook, dashed, "\subseteq"] \\
X \arrow[r] & X' \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

into a pushout. Indeed, we take $Z(\star) = X'(\star)$, $Z[n] = X'[n] \smallsetminus \mathrm{Im}(h_{[n]})$ for all n, and $I_X \to Z$ is uniquely determined by $h_\star$. In passing, we have:

Lemma 3.28. *This pushout is uniquely determined up to canonical isomorphism by h alone.*

Then, we observe that, because U is a trace, $I_X \to X \to U$ factors through $Y \to U$. Hence, we may push the whole cospan along $I_X \to Z$ exactly as in (3.1). All horizontal maps are 1-injective by construction, and we have:

Lemma 3.29. *The obtained cospan* $Y' \to U' \leftarrow X'$ *is a trace.*

Proof. We start by showing that U' is an action if U is. So assume U is obtained by pushing a seed $Y_0 \to M_0 \leftarrow X_0$ along some $I_{X_0} \to Z_0$. Then, $Z_0 \to X$ is surjective on $\star$ because $I_{X_0} \to X_0$ is and epis are stable under pushout. Thus, $I_X \to X$ factors through $Z_0 \to X$. Let Z'' denote the pushout

$$\begin{tikzcd}
I_X \arrow[r] \arrow[d] & Z_0 \arrow[d] \\
Z \arrow[r] & Z''. \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

By the pushout lemma, $Y' \to U' \leftarrow X'$ is isomorphic to the cospan obtained by pushing $Y \to U \leftarrow X$ along $Z_0 \to Z''$. By the pushout lemma again, it is isomorphic to the cospan obtained by pushing $Y_0 \to M_0 \leftarrow X_0$ along $I_{X_0} \to Z_0 \to Z''$. Thus, it is indeed an action.

We now prove the general case by induction on U. This is trivial if U is isomorphic to an identity. If now U is a composite $Z \xrightarrow{V} Y \xrightarrow{M} X$, then we compute the oplifting of M along h to obtain a double cell, say

$$\begin{tikzcd} Y \arrow[r, "h'"] \arrow[d, "M"'] \arrow[dr, phantom, "\xRightarrow{\alpha_M}"] & Y' \arrow[d, "M'"] \\ X \arrow[r, "h"'] & X', \end{tikzcd}$$

by pushing M along some morphism $I_X \to Z_1$ making

$$\begin{tikzcd} I_X \arrow[r] \arrow[d] & Z_1 \arrow[d] \\ X \arrow[r, "h"'] & X' \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

into a pushout.

The crucial insight is then that by computing the following pushout Z_2 and applying its universal property, we have a diagram

$$\begin{tikzcd} I_X \arrow[r] \arrow[d] & I_Y \arrow[r] \arrow[d] & Y \arrow[d] \\ Z_1 \arrow[r] & Z_2 \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & Y' \end{tikzcd} \tag{3.4}$$

whose exterior is a pushout by construction. So by the pushout lemma the right-hand square is again a pushout, which by Lemma 3.28 is the unique pushout along which to push V to compute the oplifting of V along h', say

$$\begin{tikzcd} Z \arrow[r, "h'"] \arrow[d, "V"'] \arrow[dr, phantom, "\xRightarrow{\alpha_V}"] & Z' \arrow[d, "V'"] \\ Y \arrow[r, "h"'] & Y'. \end{tikzcd}$$

By induction hypothesis, V' is again a trace. But by the pushout lemma again, α_V is also what we obtain by pushing V along the exterior rectangle of (3.4). A bunch of applications of the pushout lemma finally yields that $\alpha_M \bullet \alpha_V$ is the oplifting of $M \bullet V$ along h, whose horizontal codomain is thus a trace by construction. □

Remark 3.30. Such opliftings are not opcartesian in general, and moreover opcartesian liftings do not exist in general.

We now show that restrictions of seeds are traces.

Lemma 3.31. *Consider any diagram $X' \xrightarrow{h} X \xrightarrow{t} M$, where $t \in \mathcal{V}_0$ and $h \in \mathbb{D}_h(X', X)$. Its factorisation $X' \xrightarrow{t'} M' \xrightarrow{h'} M$ with $t' \in \mathcal{V}$ and $h' \in \mathcal{H}$ is such that h' is 1-injective and the obtained restriction is a trace of length at most 2. If X' is an* individual*, i.e., a configuration of the shape $[n]$, then it is a seed.*

Remark 3.32. If X' is an interface, then the restriction is automatically an equivalence (in $\mathbb{D}_v$).

Proof. We proceed by a stupid case analysis. In each case, one has to check that h' is 1-injective, that X' individual implies t' seed, that X' interface implies that t' is an equivalence, and that the upper leg of the obtained cospan is as expected: this is routine so we mention it here once and for all. If $M = \mathsf{y}_c$, for c not of the shape $\tau_{n,i,m,j,k}$, then we have $X = [n]$. If $id_c \in \mathrm{Im}(h)$, then $X' = [n] + I$ for some interface I (since h is 1-injective). Consider the diagram

$$\begin{tikzcd} M + I \arrow[r, "{[id,tk]}"] & M \\ {[n] + I} \arrow[u, "t+id_I"] \arrow[r, "{h=[id,k]}"'] & {[n].} \arrow[u, "t"'] \end{tikzcd}$$

The map $t + id_I$ is in $\mathcal{V}$ by Lemma 3.23, so we just have to prove that $[id, tk]$ is in $\mathcal{V}_0^{\perp}$, which is a simple verification.

If now $id_c \notin \mathrm{Im}(h)$, then X' is an interface, and the relevant factorisation is

$$\begin{tikzcd} X' \arrow[r, "t \circ h"] & M \\ X' \arrow[u, "id"] \arrow[r, "h"'] & {[n],} \arrow[u, "t"'] \end{tikzcd} \tag{3.5}$$

because $t \circ h$ is easily checked to be in $\mathcal{V}_0^{\perp}$.

The case of $\tau_{n,i,m,j,k}$ is a bit more complicated. Here, t is actually $t_0 = [\rho t, \epsilon t]$. First of all, if X' is an interface, then we obtain a factorisation analogous to (3.5). Consider now the case where $\mathrm{Im}(h)$ contains both agents of $[n]\,{}_i|_j\,[m]$. Let x denote the n-ary one and y denote the m-ary one (in X'). If $x \cdot s_i = y \cdot s_j$, then $X' = ([n]\,{}_i|_j\,[m]) + I$ for some interface I and the required factorisation is easily seen to be

$$\begin{tikzcd} \tau_{n,i,m,j,k} + I \arrow[r, "{[id,t_0 k]}"] & \tau_{n,i,m,j,k} \\ {([n]\,{}_i|_j\,[m]) + I} \arrow[u, "t_0+id_I"] \arrow[r, "{h=[id,k]}"'] & {[n]\,{}_i|_j\,[m].} \arrow[u, "t_0"'] \end{tikzcd}$$

Consider now the case where X' still contains both agents but $x \cdot s_i \neq y \cdot s_j$. Then $X' = [n] + [m] + I$ for some interface I, and the required factorisation is

$$\begin{tikzcd} \iota_{n,i} + o_{m,j,k} + I \arrow[r, "{[\rho,\epsilon,t_0 k]}"] & \tau_{n,i,m,j,k} \\ {[n] + [m] + I} \arrow[u, "t+t+id_I"] \arrow[r, "{[x,y,k]}"'] & {[n]\,{}_i|_j\,[m].} \arrow[u, "t_0"'] \end{tikzcd}$$

The only non-trivial point here is to show that $[\rho, \epsilon, t_0 k]$ is in $\mathcal{V}_0^{\perp}$, which easily reduces to showing that there is no commuting square

$$\begin{tikzcd} {[n]\,{}_i|_j\,[m]} \arrow[r, "u"] \arrow[d, "t_0"'] & \iota_{n,i} + o_{m,j,k} + I \arrow[d, "{[\rho,\epsilon,t_0 k]}"] \\ \tau_{n,i,m,j,k} \arrow[r, "v"'] & \tau_{n,i,m,j,k}, \end{tikzcd}$$

which is true because there is no such u.

The cases where X' only contains one agent of $[n]\,{}_i|_j\,[m]$ are similar to the latter case: if it contains x then the factorisation is through $\iota_{n,i}$, and otherwise it is through $o_{m,j,k}$. □

We now scale the previous lemma from seeds to actions, for which we need the following lemmas.

Definition 3.33. Let $\mathrm{Agents}(X) = \sum_{n\in\mathbb{N}} X[n]$ denote the set of agents of any configuration X.

Lemma 3.34. *For any seed $Y \to C \leftarrow X$, the morphism*

$$\sum_{(n,x)\in\mathrm{Agents}(X)} [n] \xrightarrow{[x]_{(n,x)}} X$$

is epi.

Proof. This is trivial except in dimension 0. There, if $C \neq \tau_{n,i,m,j,k}$ then X is representable so the result is again trivial. If finally $C = \tau_{n,i,m,j,k}$, then it's easy. □

Corollary 3.35. *For all arrows as in*

$$\begin{tikzcd} X \arrow[r, bend left, "h"] \arrow[r, bend right, "h'"'] \arrow[d, "t"'] & U \arrow[d, "f"] \\ C \arrow[r, "g"'] & U' \end{tikzcd}$$

in $\widehat{\mathbb{C}}$ such that f is 1-injective, $t \in \mathcal{V}_0$, and $fh = gt = fh'$, we have $h = h'$.

Proof. We construct the diagram

$$\begin{tikzcd} \sum_{(n,x)\in\mathrm{Agents}(X)}[n] \arrow[r, bend left, "he"] \arrow[r, bend right, "h'e"'] \arrow[d, two heads, "e"'] & U \arrow[d, equal] \\ X \arrow[r, bend left, "h"] \arrow[r, bend right, "h'"'] \arrow[d, "t"'] & U \arrow[d, "f"] \\ C \arrow[r, "g"'] & U', \end{tikzcd}$$

and observe that $h'e = he$ by 1-injectivity of f, hence $h = h'$ because e is epi by Lemma 3.34. □

Lemma 3.36. *Opliftings of traces are cartesian.*

Proof. Consider any oplifting $U \xrightarrow{f} U'$ of $Y \to U \leftarrow X$ along, say $X \to X'$. By Lemma 3.27, it is enough to show that the middle arrow $U \to U'$ is in $\mathcal{H}$ and that its upper square is a pullback. The latter follows from Lemma 3.18. So we just have to show that any square

$$\begin{tikzcd} Z \arrow[r, "u"] \arrow[d, "t"'] & U \arrow[d] \\ C \arrow[r, "v"'] & U' \end{tikzcd}$$

with $t \in \mathcal{V}_0$ admits a unique lifting $C \to U$. By the Yoneda lemma, v amounts to an element of $U'(C)$ of dimension > 1, but by construction U and U' have exactly the same such elements. This yields a candidate lifting, say k, which makes the bottom triangle commute by construction. The top one finally commutes by Corollary 3.35 with $h = u$ and $h' = kt$. □

Lemma 3.37. *For any action $Y \xrightarrow{s} M \xleftarrow{t} X$ and $h \in \mathbb{D}_h(X', X)$, the factorisation $X' \xrightarrow{t'} P' \xrightarrow{h'} M$ of th with $t' \in \mathcal{V}$ and $h' \in \mathcal{H}$ is such that h' is 1-injective and the obtained restriction is a trace of length at most 2. If X' is an individual then it is either a seed or an equivalence; if X' is an interface then it is an equivalence.*

Proof. Consider any action $Y \to M \leftarrow X$ obtained by pushing the following seed-with-interface along $I \to Z$:

$$\begin{tikzcd} & I \arrow[dl] \arrow[d] \arrow[dr] & \\ Y_0 \arrow[r] & M_0 & X_0. \arrow[l] \end{tikzcd}$$

By Lemma 3.36, $\sigma_0 \colon M_0 \to M$ is cartesian.

Consider the pullback of the left-hand square below along $h \colon X' \to X$ to obtain the right-hand square:

$$\begin{tikzcd} I \arrow[r] \arrow[d] & Z \arrow[d] & I' \arrow[r] \arrow[d] & Z' \arrow[d] \\ X_0 \arrow[r] & X & X'_0 \arrow[r] & X', \end{tikzcd} \tag{3.6}$$

which, because presheaf categories are *adhesive* [41] and $I \to X_0$ is mono, is again a pushout.

Furthermore, consider the pullback

$$\begin{tikzcd} X'_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_0 \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

in $\mathbb{D}_h$. By Lemma 3.31, restricting M_0 along $X'_0 \to X_0$ yields a trace, say $Y'_0 \to P'_0 \leftarrow X'_0$ with a morphism to $Y_0 \to M_0 \leftarrow X_0$ in $\mathbb{D}_H$. Since it is a trace, $I' \hookrightarrow X'_0 \hookrightarrow M'_0$ factors through $Y'_0 \hookrightarrow M'_0$. Pushing $Y'_0 \to P'_0 \leftarrow X'_0$ along $I' \to Z'$ via the injection $I' \hookrightarrow X'_0$, we obtain (because the obtained configuration is X' by the pushout (3.6)) a trace $Y' \to P' \leftarrow X'$ with a morphism from $Y'_0 \to P'_0 \leftarrow X'_0$ in $\mathbb{D}_H$, which is an oplifting, hence cartesian (by Lemma 3.36). Then, by universal property of pushout we obtain a unique morphism $f \colon P' \to M$ making the front face of the following cube commute:

$$\begin{tikzcd} I' \arrow[rr] \arrow[dr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Z' \arrow[dr] \arrow[dd] & \\ & P'_0 \arrow[rr] \arrow[dd] & & P' \arrow[dd, "f"] \arrow[ul, phantom, "\ulcorner", very near start] \\ I \arrow[rr] \arrow[dr] & & Z \arrow[dr] & \\ & M_0 \arrow[rr] & & M. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

By Corollary 3.20, f is 1-injective, which entails that the induced morphism of traces is in $\mathbb{D}_H$.

We now need to show that $P' \to M$ is cartesian, which by Lemma 3.18 amounts to showing that its middle arrow $f: P' \to M$ is in $\mathcal{H}$. To this end, consider any morphism $t: Z'' \to C$ in $\mathcal{V}_0$ and morphism $(u,v): t \to f$ in $\widehat{\mathbb{C}}^{\to}$. First of all, because $M_0 \to M$ is identity in dimensions > 1, the morphism v uniquely factors through $M_0 \to M$. Furthermore, in all cases where $f_0: P'_0 \to M_0$ is identity in dimensions > 1, the Yoneda lemma entails that $C \to M_0$ uniquely factors through f_0, which yields a diagram

$$\begin{tikzcd}
Z'' \arrow[r, dashed, "l"] \arrow[d] \arrow[rr, bend left, "u"] & P'_0 \arrow[r, "g"] \arrow[d, "f_0"] & P' \arrow[d, "f"] \\
C \arrow[r] \arrow[ur, dashed, "k"] \arrow[rr, bend right, "v"'] & M_0 \arrow[r] & M,
\end{tikzcd}$$

with the upper part marked '?', which commutes except perhaps for the upper part marked '?'. But the latter also commutes by Corollary 3.35 with $h = u$ and $h' = gl$. We thus obtain a lifting which is unique by 1-injectivity of f.

So when is f_0 non-identity in dimensions > 1? By inspection of the proof of Lemma 3.31, this is when $M_0 = \tau_{n,i,m,j,k}$ for some n, m, i, j, k, and P'_0 has one of the shapes $\iota_{n,i}+J$, $o_{m,j,k}+J$, or $\iota_{n,i} + o_{m,j,k} + J$, for some interface J. In the first two cases, $C \neq \tau_{n,i,m,j,k}$ because there can be no $u: [n]\,{}_i|_j\,[m] \to P'$ (one agent is missing in P'), so the previous argument applies. In the third case, letting x and y respectively denote the n- and m-ary initial agents in P'_0 and $a = x \cdot s_i$ and $a' = y \cdot s_j$ the corresponding channels, one easily shows that $g(a) \neq g(a')$, so again there can be no $u: [n]\,{}_i|_j\,[m] \to P'$. Thus, $C \neq \tau_{n,i,m,j,k}$ and the previous argument again applies. □

So far, we have shown that actions admitted cartesian liftings in $\mathbb{D}_H$. We now show that it is also the case for arbitrary traces.

Lemma 3.38. *In sets, for any commuting diagram*

$$\begin{tikzcd}
I \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, hook] & C \arrow[d, hook] \\
J \arrow[r] & B \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & D
\end{tikzcd}$$

whose exterior rectangle is a pullback, with the marked pushout and monos, the right-hand square is also a pullback.

Proof. We check the universal property of pullback for A, relative to 1, which is enough in sets. So consider any commuting square

$$\begin{tikzcd}
1 \arrow[r, "c"] \arrow[d, "b"'] & C \arrow[d] \\
B \arrow[r] & D.
\end{tikzcd}$$

First, we observe that there is at most one mediating arrow $1 \to A$, because $A \to B$ is mono.

If b does not have any antecedent in J, then because $A + J \to B$ is surjective, it has one in A, say a. But then, because $C \to D$ is mono, a makes both required triangles commute and we are done.

If b admits an antecedent in J, i.e., there exists $j{:}\,1 \to B$ such that b is $1 \xrightarrow{j} J \to B$, then we have a cone to $C \to D \leftarrow J$, so we apply the universal property of pullback to obtain i as in

$$\begin{tikzcd}
1 \arrow[dr, "i"] \arrow[drrr, "c"] \arrow[ddr, bend right, "j"'] \arrow[ddrr, bend right, "b"] & & & \\
& I \arrow[r] \arrow[d, dashed] & A \arrow[r] \arrow[d, hook] & C \arrow[d, hook] \\
& J \arrow[r] & B \arrow[r] & D
\end{tikzcd}$$

making everything commute and so $1 \xrightarrow{i} I \to A$ does the job. ☐

Corollary 3.39. *In any presheaf category, in any commuting cube*

$$\begin{tikzcd}
I \arrow[rr] \arrow[dr] \arrow[dd] & & B \arrow[dr] \arrow[dd] & \\
& A \arrow[rr] \arrow[dd] & & C \arrow[dd, "f"] \\
I' \arrow[rr, hook] \arrow[dr] & & B' \arrow[dr] & \\
& A' \arrow[rr] & & C',
\end{tikzcd}$$

with the marked pushouts, pullback, and mono, the front square is also a pullback.

Proof. It suffices to show the result in sets, as all involved properties are pointwise in presheaf categories. First, as monos are stable under pullback and pushout, $I \to B$, $A \to C$, and $A' \to C'$ are also monos. Furthermore, pushouts along monos are also pullbacks, so the top and bottom faces are also pullbacks. By the pullback lemma, the rectangle

$$\begin{tikzcd}
I \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & I' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d] \\
B \arrow[r] & B' \arrow[r] & C'
\end{tikzcd}$$

is a pullback. The previous lemma thus entails the result. ☐

Lemma 3.40. *For any commuting diagram as the solid part of*

$$\begin{tikzcd}
T \arrow[r] \arrow[d, "t"'] & V' \arrow[r] \arrow[d, "h"'] & U' \arrow[d, "f"] \\
C \arrow[r] \arrow[urr, dashed, "k"] & V \arrow[r] & U
\end{tikzcd}$$

with $t \in \mathcal{V}_0$, $h \in \mathcal{H}$, and f 1-injective, there is a unique lifting k as shown.

Proof. Because $h \in \mathcal{H}$, there is a unique map from $k' : C \to V'$ making both triangles commute. By composing k' with $V' \to U'$, we obtain a lifting k for the desired square. Uniqueness follows from 1-injectivity of f. □

Lemma 3.41.

(P1) The codomain functor $\mathrm{cod} : \mathbb{D}_H \to \mathbb{D}_h$ *is a subfibration of* $\mathrm{cod} : \mathbb{D}^1_H \to \mathbb{D}^1_h$.

Proof. All that remains to prove that given any trace $Y \xrightarrow{s} U \xleftarrow{t} X$ and $h \in \mathbb{D}_h(X', X)$, any cartesian lifting in $\mathbb{D}^1$ lies in $\mathbb{D}$, i.e., the obtained vertical morphism is again a trace and the double cell to U lies in $\mathbb{D}$ (i.e., all its components are 1-injective).

We proceed by induction on U. If U is an equivalence, then the result is obviously true. Otherwise, $U = M \bullet V$ for some action M and trace V. Let us call Z the final configuration of M. By Lemma 3.37, M admits a lifting P' along h, with a final configuration Z', and $Z' \to Z$ and $f_M : P' \to M$ are 1-injective. By induction hypothesis, V admits a lifting V' along $Z' \to Z$ with a double cell to V in $\mathbb{D}_H$. Therefore, considering the composite $P' \bullet V'$, we have a commuting diagram

$$\begin{tikzcd}
Y' \arrow[rrrr] \arrow[dr] & & & & Y \arrow[dr, "s_1"] & & \\
& V' \arrow[rrrr, "f_V"] \arrow[dr, "s'_2"'] & & & & V \arrow[dr, "s_2"] & \\
& & P' \bullet V' \arrow[rrrr, "f"] & & & & M \bullet V \\
Z' \arrow[uur] \arrow[rrrr] \arrow[dr] & & & & Z \arrow[uur, "t_1"] \arrow[dr, "s_0"] & & \\
& P' \arrow[uur, "t'_2"] \arrow[rrrr, "f_M"] & & & & M \arrow[uur, "t_2"] & \\
X' \arrow[ur] \arrow[rrrr] & & & & X, \arrow[ur, "t_0"] & &
\end{tikzcd}$$

where f is obtained by universal property of pushout.

Because pushouts along monos are also pullbacks in presheaf categories, both marked pushouts are also pullbacks. Furthermore, by Lemma 3.18, $Z' = P' \times_M Z$ as shown. Also, by Corollary 3.39, $V' = (P' \bullet V') \times_{M \bullet V} V$ as shown. Furthermore, by Corollary 3.20, f is 1-injective.

By Lemmas 3.27 and 3.18, it suffices to show that f is in $\mathcal{H}$, i.e., that it is right-orthogonal to any $T \xrightarrow{t} C$ in $\mathcal{V}_0$. Consider any commuting square

$$\begin{tikzcd}
T \arrow[r, "u"] \arrow[d, "t"'] & P' \bullet V' \arrow[d, "f"] \\
C \arrow[r, "v"'] & M \bullet V.
\end{tikzcd}$$

Since $M \bullet V$ is the coproduct of M and V in dimensions greater than 1 and C is a representable of dimension greater than 1, we have that v factors either through t_2 or s_2.

If v factors through s_2, then by universal property of pullback we find a map $u' : T \to V'$ making

$$\begin{tikzcd}
T \arrow[r, "u'"] \arrow[d, "t"'] \arrow[rr, bend left, "u"] & V' \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & P' \bullet V' \arrow[d] \\
C \arrow[r] \arrow[rr, bend right, "v"'] & V \arrow[r, "s_2"'] & M \bullet V
\end{tikzcd}$$

commute. Then, by Lemma 3.40, we find a unique lifting as desired.

If v factors as $t_2 v'$, then by Lemma 3.40, it is sufficient to show that there is a map $u' : T \to P'$ making

$$\begin{tikzcd}
T \arrow[r, "u'"] \arrow[d, "t"'] \arrow[rr, bend left, "u"] & P' \arrow[r] \arrow[d] & P' \bullet V' \arrow[d] \\
C \arrow[r, "v'"'] \arrow[rr, bend right, "v"'] & M \arrow[r, "t_2"'] & M \bullet V
\end{tikzcd}$$

commute. To that end, it is sufficient to show that for every $[n] \xrightarrow{x} T$, there is a map $[n] \xrightarrow{f_x} P'$ such that

$$\begin{tikzcd}
{[n]} \arrow[r, "x"] \arrow[d, "f_x"'] & T \arrow[d] \\
P' \arrow[r] & P' \bullet V'
\end{tikzcd} \tag{3.7}$$

commutes. Indeed, if that is the case, then the square

$$\begin{tikzcd}
\sum_{(n,x)\in \mathrm{Agents}(T)} [n] \arrow[r, "{[x]_{(n,x)\in \mathrm{Agents}(T)}}"] \arrow[d, "{[f_x]_{(n,x)\in \mathrm{Agents}(T)}}"'] & T \arrow[d] \\
P' \arrow[r] & P' \bullet V'
\end{tikzcd}$$

also commutes, and since its top map is epi and its bottom map is mono, there is a unique lifting $u' : T \to P'$ making both triangles commute. The square

$$\begin{tikzcd}
T \arrow[r, "u'"] \arrow[d, "t"'] & P' \arrow[d] \\
C \arrow[r, "v'"'] & M
\end{tikzcd}$$

also commutes because it commutes when composed with t_2, which is mono.

So we now need to show that for every $[n] \xrightarrow{x} T$, there is a map $[n] \xrightarrow{f_x} P'$ making the square (3.7) commute. First of all, we notice that, since C is a seed, M an action, and T and X are their respective initial configurations, there is a map $T \to X$ that makes

$$\begin{tikzcd}
T \arrow[r, "t"] \arrow[d] & C \arrow[d, "v'"] \\
X \arrow[r] & M
\end{tikzcd}$$

commute.

Because $P' + V' \xrightarrow{[t'_2, s'_2]} P' \bullet V'$ is epi and $[n]$ is a representable presheaf, $[n] \xrightarrow{ux} P' \bullet V'$ factors through either t'_2 or s'_2.

If it factors through s'_2, say as $s'_2 x'$, then we have

$$t_2(v'tx) = vtx = fux = fs'_2x' = s_2 f_V x',$$

so by universal property of the pullback Z there exists a unique $x'' : [n] \to Z$ such that $s_0 x'' = v'tx$ and $t_1 x'' = f_V x'$. We thus obtain a commuting diagram

$$\begin{tikzcd}
{[n]} \arrow[rr, "x''"] \arrow[d, "x"'] & & Z \arrow[d] \\
T \arrow[r, "t"] & C \arrow[r, "v'"] & M,
\end{tikzcd}$$

which is impossible because Z is the final configuration of M.

Thus, ux factors through t'_2 and we are done. □

Remark 3.42. Restrictions of actions to individuals are either seeds or equivalences. Restrictions of traces to interfaces are equivalences.

3.3. **A candidate playground.** In this section, we define additional structure on $\mathbb{D}$ to make it into a candidate playground, and prove that it satisfies all the needed axioms but two, which are treated in the next sections.

Remark 3.43. We make just one slight modification of the playground axioms, in not imposing that the class $\mathbb{B}$ of basic actions is replete, i.e., closed under isomorphism in $\mathbb{D}_H$. This does not change anything significant.

Definition 3.44. We recall from Lemma 3.31 that $\mathbb{I}$, the set of *individuals*, consists of representable configurations $[n]$. Let $\mathbb{B}$, the set of *basic* actions, consist of all seeds of shape τ_n, π^r_n, π^l_n, ν_n, $\heartsuit_n$, $\iota_{n,i}$, or $o_{m,j,k}$. *Full* actions (notation $\mathbb{F}$) are all actions obtained from seeds of shape τ_n, π_n, ν_n, $\heartsuit_n$, $\iota_{n,i}$, $o_{m,j,k}$, or $\tau_{n,i,m,j,k}$. *Closed-world* actions (notation $\mathbb{W}$) are all actions obtained from seeds of shape τ_n, π_n, ν_n, $\heartsuit_n$, or $\tau_{n,i,m,j,k}$. Finally, all decompositions of any trace U into actions have the same length which we denote by $|U|$.

Definition 3.45. In a pseudo double category, a cell α is *special* when its vertical domain and codomain $\mathrm{cod}(\alpha)$ and $\mathrm{dom}(\alpha)$ are identities.

Lemma 3.46.

(P2) *$\mathbb{I}$, viewed as a subcategory of $\mathbb{D}_h$, is discrete. Basic actions have no non-trivial automorphisms in $\mathbb{D}_H$. Vertical identities on individuals have no non-trivial automorphisms.*

(P3) *(Individuality) Basic actions have individuals as both domain and codomain.*

(P4) *(Atomicity) For any cell $\alpha : U \to U'$, if $|U'| = 0$ then also $|U| = 0$. Up to a special isomorphism in $\mathbb{D}_H$, all traces of length $n > 0$ admit decompositions into n actions. For any $U : X \to\!\!\bullet\!\!\to Y$ of length 0, there is an isomorphism $id^\bullet_X \to U$ as on the right in $\mathbb{D}_H$.*

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal, "\bullet"] & X \arrow[d, "U"] \\
X \arrow[r, "\bar{U}"'] \arrow[ur, phantom, "\Rightarrow \alpha^U"] & Y
\end{tikzcd}$$

(P5) *(Fibration, continued) Restrictions of actions (resp. full actions) to individuals either are actions (resp. full actions), or have length 0.*

Proof. (P2) and (P3) are easy. (P5) is also easy in view of Lemma 3.41 and its proof. For (P4), any vertical $X \to U \leftarrow Y$ of length 0, being a trace, is isomorphic to an identity cospan, say $Z \to Z \leftarrow Z$. To construct α^U, just take the composite $id^\bullet_X \xrightarrow{\cong} id^\bullet_Z \xrightarrow{\cong} U$. ☐

The axiom for views is really easy. It actually becomes stronger because of Remark 3.43, though this does not affect the rest of the construction:

Definition 3.47. Let $\mathbb{B}_0$ be the full subcategory of $\mathbb{D}_H$ having as objects basic actions and vertical identities between individuals.

Lemma 3.48.

(P6) for any action $M : Y \to X$ and $y : d \to Y$ in $\mathbb{D}_h$ with $d \in \mathbb{I}$, there exists a unique cell

$$\begin{tikzcd} d \arrow[r, "y"] \arrow[d, dashed, "v^{y,M}"'] \arrow[dr, phantom, "\alpha^{y,M}"] & Y \arrow[d, "M"] \\ d^{y,M} \arrow[r, dashed, "y^M"'] & X, \end{tikzcd}$$

with $v^{y,M} \in \mathbb{B}_0$.

Proof. A straightforward case analysis. ☐

Remark 3.49. The important result that Axiom (P6) entails [34, Proposition 4.24] says that when we replace M with any trace u, we get a double cell $\alpha^{y,u}$ which is only unique up to isomorphism. Below, we still define views up to isomorphism, so our modified Axiom (P6) does not make this any stronger.

Lemma 3.50.

(P9) For any X, $\mathbb{I}/X$ is finite.

Proof. Easy. ☐

Definition 3.51. We recall that $\mathrm{Agents}(X)$ denotes the set $\sum_n X[n]$ of agents of any configuration X. Furthermore, for any action $M : X' \to X$, let $\mathrm{Agents}_M(X') = \{(d', x') \in \mathrm{Agents}(X') \mid |v^{x',M}| \neq 0\}$.

For any X, let $[\mathbb{F}]_X$ be the set of isomorphism classes of full actions with codomain X, in $\mathbb{D}_H(X)$ (i.e., with identity vertical codomain). Similarly, let $\mathbb{B}_d$ denote the set of all basic actions with codomain d. Let then χ denote the map

$$\begin{array}{rcl} [\mathbb{F}]_d & \to & \mathcal{P}_f(\mathbb{B}_d) \\ [M] & \mapsto & \{B \in \mathbb{B}_d \mid \exists \alpha \in \mathbb{D}_H(B, M)\}. \end{array}$$

The map χ is easily checked to be well-defined. The next axiom to check demands that basic sub-actions of a full action $[M] \in [\mathbb{F}]_d$ may not be sub-actions of other full actions.

Lemma 3.52.

(P10) (Basic vs. full) For any $d \in \mathbb{I}$ and $M, M' \in [\mathbb{F}]_d$, if $M \neq M'$, then $\chi(M) \cap \chi(M') = \varnothing$.

Proof. Straightforward. ☐

The last lemma is straightforward, but observe that it is crucial here that $[n]$ alone cannot receive one of its already known channels. More precisely, let $\iota_{n,i,j}$ be $\iota_{n,i}$, quotiented by $s \circ s_{n+1} = s \circ s_j$, for $j \in n$. Further let $[n]/\{i = j\}$ denote $[n]$ quotiented by $s_i = s_j$. The cospan

$$[n+1]/\{n+1 = j\} \xrightarrow{s} \iota_{n,i,j} \xleftarrow{t} [n]$$

is not an action. Hence, consider, e.g., the synchronisation on $[n]_{i,l}|_{j,k}[m]$ where $[m]$ sends k on j. Its restriction to the receiver is $\iota_{n,i}$, not $\iota_{n,i,l}$.

There only remain two axioms for playgrounds, called *right* and *left* decomposition, respectively. These require the development of more machinery, which we undertake in the next section.

3.4. **Correctness criterion.** In order to prove the remaining playground axioms for $\mathbb{D}$, we set up a combinatorial characterisation of traces among cospans.

Given a trace $X \hookrightarrow U \hookleftarrow Y$, we start by forgetting the cospan structure and exhibiting some properties of U alone.

Definition 3.53. A *core* of a presheaf $U \in \widehat{\mathbb{C}}_f$ is an element of dimension > 1 which is not the image of any element of higher dimension.

Here is a first easy property of traces. Before the next definition, observe that for all seeds $Y \hookrightarrow M \hookleftarrow X$, M is a representable presheaf.

Definition 3.54. A presheaf U is *locally 1-injective* iff for any seed $Y \hookrightarrow M \hookleftarrow X$ with interface I and core $\mu \in U(M)$, if two elements of M are identified by the Yoneda morphism $\mu: M \to U$, then they are in (the image of) $I(\star)$.

Proposition 3.55. *Any trace U is locally 1-injective.*

Proof. Choose a decomposition of U into actions; μ corresponds to precisely one such action, say M', obtained, by definition, from some seed M as a pushout (3.1). By construction of pushouts in presheaf categories, M' is obtained from M by identifying some channels according to $I \to Z$. □

Observe that, because local 1-injectivity only is about cores, an input which is part of a synchronisation may receive an already known channel, even if its $n+1$th channel is not part of its interface — because it is not a core.

We now extract from any presheaf a graph, which represents its candidate causal structure. Let, for any presheaf U, $\mathrm{el}(U)$ denote its category of elements.

Definition 3.56. In any U, the *sources* of a core μ are the agents x with a morphism, in $\mathrm{el}(U)$, of the shape $x \xrightarrow{f \circ s \circ f'} \mu$ to μ; its *targets* are the agents y with a morphism of the shape $y \xrightarrow{f \circ t \circ f'} \mu$.

Example 3.57. In the representable π_n, there is one target, $l \circ t$ (or equivalently $r \circ t$), and two sources, $s_1 = l \circ s$ and $s_2 = r \circ s$. Another example is $\tau_{n,i,m,j,k}$, which has two targets, $\epsilon \circ t$ and $\rho \circ t$, and two sources.

Definition 3.58. For any presheaf U and core $\mu \in U(M)$, we say that $a \in \mathrm{Im}(M(\star))$ is *created* by μ iff M is either ν_n or $\iota_{n,i}$ for some (n,i), and a is the image of $s \circ s_{n+1}$.

Given a presheaf U, we construct its *causal (simple) graph* G_U as follows:

- its vertices are all channels, agents, and cores in U;
- there are edges to each core from its sources and from each core to its targets, as in

$$\begin{tikzcd} \text{source}_1 \arrow[dr] & & \text{source}_2 \arrow[dl] \\ & \text{core} \arrow[dl] \arrow[dr] & \\ \text{target}_1 & & \text{target}_2; \end{tikzcd}$$

- there is an edge $x \to x \cdot s_i$ for all $x \in U[n]$ and $i \in n$;
- there is an edge $a \to \mu$ for each channel a created by μ.

Note that edges $a \to \mu$ from a channel to an input action exist only if the involved action is not part of a synchronisation; for in this case the synchronisation is a core, not the input.

The obtained graph is actually a binary relation, since there is at most one edge between any two vertices. It is also a colored graph, in the sense that it comes equipped with a morphism to the graph L:

$$\begin{tikzcd} \infty \arrow[r, bend left] & 1 \arrow[l, bend left] \arrow[r] & 0, \arrow[ll, bend left] \end{tikzcd}$$

mapping cores to ∞, agents to 1, and channels to 0. (Observe in particular that there are no edges from channels to agents or cores to channels.)

For any simple graph G, equipped with a morphism $l: G \to L$, we call vertices of G channels, agents, or cores, according to their label.

Definition 3.59. An agent x in U is *final* iff it is not the target of any action, i.e., for no action $\mu \in U$, $x = \mu \cdot t$. All channels are final.

Lemma 3.60. *An agent is final in U iff it has no edge from any core in G_U.*

Definition 3.61. An agent is *initial* in U when it is not the source of any action, i.e., for no action $\mu \in U$, $x = \mu \cdot s$. A channel is initial when it is not created by any core.

Lemma 3.62. *An agent is initial in U iff it has no edge to any core in G_U.*

Definition 3.63. Any $G \in \mathsf{Gph}/L$ is *source-linear* iff for any cores μ, μ', and other vertex (necessarily an agent or a channel) x, $\mu \leftarrow x \to \mu'$ in G, then $\mu = \mu'$. G is *target-linear* iff for any cores μ, μ' and agent x, if $\mu \to x \leftarrow \mu'$ in G, then $\mu = \mu'$. G is *linear* iff it is both source-linear and target-linear.

Proposition 3.64. *For any trace $Y \xhookrightarrow{s} U \xhookleftarrow{t} X$, G_U is linear.*

Proof. Straightforward, by induction on any decomposition of U into actions, observing that we glue along agents and channels which are initial on one side and final on the other. □

Proposition 3.65. *For any trace as above, G_U is acyclic (in the directed sense).*

Proof. By induction on any decomposition of U, similar to the previous proof. □

Now, here is the expected characterisation:

Theorem 3.66. *A cospan $Y \xhookrightarrow{s} U \xhookleftarrow{t} X$ of finite presheaves is a trace iff*

(1) U is locally 1-injective,
(2) X contains exactly the initial agents and channels in U,
(3) Y contains exactly the final agents and channels in U,
(4) and G_U is linear and acyclic.

Of course, we have almost proved the 'only if' direction, and the rest is easy, so only the 'if' direction remains to prove. The rest of this section is devoted to this. First, let us familiarise ourselves with removing elements from a presheaf. For two morphisms of presheaves $U \xrightarrow{f} V \xleftarrow{g} W$, we denote by $U \smallsetminus W$ the topos-theoretic difference $U \cap \neg W$ of (the images of) f and g in the lattice $\mathrm{Sub}(V)$ of subobjects of V. This differs in general from what we denote $U - W$, which is the set of elements in V which are in the image of U but not that of W, i.e., $\sum_{c \in \mathbb{C}} U(c) \smallsetminus W(c)$. More generally, for any morphism of presheaves $f: U \to V$ and set W, let $U - W = \sum_{c \in \mathbb{C}} \mathrm{Im}(U(c)) \smallsetminus W$.

Remark 3.67. We observe that $U - W$ is generally just a set, not a presheaf; i.e., its elements are not necessarily stable under the action of morphisms in $\mathbb{C}$. Consider for example $U = [1] \mid [1]$ and let $W: [1] \to [1] \mid [1]$ denote the injection of the first agent into U. Then $U - W$ does not contain the unique channel of U, so the action of s_1 on the second agent steps outside $U - W$.

But there is one useful case where $U - W$ is indeed a subpresheaf of U, as we show below in Lemma 3.69.

Definition 3.68. For any seed $Y \hookrightarrow M \hookleftarrow X$, let the *past* $\mathrm{past}(M) = M - Y$ of M be the set of its elements not in the image of Y. For any such M, presheaf U, and core $\mu \in U(M)$, let $\mathrm{past}(\mu) = \mathrm{Im}(\mathrm{past}(M))$ consist of all images of $\mathrm{past}(M)$.

To explain the statement a bit more, by Yoneda, we see μ as a map $M \to U$, so we have a set-function

$$\mathrm{past}(M) \hookrightarrow \mathrm{el}(M) \to \mathrm{el}(U).$$

We observe that $\mathrm{past}(\mu)$ is always a set of agents and actions only, since channels present in X always are in Y too.

Given a core $\mu \in U$, an important operation for us will consist of considering

$$U \setminus\!\!\setminus \mu = \bigcup \{ V \hookrightarrow U \mid \mathrm{el}(V) \cap \mathrm{past}(\mu) = \varnothing \}.$$

$U \setminus\!\!\setminus \mu$ is thus the largest subpresheaf of U not containing any element of the past of μ. The good property of this operation is:

Lemma 3.69. *If μ is a maximal core in G_U (i.e., there is no path to any further core) and G_U is target-linear, then $(U \setminus\!\!\setminus \mu)(c) = U(c) \smallsetminus \mathrm{past}(\mu)$ for all c.*

Proof. The direction $(U \setminus\!\!\setminus \mu)(c) \subseteq U(c) \smallsetminus \mathrm{past}(\mu)$ is by definition of $\setminus\!\!\setminus$. Conversely, it is enough to show that $c \mapsto U(c) \smallsetminus \mathrm{past}(\mu)$ forms a subpresheaf of U, i.e., that for any $f: c \to c'$ in $\mathbb{C}$, and $x \in U(c') \smallsetminus \mathrm{past}(\mu)$, $x \cdot f \notin \mathrm{past}(\mu)$. Assume on the contrary that $x' = x \cdot f \in \mathrm{past}(\mu)$. Then, of course f cannot be the identity. Furthermore, x' is either an agent or an action; so, up to pre-composition of f with a further morphism, we may assume that x' is an agent and $\mu \to x'$ in G_U. But then, since f is non-identity, x must be an action, with x' being one of its sources or targets. Now, up to post-composition of f with a further morphism, we may assume that x is a core. So, there is either an edge $x \to x'$ or an edge $x' \to x$ in G_U. However, $x \neq \mu$, so $x \to x'$ is impossible by target-linearity of G_U, and $x' \to x$ is impossible by maximality of μ. $\square$

Proof of Theorem 3.66. We proceed by induction on the number of actions in U. If it is zero, then U is a configuration; by (2), t is an iso, and by (3) so is s, hence the cospan is a trace. For the induction step, we first decompose U into

$$Y \xhookrightarrow{s_2} U' \xhookleftarrow{t_2} Z \xhookrightarrow{s_1} M' \xhookleftarrow{t_1} X,$$

and then show that M' is an action and U' satisfies the conditions of the theorem.

So, first, by acyclicity, pick a maximal core μ in G_U, i.e., one with no path to any other core. Let

$$\begin{tikzcd}
 & I_0 \arrow[dl] \arrow[d] \arrow[dr] & \\
Y_0 \arrow[r] & M_0 & X_0 \arrow[l]
\end{tikzcd}$$

be the seed with interface corresponding to μ, so we have the Yoneda morphism $M_0 \to U$.

Let $U' = (U \setminus\setminus \mu)$, and $X_1 = X - \mathrm{Agents}(X_0)$. X_1 is a subpresheaf of X, since it contains all channels. The square

$$\begin{tikzcd}
I_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\
X_0 \arrow[r] & X \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

is a pushout, since it just adds the missing agents to X_1. Define now Z, M', s_1, and t_1 by the pushouts

$$\begin{tikzcd}
 & Y_0 \arrow[rr] \arrow[d] & & Z \arrow[d, dashed, "s_1"] \arrow[dr] & \\
 & M_0 \arrow[rr] & & M' \arrow[r] & U \\
I_0 \arrow[uur] \arrow[ur] \arrow[rr] \arrow[dr] & & X_1 \arrow[uur] \arrow[ur] \arrow[dr] \arrow[urr] & & \\
 & X_0 \arrow[uu] \arrow[rr] & & X \arrow[uu, dashed, "t_1"'] \arrow[uur] &
\end{tikzcd}$$

and the induced arrows. We further obtain arrows to U by universal property of pushout. We show that the arrow $f \colon M' \to U$ is mono.

First, it is obviously mono in dimensions > 1. It is also mono in dimension 1, because $M'[n] = X[n] + Y_0[n]$ for all n, $X \to U$ is mono with image consisting only of initial agents, which are thus dijoint from the image of Y_0. Finally, for dimension 0, i.e., at $\star$, the pushout defining M' is isomorphic to

$$\begin{tikzcd}
I_0(\star) = X_0(\star) \arrow[r] \arrow[d, "\mathrm{inj}_l"'] & X_1(\star) = X(\star) \arrow[d, "\mathrm{inj}_l"] \\
M_0(\star) = X_0(\star) + I \arrow[r] & M'(\star) = X(\star) + I \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

where $I = M_0(\star) \setminus X_0(\star)$ is the set of channels 'created' by the action. Consider any $a, b \in M'(\star)$ such that $a \neq b$. Because $X \to U$ is mono, if $a, b \in X(\star)$ then $f(a) \neq f(b)$. By local 1-injectivity of U, if $a, b \in I$ then $f(a) \neq f(b)$. Finally, if $a \in X(\star)$ and $b \in I$, then we have an edge $f(b) \to \mu$ in G_U, whereas $f(a)$ is initial by (2) (in the statement of Theorem 3.66), so $f(a) \neq f(b)$. This entails that $Z \to U$ is a mono, because s_1 is a pushout of the mono $Y_0 \to M_0$.

By (1) and Lemma 3.69, $U = M' \cup U'$, i.e., the square

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & U' \arrow[d] \\ M' \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & U \end{tikzcd}$$

is a pushout, so U is indeed a composite as claimed, with $Z \hookrightarrow M' \hookleftarrow X$ an action by construction. So, it remains to prove that $Y \hookrightarrow U' \hookleftarrow Z$ satisfies the conditions. First, as a subpresheaf of U, U' is locally 1-injective and has a linear and acyclic causal graph, so satisfies (1) and (4). U' furthermore satisfies (2) by construction of Z and source-linearity of G_U, and (3) because removing past(μ) cannot make any non-final agent final. □

We conclude this section with a helpful corollary:

Corollary 3.70. *There is at most one cell filling any diagram*

$$\begin{tikzcd} Y' \arrow[r, "k"] \arrow[d, "u'"'] & Y \arrow[d, "u"] \\ X' \arrow[r, "h"'] & X \end{tikzcd}$$

in $\mathbb{D}$.

Proof. By definition, we have cospans $Y' \xrightarrow{s'} u' \xleftarrow{t'} X'$ and $Y \xrightarrow{s} u \xleftarrow{t} X$. Suppose we are given $l, l' : u' \to u$ making (k, l, h) and (k, l', h) into cells. By naturality, l and l' are determined by their images on channels, agents, and cores. We show by induction on the ordering induced by Theorem 3.66 that they have to coincide on these. For the base case: they have to coincide on initial agents and channels by definition of cells. For the induction step, we proceed by case analysis on the kind of element to consider. The image of any channel or agent created by a core μ is uniquely determined by naturality, which leaves the case of a core μ, of which we assume that there is an agent x such that $\mu \to x$ in $G_{u'}$ and $l(x) = l'(x)$. The former yields a morphism, say t, in $\mathbb{C}$ such that $\mu \cdot t = x$. But then by naturality we have $l(\mu) \cdot t = l(x) = l'(x) = l'(\mu) \cdot t$. By linearity of $G_{u'}$ we have $\mathrm{core}(l(\mu)) = \mathrm{core}(l'(\mu))$. Now let c_μ denote the object of $\mathbb{C}$ over which μ lies, and let c' be the one over which $\mathrm{core}(l(\mu))$ lies. By inspection of $\mathbb{C}$, there is exactly one morphism $f : c_\mu \to c'$, and so we have $l(\mu) = \mathrm{core}(l(\mu)) \cdot f = \mathrm{core}(l'(\mu)) \cdot f = l(\mu)$, as desired. □

3.5. **A playground.**

Theorem 3.71. $\mathbb{D}$ *forms a playground.*

This is proved by Lemmas 3.72 and 3.73 below.

Lemma 3.72.

(P8) Any double cell as in the center below, where B is a basic action and M is an action, decomposes in exactly one of the forms on the left and right:

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "\bullet"] \arrow[dr, phantom, "\alpha_1"] & X \arrow[d, "\bullet"] \\
C \arrow[r] \arrow[d, "\bullet"] \arrow[dr, phantom, "\alpha_2"] & Y \arrow[d, "\bullet"] \\
D \arrow[r] & Z
\end{tikzcd}
\quad \leftsquigarrow \quad
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "U\bullet"'] \arrow[ddr, phantom, "\alpha"] & X \arrow[d, "\bullet V"] \\
C \arrow[d, "B\bullet"'] & Y \arrow[d, "\bullet M"] \\
D \arrow[r, "k"'] & Z
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
A \arrow[r] \arrow[d, "\bullet"] \arrow[dr, phantom, "\alpha_1"] & X \arrow[d, "\bullet"] \\
C \arrow[d, "\bullet"] & Y \arrow[d, "\bullet"] \\
D \arrow[r] \arrow[ur] \arrow[ur, phantom, "\alpha_2"'] & Z.
\end{tikzcd}
$$

Proof. For any element a ovec $c \in \mathbb{C}$ of any presheaf $F \in \widehat{\mathbb{C}}$, let its *neighbourhood* consist of all elements in the image of $a : c \to F$.

Let $b \in B$ and $m \in M$ be the unique cores of B and M, respectively. Let V_m be the neighbourhood of m in $M \bullet V$.

If $\alpha(b) \in V_m$, we show that the whole of U is mapped to V, and we are in the left-hand case. It is clear for channels. If there exists an element x of U of dimension ≥ 1 mapped to y in $M \setminus V$, i.e., $M \setminus Y$, then we obtain a path $x \to x'$ to an agent x' of C, in $G_{B \bullet U}$. Via α, this yields a path $M \setminus Y \to Y$ in $G_{M \bullet V}$ between elements of dimension ≥ 1, a contradiction.

If now $\alpha(b) \notin V_m$, we show similarly that the whole of $B \bullet U$ is mapped to V, because the contrary would imply the existence of a path $M \setminus Y \to V \setminus Y$ in $G_{M \bullet V}$, which also is a contradiction. Hence, we are in the right-hand case. □

Lemma 3.73.

(P7) Any double cell

$$
\begin{tikzcd}
A \arrow[r, "h"] \arrow[dd, "U\bullet"'] \arrow[ddr, phantom, "\alpha"] & X \arrow[d, "\bullet W_1"] \\
 & Y \arrow[d, "\bullet W_2"] \\
B \arrow[r, "k"'] & Z
\end{tikzcd}
\quad \textit{decomposes as} \quad
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "\bullet U_1"] \arrow[dd, bend right=60, "U\bullet"'] \arrow[dr, phantom, "\alpha_1"] \arrow[d, phantom, "\alpha_3"', near start] & X \arrow[d, "\bullet W_1"] \\
C \arrow[r, "l"] \arrow[d, "\bullet U_2"] \arrow[dr, phantom, "\alpha_2"] & Y \arrow[d, "\bullet W_2"] \\
B \arrow[r, "k"'] & Z
\end{tikzcd}
$$

with α_3 an isomorphism, in an essentially unique way.

Proof. We first treat the case where W_2 is an action M.

Definition 3.74. Let, first, for any action $x \in \mathrm{el}(U)$, the *core associated to x* $\mathrm{core}(x)$ be the unique core $\mu \in \mathrm{el}(U)$ for which there exists f in $\mathbb{C}$ such that $\mu \cdot f = x$. If x is an agent or a channel, then by definition $\mathrm{core}(x) = x$.

Let $U_1 = U \times_W W_1$ and let $A \to U_1$ denote the induced arrow. By construction, all of $A \to U_1 \to U$ are monos and, by Lemma 3.18 and the pullback lemma, $A = U_1 \times_{W_1} X$.

Furthermore, consider any channel or agent $x \in \mathrm{el}(U_1)$. If $x' = f_1(x)$ is not initial in W_1, then by Theorem 3.66 we have an edge $x' \to \mu'$ for some core μ' of W_1. But, since f is a morphism between traces, it preserves initiality, so x cannot be initial in U, hence we find $x \to \mu$ in G_U. By source linearity of G_W, $\mathrm{core}(f(\mu)) = \mu'$, so we find a action $m' \in W$ with antecedents both in U and W_1, which entails that x is not initial in G_{U_1}. Thus, $f_1 : U_1 \to W_1$ preserves initiality of channels and agents.

$$\begin{tikzcd}
 & & A \arrow[rrrr, "f_s"] \arrow[d] \arrow[dddr, bend left] & & & & X \arrow[d] \arrow[dddr, bend left] & \\
 & & U_1 \arrow[rrrr, "f_1"] \arrow[ddr] & & & & W_1 \arrow[ddr] & \\
C \arrow[urr] \arrow[rrrr] \arrow[ddr] & & & & Y \arrow[urr] \arrow[ddr] & & & \\
 & & & U \arrow[rrrr, "f"] & & & & W \\
 & U_2 \arrow[urr] \arrow[rrrr, "f_2"'] & & & & M \arrow[urr] & & \\
 & B \arrow[u] \arrow[uurr, bend right] \arrow[rrrr, "f_t"'] & & & & Z, \arrow[u] \arrow[uurr, bend right] & &
\end{tikzcd}$$

Figure 5: Proof of Lemma 3.73

Let now $C \hookrightarrow U_1$ denote the subpresheaf of U_1 consisting of initial channels and agents (a subpresheaf because if x is an initial, n-ary agent, then $x \cdot s_i$ is an initial channel for any $i \in n$). Since f_1 preserves initiality, $C \to U_1 \to W_1$ factors through $Y \to W_1$, uniquely since the latter is mono, say as

$$\begin{tikzcd}
C \arrow[r, "f_m"] \arrow[d, hook] & Y \arrow[d, hook] \\
U_1 \arrow[r, "f_1"'] & W_1.
\end{tikzcd}$$

By construction and Theorem 3.66, $A \to U_1 \leftarrow C$ is a trace and (f_s, f_1, f_m) defines a morphism to $X \to W_1 \leftarrow Y$.

Let then $U_2 \hookrightarrow U$ denote the subpresheaf of U consisting of elements below C in G_U, i.e.,

$$x \in U_2 \Leftrightarrow \exists c \in C. c \to^{\star}{}_{G_U} \operatorname{core}(x).$$

A first observation is that all initial channels and agents of U are in U_2. Indeed, consider any such initial x. Let S denote the set of all $y \in U$ for which there is a path $y \to^{\star}{}_{G_U} x$. By finiteness of G_U and Theorem 3.66, this intersects A, hence U_1, hence C (by source-linearity), so we find a path $c \to^{\star} x$ for some $c \in C$, as desired.

Now, because $U_2 \to U$ and $M \to W$ are monos, showing that all elements x of U_2 map to M will imply that $U_2 \to U \to W$ uniquely factors through $M \to W$. Let us do this by case analysis:

- If x is not a channel, then f preserves paths from agents to x, so we find some path $y \to^{\star}{}_{G_W} \operatorname{core}(f(x))$ with $y \in Y$, which implies that $f(x) \in M$ ($f(x) \in W_1 - M$ would contradict initiality of Y in W_1).
- If x is some channel initial in U, then since f preserves initiality x is mapped to Z hence to M.

- If finally x is some non-initial channel, then $x \to \mu$ for some core $\mu \in U$. Now $\mu \in U_2$, as witnessed by the path $c \to^\star x \to \mu$. But then $x = \mu \cdot u$ for some morphism u of $\mathbb{C}$, so since by the above $f(\mu) \in M$, we have that $f(x) = f(\mu) \cdot u$ is in M too, as desired.

This shows that we have a commuting square

$$\begin{tikzcd} U_2 \arrow[r, "f_2"] \arrow[d, hook] & M \arrow[d, hook] \\ U \arrow[r, "f"'] & W. \end{tikzcd}$$

Finally, since all initial elements of U are in U_2 by construction, $B \to U$ factors through $U_2 \to U$, and we get a diagram as in Figure 5, which commutes thanks to all arrows in the right-hand square being mono.

By Theorem 3.66, $C \to U_2 \leftarrow B$ is a trace, and $U = U_2 \bullet U_1$, which shows existence of the desired decomposition.

For any decomposition as in Figure 5, we have $C = U_2 \times_M Y$ by Lemma 3.18, so by Corollary 3.39, we also have $U_1 = U \times_W W_1$. Thus, U_1 is uniquely determined up to canonical isomorphism. But by Theorem 3.66, $C \to U_1$ is so too, as the subobject of initial agents and channels. But then U_2 precisely consists of elements below C. Indeed, by finiteness of G_{U_2} and (3) in Theorem 3.66, all of U_2 clearly lies below C. Conversely, any $x \notin U_2$ is in $U_1 - U_2$ so by finiteness of G_{U_1} and (3) in Theorem 3.66, we have a path $x \to^+ c$ to some $c \in C$ which so x cannot lie below C. Our decomposition is thus unique up to canonical isomorphism. □

4. A sheaf model

In this section, we define our sheaf model for π, using constructions from [34] which are recalled along the way.

4.1. **Strategies and behaviours.** We first recall notions of strategies. As announced in the introduction, we define a category $\mathbb{T}(X)$ combining prefix ordering and isomorphism of traces: $\mathbb{T}(X)$ has traces $u: Y \multimap X$ as objects, and as morphisms $u \to u'$ all pairs (w, α) with $w: Y' \multimap Y$ and α an isomorphism $u \bullet w \to u'$ in the hom-category $\mathbb{D}_v(Y', X)$, as on the right[2]. Thus, u' is an extension of u by w.

$$\begin{tikzcd} & Y' \arrow[dl, "w"'] \arrow[dd, bend left, "u'"] \\ Y \arrow[dr, "u"'] \arrow[r, phantom, "\overset{\cong}{\underset{\alpha}{\Longrightarrow}}"] & {} \\ & X \end{tikzcd}$$

Definition 4.1. Let the category of *(naive) strategies* on X be $\widehat{\mathbb{T}(X)}$.

Strategies do not yield a satisfactory model for π:

Example 4.2. Consider the configuration X with three agents x, y, z sharing a channel a, and the following traces on it: in $u_{x,y}$, x sends a on a, and y receives it; in $u_{x,z}$, x sends a on a, and z receives it; in i_z, z inputs on a. One may define a strategy S mapping $u_{x,y}$ and i_z to a singleton, and $u_{x,z}$ to $\varnothing$. Because $u_{x,y}$ is accepted, x accepts to send a on a; and because i_z is accepted, z accepts to input on a. The problem is that S rejecting $u_{x,z}$ roughly amounts to x refusing to synchronise with z, or conversely.

[2]There is a small problem, however: morphisms should only describe how u maps to u', not w. We actually quotient them out just as in [34] to rectify this.

We want to rule out this kind of strategy from our model, by adapting the idea of innocence. We start by extending $\mathbb{T}(X)$ with objects representing traces on sub-configurations of X. For this, we consider the following category $\mathbb{T}_X$. It has as objects pairs (u,h) of a trace $u{:}\,Z \multimap Y$ and a morphism $h{:}\,Y \to X$ in $\mathbb{D}_h$. A morphism $(u,h) \to (u',h')$ consists of a trace $w{:}\,T \multimap Z$ and a cell making the diagram on the right commute[2].

$$\begin{tikzcd}
T \arrow[r, "s"] \arrow[d, "w"'] & Z' \arrow[dd, "u'"] \\
Z \arrow[r, phantom, "\overset{\alpha}{\Longrightarrow}"] \arrow[d, "u"'] & \\
Y \arrow[r, "r"] \arrow[dr, "h"'] & Y' \arrow[d, "h'"] \\
& X
\end{tikzcd}$$

Example 4.3. Recalling the right-hand trace of Figure 4, say $u{:}\,Y \multimap X$, y's first action is an input on its unique channel b. This yields a trace $\iota_{1,1}{:}\,[2] \multimap [1]$. There is a morphism $(\iota_{1,1}, y) \to (u, id_X)$ in $\mathbb{T}_X$, pictured as the right-hand diagram, which we think of as an occurrence of the trace $\iota_{1,1}$ in u. Thus, morphisms in $\mathbb{T}_X$ account both for prefix inclusion and for 'spatial' inclusion, i.e., inclusion of a trace into some other trace on a larger configuration.

$$\begin{tikzcd}
{[3]} \arrow[r, "y''"] \arrow[d, "\iota_{2,2}"'] & Y \arrow[dd, "u"] \\
{[2]} \arrow[r, phantom, "\Longrightarrow"] \arrow[d, "\iota_{1,1}"'] & \\
{[1]} \arrow[r, "y"] \arrow[dr, "y"'] & X \arrow[d, equal] \\
& X.
\end{tikzcd}$$

We now define *views* within $\mathbb{T}_X$:

Definition 4.4. A *view* is a trace isomorphic to some (possibly empty) composite of basic actions (Definition 3.44). Let $\mathbb{V}_X$ denote the full subcategory of $\mathbb{T}_X$ spanning pairs (u,h) where u is a view.

Intuitively, basic actions follow exactly one agent through an action. An object of $\mathbb{V}_X$ consists of a view, say $v{:}\,[n'] \multimap [n]$, plus a morphism $h{:}\,[n] \to X$ in $\mathbb{D}_h$, which by Yoneda is just an agent of X. So an object of $\mathbb{V}_X$ is just an agent of X and a view from it.

Definition 4.5. The inclusion $\mathsf{j}_X{:}\,\mathbb{V}_X \hookrightarrow \mathbb{T}_X$ induces a Grothendieck topology, for which a family $((u_i,h_i) \xrightarrow{(w_i,\alpha_i)} (u,h))_{i\in I}$ of morphisms to some trace u is *covering* iff it contains all morphisms from views into u. Let the category $\mathsf{S}_X \hookrightarrow \widehat{\mathbb{T}_X}$ of *innocent* strategies be the category of sheaves of finite sets for this topology. Let the category B_X of *behaviours* over X be $\widehat{\mathbb{V}_X}$.

As announced in the introduction, we have:

Proposition 4.6. *The embedding* $\mathrm{ran}_{\mathsf{j}_X^{op}}$ *induces an equivalence of categories* $\mathsf{B}_X \simeq \mathsf{S}_X$.

We thus obtain the innocent strategy S_B associated to a behaviour $B \in \mathsf{B}_X$ by taking its *right Kan extension* [50] along the inclusion $\mathsf{j}_X^{op}{:}\,\mathbb{V}_X^{op} \hookrightarrow \mathbb{T}_X^{op}$, as on the right. Explicitly, using standard results, we obtain the *end* $S_B(u,h) = \int_{(v,x)\in\mathbb{V}_X} B(v,x)^{\mathbb{T}_X((v,x),(u,h))}$, which is a kind of generalised product. In the boolean setting (functors to 2), this end reduces to a conjunction $\bigwedge_{\{(v,x)\in\mathbb{V}_X \mid \exists\alpha:(v,x)\to(u,h)\}} B(v,x)$, demanding precisely that all views of u are accepted by B. In the general case, the intuition is that a way of accepting u for S_B is a compatible family of ways of accepting the views of u for B. The forgetful functor $\mathcal{U}$ to naive strategies is then given by restricting along $\mathbb{T}(X)^{op} \hookrightarrow \mathbb{T}_X^{op}$ as above right. Some local information may be forgotten by $\mathcal{U}$, which is neither injective on objects, nor full, nor faithful. E.g., if two behaviours differ, but are both empty on the views of some agent, then both are mapped to the empty naive strategy.

$$\begin{tikzcd}
\mathbb{V}_X^{op} \arrow[r, hook, "\mathsf{j}_X^{op}"] \arrow[dr, bend right, "B"'] & \mathbb{T}_X^{op} \arrow[d, "S_B"] & \mathbb{T}(X)^{op} \arrow[l, hook'] \arrow[dl, bend left, "\mathcal{U}(S_B)"] \\
& \mathsf{set} &
\end{tikzcd}$$

Example 4.7. Recalling X and S from Example 4.2, let us show that for any $B \in \mathsf{B}_X$, the associated strategy $\mathcal{U}(S_B) \in \widehat{\mathbb{T}(X)}$ cannot be S. Indeed, if $\mathcal{U}(S_B)$ was S, then because S accepts $u_{x,y}$ and i_z, B has to accept the following views: (1) i_z, (2) o_x, in which x sends a on a (without any matching input), (3) i_y, in which y inputs on a, and (4) all identity views on x, y, and z. But then $\mathcal{U}(S_B)$ has to accept both $u_{x,y}$ and $u_{x,z}$, because B accepts all views mapping into them.

The assignment $X \mapsto \mathsf{B}_X$ is further studied in [34], and extended to a pseudo double functor $\mathbb{D}^{op} \to \mathbb{Q}\mathsf{Cat}$, where $\mathbb{Q}\mathsf{Cat}$ denotes Ehresmann's double category of *quintets* over Cat. Let us briefly explain how this works.

First, the action of a horizontal morphism $k: X' \to X$ on a behaviour $B \in \mathsf{B}_X$ yields the behaviour $B \cdot k$ such that for all $(v, h) \in \mathbb{V}_{X'}$, $(B \cdot k)(v, h) = B(v, k \circ h)$.

Proposition 4.8. *The functor* $\mathsf{B}_X \cong \prod_{n,x:[n]\to X} \mathsf{B}_{[n]}$ *given at* (n, x) *by horizontal action of* x*, i.e.,* $B \mapsto B \cdot x$*, is an isomorphism.*

Proof. We have $\mathbb{V}_X \cong \sum_{n,x:[n]\to X} \mathbb{V}_{[n]}$. □

Notation 4.9. If $(B_x)_{x:[n]\to X}$ is a family of behaviours indexed by the agents of X, we denote its unique antecedent by $[B_x]_{x:[n]\to X}$.

Vertical action is a bit harder. The easiest way to get to it is perhaps to realise that there is a very simple syntax for behaviours on individuals. Indeed, consider any n and behaviour $B \in \mathsf{B}_{[n]}$. B is a coproduct $B = \sum_{i\in\gamma} B_i$ of *definite* behaviours, i.e., ones with a single initial state. Here, γ is of course $B(id^{\bullet}_{[n]}, id_{[n]})$. Now each term B_i of the above sum is determined up to isomorphism by giving for any basic action $b: [n_b] \multimap [n]$, its *residual* $B_i \cdot b$ along b, which is the unique behaviour B' on $[n_b]$ such that $B'(v, id_{[n_b]}) = B_i(b \bullet v, id_{[n]})$ for all views $v: [n'] \multimap [n_b]$. (This rests on the fact that there is exactly one agent in $[n]$, namely $(n, id_{[n]})$.) Each $B_i \cdot b$ may in turn be characterised in this way. We thus pose:

Definition 4.10. Let *behaviour-terms* and *definite behaviour-terms* respectively denote infinite terms generated by the judgements $\vdash$ and $\vdash_{\mathsf{D}}$ of the grammar in Figure 6.

$$\frac{\ldots \quad n \vdash_{\mathsf{D}} E_k \quad \ldots \quad (\forall k \in \gamma)}{n \vdash \oplus_{k\in\gamma} E_k} \qquad \frac{\ldots \quad n_b \vdash T_b \quad \ldots \quad (\forall (b: [n_b] \multimap [n]) \in \mathbb{B}_{[n]})}{n \vdash_{\mathsf{D}} \langle T_b \rangle_{b\in\mathbb{B}_{[n]}}} .$$

Figure 6: Syntax for behaviours

The intuition is that the first judgement $\vdash$ describes behaviours and that $\vdash_{\mathsf{D}}$ describes definite behaviours.

In [34], we impose two additional restrictions:

- the rule for forming behaviour-terms requires γ to be a finite ordinal, and
- behaviours have to be presheaves of finite ordinals (with monotone maps between them).

When both restrictions are imposed, the indexed family $[n] \mapsto \mathsf{D}_{[n]}$ of definite behaviours over $[n]$ is the final coalgebra for the functor mapping any $\mathbb{I}$-indexed family U to

$$\prod_{b:[n_b]\to[n]} (U_{[n_b]})^{\star}.$$

Without the restrictions, we need to work a bit more in order to precisely state the sense in which behaviour-terms correspond to behaviours. We first treat the case of behaviours on individuals. We consider the relations $\leftrightarrow$ and $\leftrightarrow_{\mathsf{D}}$ defined by mutual coinduction by the rules in Figure 7, where the subtlety is that the coproduct $\sum_{i\in\gamma} D_i$ is only defined up to

$$\frac{\dots \qquad (n \vdash_{\mathsf{D}} E_i) \leftrightarrow_{\mathsf{D}} (n \vdash_{\mathsf{D}} D_i) \qquad \dots \qquad (\forall i \in \gamma)}{(n \vdash \oplus_{i \in \gamma} E_i) \leftrightarrow (n \vdash \sum_{i \in \gamma} D_i)}$$

$$\frac{\dots \qquad (n_b \vdash T_b) \leftrightarrow (n_b \vdash D \cdot b) \qquad \dots \qquad (\forall (b \colon [n_b] \multimap [n]) \in \mathbb{B}_{[n]})}{(n \vdash_{\mathsf{D}} \langle (T_b)_b \rangle) \leftrightarrow_{\mathsf{D}} (n \vdash_{\mathsf{D}} D)}$$

Figure 7: Correspondence between behaviour-terms and behaviours

canonical isomorphism. Both relations are easily seen to be total and surjective but neither of them is functional or injective. They are still sensible, as witnessed by

Lemma 4.11. *If any (resp. definite) behaviour-term is related to two (resp. definite) behaviours B_1 and B_2, then $B_1 \cong B_2$.*

Proof. We prove both statements by mutual coinduction on the proofs of $T \leftrightarrow B_1$ and $T \leftrightarrow B_2$ (resp. $E \leftrightarrow_{\mathsf{D}} D_1$ and $E \leftrightarrow_{\mathsf{D}} D_2$).

If we are given proofs of $E \leftrightarrow D_1$ and $E \leftrightarrow D_2$, then because E is a definite behaviour-term, the last applied inference rule is the second in both cases, so $E = \langle (T_b)_b \rangle$, $D_1 = \langle (B^1_b)_b \rangle$ and $D_2 = \langle (B^2_b)_b \rangle$ with $B^i_b = D_i \cdot b$ for all b and $i = 1, 2$. By coinduction hypothesis, we obtain, an isomorphism $B^1_b \cong B^2_b$ for all b, which trivially induces an isomorphism $D_1 \cong D_2$.

If we are given proofs of $T \leftrightarrow B_1$ and $T \leftrightarrow B_2$, then because T is a behaviour-term, the last applied inference rule is the first in both cases, so $T = \oplus_{i \in \gamma} E_i$, $B_1 = \sum_{j \in \gamma} D^1_j$, $B_2 = \sum_{j \in \gamma} D^2_j$, for some γ. By coinduction hypothesis this yields a family of isomorphisms $D^1_j \cong D^2_j$, hence $B_1 \cong B_2$ as desired. □

In conclusion, all (definite) behaviours over individuals are uniquely described up to isomorphism by (definite) behaviour-terms. This brings us both a syntax and a decomposition (coinduction) principle for (definite) behaviours. Using this, we may now define the action of views over behaviours on individuals:

Definition 4.12. We define $B \cdot u$ and $D \cdot u$ by mutual coinduction:

- $(\sum_{i \in \gamma} D_i) \cdot u = \sum_i (D_i \cdot u)$,
- $D \cdot id^{\bullet} = D$,
- $\langle (B_b)_b \rangle \cdot (b \bullet u') = B_b \cdot u'$.

(This is only defined up to isomorphism of behaviours, which is enough for our purposes.)

Let us now consider arbitrary traces and not-necessarily-individual configurations. We use $\cdot$ for denoting both horizontal and vertical action, but that there should be no ambiguity since we use different letters to range over horizontal $(h, k, x, y, \dots)$ and vertical $(v, w, u, \dots)$ morphisms. E.g., for any $B \in \mathsf{B}_X$, $x \colon [n] \to X$, and $b \colon [n_b] \multimap [n]$, $B \cdot x \in \mathsf{B}_{[n]}$ denotes the corresponding behaviour over $[n]$, and $B \cdot x \cdot b$ denotes its residual along b.

Recalling Proposition 4.8 and the notation of Lemma 3.48 (or rather [34, Proposition 4.24]), we state:

Definition 4.13. The vertical action of $u \colon Y \multimap X$ on $B \in \mathsf{B}_X$ is uniquely defined up to isomorphism by

$$B \cdot u \cdot y \cong B \cdot y^u \cdot v^{y,u}.$$

As announced, this further extends to a pseudo double functor $\mathbb{D}^{op} \to \mathbb{Q}\mathsf{Cat}$. In our case, Corollary 3.70 almost reduces this to

Lemma 4.14. *For any $B \in \mathsf{B}_X$ and cell*

$$\begin{tikzcd} Y' \arrow[r, "k"] \arrow[d, "u'"'] \arrow[dr, phantom, "\overset{\alpha}{\Longrightarrow}"] & Y \arrow[d, "u"] \\ X' \arrow[r, "h"'] & X, \end{tikzcd}$$

we have $D \cdot h \cdot u' \cong D \cdot u \cdot k$.

The result is stated (and proved) in full generality as [34, Proposition 4.31].

4.2. **Semantic fair testing.**

We now define our semantic analogue of fair testing equivalence. It rests on two main ingredients: a notion of *closed-world* trace, and an analogue of *parallel composition* in game semantics.

$$\begin{tikzcd} \mathbb{V}^{op}_{X_1} \arrow[r, hook] \arrow[dr, "B_1"'] & \mathbb{V}^{op}_{X} \arrow[d, dashed, "{[B_1,B_2]}"] & \mathbb{V}^{op}_{X_2} \arrow[l, hook'] \arrow[dl, "B_2"] \\ & \mathsf{set} & \end{tikzcd}$$

The intuitive purpose of parallel composition is to let behaviours interact. If we partition the agents of a configuration X into two teams, we obtain two subconfigurations $X_1 \hookrightarrow X \hookleftarrow X_2$, each agent of X belonging to X_1 or X_2 according to its team. The crucial fact is that the category $\mathbb{V}_X$ of views on X is isomorphic to the coproduct category $\mathbb{V}_{X_1} + \mathbb{V}_{X_2}$. Parallel composition of any $B_1 \in \widehat{\mathbb{V}_{X_1}}$ and $B_2 \in \widehat{\mathbb{V}_{X_2}}$ is then simply given by copairing $[B_1, B_2]$ (following Notation 4.9).

We now describe closed-world actions and traces, which are then used as a criterion for success of tests. *Closed-world* actions were defined (Definition 3.44) as those not involving any interaction with the environment, i.e., formally, pushouts of a seed of any shape among ν_n,τ_n,$\heartsuit_n$,π_n, and $\tau_{n,a,m,c,d}$. A trace is *closed-world* when it is a composite of closed-world actions. Let $\mathbb{W}(X) \xhookrightarrow{\mathsf{i}_X} \mathbb{T}(X)$ denote the full subcategory of $\mathbb{T}(X)$ consisting of closed-world traces, and let the category of *closed-world strategies* be $\widehat{\mathbb{W}(X)}$. Further, denote by $B \mapsto \overline{B}$ the composite functor $\widehat{\mathbb{V}_X} \to \widehat{\mathbb{T}(X)} \xrightarrow{\Delta_{\mathsf{i}_X^{op}}} \widehat{\mathbb{W}(X)}$, where $\Delta_{\mathsf{i}_X^{op}}$ denotes restriction along i_X^{op}.

A closed-world trace is *successful* when it contains a $\heartsuit$ action, and *unsuccessful* otherwise. A state $\sigma \in S(u)$ of a strategy $S \in \widehat{\mathbb{W}(Z)}$ over a closed-world trace $u: Z' \to Z$ is successful iff u is. Define $⫫_Z$ as the set of closed-world strategies $S \in \widehat{\mathbb{W}(Z)}$ such that any unsuccessful closed-world state admits a successful extension, i.e. $S \in ⫫_Z$ iff for all unsuccessful $u \in \mathbb{W}(Z)$ and $\sigma \in S(u)$, there exists a successful $u' \in \mathbb{W}(Z)$, a morphism $f: u \to u'$, and a state $\sigma' \in S(u')$ such that $\sigma' \cdot f = \sigma$. Finally, in order to compare behaviours for semantic fair testing equivalence, we specify what a test is for a given behaviour $B \in \mathsf{B}_X$. A *test* consists of a configuration Y and a behaviour $T \in \mathsf{B}_Y$. Recalling Definition 3.8, we say that the behaviour B *should pass* the test (Y, T) iff $I_X = I_Y$ and $\overline{[B,T]} \in ⫫_Z$, where Z is the pushout $X +_{I_X} Y$ (X and Y thus form two teams on Z). At last, we define *semantic fair testing equivalence*, for any $B \in \mathsf{B}_X$ and $B' \in \mathsf{B}_{X'}$:

Definition 4.15. Let $B \sim_f B'$ iff B and B' should pass the same tests.

5. Intensional full abstraction

5.1. **Labelled transition system for behaviours.** We now work towards taming the previous semantic definition, characterising it as fair testing equivalence on an *ad hoc* graph with testing. In [34], an LTS is constructed for definite behaviours, with full *quasi-actions* as an alphabet, i.e., the graph with configurations as objects, and as edges all full traces whose restrictions to agents have length ≤ 1. Formally, $u : Y \multimap X$ is a quasi-action iff for all agents $x : [n] \to X$, the restriction $u_{|x}$ has length ≤ 1. Such a quasi-action is *full* iff for all agents, $u_{|x}$ is full (Definition 3.44).

Definition 5.1. Let $\mathbb{Q}$ denote the graph of full quasi-actions.

Definition 5.2. Let $\mathbb{D}_v^{\mathbb{W}}$ denote the smallest locally full subbicategory of $\mathbb{D}_v$ containing all closed-world actions in $\mathbb{W}$ (Definition 3.44). The graph morphism $\mathbb{W} \to \Sigma$ extends to a pseudo functor $\mathsf{p}^{\mathbb{W}} : \mathbb{D}_v^{\mathbb{W}} \to \mathsf{fc}(\Sigma)$, which essentially counts the number of ticks.

The LTS that we use here differs slightly from that of [34], in that it is up-to-iso. This means that, e.g., if there is a transition $S_1 \leftarrow S_2$ and $S_2 \cong S_2'$, then also $S_1 \leftarrow S_2'$.

Notation 5.3. For any behaviour $B = \sum_{k \in \gamma} D_k$, let $|B| = \gamma$ and, for any $k \in \gamma$, $B_{|k} = D_k$.

In order to define our LTS, we need to extend the notion of a definite behaviours from individuals to arbitrary configurations.

Definition 5.4. For any configuration X, let $\mathsf{D}_X \subseteq \mathsf{B}_X$, the category of *definite* behaviours on X, denote the full subcategory consisting of all behaviours $B \in \mathsf{B}_X$ such that for all agents $x : [n] \to X$, $B \cdot x$ is definite.

Definition 5.5. The graph $\mathbb{S}^{\mathbb{D}}$ underlying the LTS of definite behaviours has as objects all pairs (X, D) with $D \in \mathsf{D}_X$, and as edges $(X, D) \leftarrow (Y, E)$ all full quasi-actions $M : Y \multimap X$ such that there exists a family of states $\sigma_y \in |D \cdot M \cdot y|$ indexed by all agents $y : [n_y] \to Y$, satisfying

$$E \cdot y \cong (D \cdot M \cdot y)_{|\sigma_y}$$

for all y.

The morphism $p^{\mathbb{D}} : \mathbb{S}^{\mathbb{D}} \to \mathbb{Q}$ maps (X, D) to X and (M, σ) to M.

We'll here only need the following closed-world restriction of $\mathbb{S}^{\mathbb{D}}$:

Definition 5.6. Let $\mathbb{S}$ denote the identity-on-objects subgraph of $\mathbb{S}^{\mathbb{D}}$ with edges restricted to closed-world actions and identities.

Moreover, let $\mathsf{p}^{\mathbb{S}} : \mathbb{S} \to \Sigma$ denote the map sending $(X, D) \xleftarrow{(M,\sigma)} (X', D')$ to the $\heartsuit$ edge in Σ if M is a tick action and to τ otherwise.

Definition 5.7. We now define the relation $|_{\mathbb{S}}$ by $(Z, D) \in ((X_1, D_1) |_{\mathbb{S}} (X_2, D_2))$ iff $X_1(\star) = X_2(\star)$ and there is a pushout square

$$\begin{tikzcd} I_{X_1} \arrow[r] \arrow[d] & X_2 \arrow[d, "\mathrm{inj_r}"] \\ X_1 \arrow[r, "\mathrm{inj_l}"'] & Z \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

such that $D_1 = D \cdot \mathrm{inj_l}$ and $D_2 = D \cdot \mathrm{inj_r}$.

Lemma 2.5 entails:

Proposition 5.8. *The morphism* $\mathsf{fc}(\mathsf{p}^{\mathcal{S}}) \colon \mathsf{fc}(\mathcal{S}) \to \mathsf{fc}(\Sigma)$*, with* $|_{\mathcal{S}}$ *as testing relation, forms a free graph with testing.*

Proof. We exhibit a weak bisimulation relating any two such pushouts (Z, D) and (Z', D'). The relation containing two such pairs as soon as there exists a horizontal isomorphism $h \colon Z \to Z'$ such that $D = D' \cdot h$ does the job. □

We have:

Lemma 5.9. *For any definite behaviour* $D \in \mathsf{D}_X$*, we have* $D \in \perp\!\!\!\perp_X$ *iff* $(X, D) \in \perp^{\mathcal{S}}$*.*

Proof. We have $D \in \perp\!\!\!\perp_X$ iff $(X, D) \in \perp^{\mathcal{C}}$ iff $(X, D) \in \perp^{\mathcal{S}}$ by Corollary 5.25 and Lemma 5.26 below. □

The rest of this section is devoted to the proof of these two results. We first introduce and study the graph with testing $\mathcal{C}$, which requires a bit of preliminary work. Consider any closed-world trace $W \colon X' \mathrel{\multimap\!\!\!\to} X$ and recall from [34] that we have a map

$$\psi^D_W \colon \overline{D}(W) \to \prod_{x' \colon [n_{x'}] \to X'} |D \cdot W \cdot x'|.$$

(From now on, we omit the superscript D which is clear from context.)

This map is constructed by first recalling that the domain is an end:

$$\overline{D}(W) \cong \int_{(v,x) \in \mathbb{V}_X} D(v, x)^{\mathbb{T}_X((v,x),(W, id_X))}.$$

Furthermore, by definition, the codomain is

$$\prod_{x' \colon [n_{x'}] \to X'} D(v^{x',W}, (x')^W),$$

where $v^{x',W}$ denotes the view of x' in W, and $(x')^W$ denotes the corresponding agent in X, as in

$$\begin{tikzcd} {[n_{x'}]} \arrow[r, "x'"] \arrow[d, "v^{x',W}"'] \arrow[dr, phantom, "\xRightarrow{\alpha^{x',W}}"] & X' \arrow[d, "W"] \\ d^{x',W} \arrow[r, "(x')^W"'] & X. \end{tikzcd} \tag{5.1}$$

So, viewing the above end as a subset of $\prod_{(v,x) \in \mathbb{V}_X} D(v, x)^{\mathbb{T}_X((v,x),(W, id_X))}$, $\psi_W(\sigma)$ is defined at x' to be

$$\sigma(v^{x',W}, (x')^W)(id^\bullet_{[n_{x'}]}, \alpha^{x',W}).$$

Here, $\sigma(v^{x',W}, (x')^W)$ is in $D(v^{x',W}, (x')^W)^{\mathbb{T}_X((v^{x',W},(x')^W),(W, id_X))}$. So by applying it to (5.1) viewed as a morphism in $(v^{x',W}, (x')^W) \to (W, id_X)$ in $\mathbb{T}_X$, we obtain an element of $D(v^{x',W}, (x')^W)$, as desired.

The proof of [34, Proposition 5.23] actually applies to show:

Lemma 5.10. *If* W *is a quasi-action, then* ψ_W *is bijective.*

Furthermore, ψ_W is always injective, but not generally surjective if W is not a quasi-action.

Example 5.11. Consider the trace $W = (\tau_0 \bullet \pi_0)$ consisting of a nullary agent performing a τ action and then forking. Consider now any definite behaviour D such that $D(\tau_0) = D(\tau_0 \bullet \pi_0^l) = D(\tau_0 \bullet \pi_0^r) = 2$, which maps both inclusions $\tau_0 \bullet \pi_0^l \hookleftarrow \tau_0 \hookrightarrow \tau_0 \bullet \pi_0^r$ to the identity. Then $\overline{D}(W) = 2$: it consists of pairs (σ^l, σ^r) in $D(\tau_0 \bullet \pi_0^l) \times D(\tau_0 \bullet \pi_0^r)$ whose restrictions to $D(\tau_0)$ coincide. This leaves just $(1,1)$ and $(2,2)$. On the other hand, the set $\prod_{x':[0]\to[0]|[0]} D(v^{x',W}, (x')^W)$ is 2×2, hence has four elements.

Notation 5.12. We extend Notation 5.3: if $B \in \mathsf{B}_X$ and $\sigma \in \prod_{n,x:[n]\to X} |B|$, let $B_{|\sigma}$ be defined up to isomorphism by

$$B_{|\sigma} \cdot x = (B \cdot x)_{|\sigma(X)}.$$

Definition 5.13. Let $\mathcal{C}$ denote the graph with $\mathrm{ob}(\mathcal{C}) = \mathrm{ob}(\mathcal{S})$, and where $\mathcal{C}((X',D'),(X,D))$ is the set of closed-world traces $W: X' \multimap X$ such that there exists a state $\sigma \in \overline{D}(W)$ satisfying $(D \cdot W)_{|\psi_W(\sigma)} \cong D'$.

Thus, $\mathcal{C}$ is a generalisation of $\mathcal{S}$ from closed-world actions to closed-world traces with the fancy notation just introduced.

Lemma 5.14. *For all edges $(X,D) \xleftarrow{W} (X',D') \xleftarrow{W'} (X'',D'')$ in $\mathcal{C}$, there is an edge $(X,D) \xleftarrow{W \bullet W'} (X'',D'')$.*

Proof. Consider $\sigma \in \overline{D}(W)$ such that $(D \cdot W)_{|\psi_W(\sigma)} \cong D'$ and $\sigma' \in \overline{D'}(W')$ such that $(D' \cdot W')_{|\psi_{W'}(\sigma')} \cong D''$. We want to construct an edge $W \bullet W': (X'',D'') \to (X,D)$, i.e., find $\sigma'' \in \overline{D}(W \bullet W')$ such that $(D \cdot (W \bullet W'))_{|\psi_{W \bullet W'}(\sigma')} \cong D''$. Now, the isomorphism $\varphi: (D \cdot W)_{|\psi_W(\sigma)} \xrightarrow{\sim} D'$ yields a state $\sigma_1 = \overline{\varphi}^{-1}_{W'}(\sigma') \in \overline{(D \cdot W)_{|\psi_W(\sigma)}}(W')$ such that

$$((D \cdot W)_{|\psi_W(\sigma)} \cdot W')_{|\psi_{W'}(\sigma_1)} \cong (D' \cdot W')_{|\psi_{W'}(\sigma')}.$$

Now we have

$$\overline{(D \cdot W)_{|\psi_W(\sigma)}}(W') \cong \{\sigma'' \in \overline{D}(W \bullet W') \mid \sigma''_{|W} = \sigma\},$$

where $\sigma''_{|W}$ denotes restriction of σ'' along the prefix inclusion $W \hookrightarrow W \bullet W'$. So the left-hand side above is just $(D \cdot (W \bullet W'))_{|\psi_{W \bullet W'}(\sigma_1)}$, which yields the desired transition. □

Lemma 5.15. *The obvious projection $\mathsf{p}^{\mathcal{C}}: \mathcal{C} \to \mathbb{D}_v^{\mathbb{W}}$ is faithful.*

Proof. By construction. □

Recalling Definition 5.2, we have:

Proposition 5.16. *The projection $\mathcal{C} \xrightarrow{\mathsf{p}^{\mathcal{C}}} \mathbb{D}_v^{\mathbb{W}} \xrightarrow{\mathsf{p}^{\mathbb{W}}} \mathsf{fc}(\Sigma)$, with $|_{\mathcal{S}}$ as testing relation, forms a graph with testing.*

Proof. Just as Lemma 5.8. □

We now show that fair testing equivalence in the sense of $\mathcal{C}$ and $\mathcal{S}$ both coincide with semantic fair testing equivalence.

$\mathcal{C}$ does not quite form a category, because composition of traces is not associative. We could quotient out morphisms to rectify this, or take 2-cells into account to make $\mathcal{C}$ into a bicategory. However, we will here only need the following bit of 2-dimensionality.

Notation 5.17. For any morphism $p: G \to H$ in Gph, we denote by $p_{A,B}: G(A,B) \to H(p(A), p(B))$ the component of p at A and B.

Lemma 5.18. *For any $(X,D),(X',D') \in \mathcal{C}$, if there exists any special isomorphism $W_1 \cong W_2$ in $\mathbb{D}_v^{\mathbb{W}}(X',X)$, we have*

$$(\mathsf{p}^{\mathcal{C}})^{-1}_{(X,D)(X',D')}(W_1) \cong (\mathsf{p}^{\mathcal{C}})^{-1}_{(X,D)(X',D')}(W_2).$$

Proof. The given special isomorphism induces by pseudo double functoriality of B an isomorphism $\varphi: D \cdot W_1 \Rightarrow D \cdot W_2$, hence an isomorphism

$$\overline{\varphi}_{id_{X'}}: \overline{D}(W_1) \Rightarrow \overline{D}(W_2).$$

This isomorphism is such that for any σ,

$$(D \cdot W_1)_{|\psi_{W_1}(\sigma)} \cong (D \cdot W_2)_{|\psi_{W_2}(\overline{\varphi}_{id_{X'}}(\sigma))}.$$

Thus, $(D \cdot W_1)_{|\psi_{W_1}(\sigma_1)} \cong D'$ iff $(D \cdot W_2)_{|\psi_{W_2}(\varphi(\sigma_1))} \cong D'$. $\square$

We now relate $\mathcal{C}$ to $\mathcal{S}$.

Proposition 5.19. *The projection $\mathsf{p}^{\mathcal{C}}: \mathcal{C} \to \mathbb{D}_v^{\mathbb{W}}$ satisfies the following weak Conduché condition: for all $X'' \xrightarrow{W_2} X' \xrightarrow{W_1} X$, if there is an edge $(X'',D'') \xrightarrow{W_1 \bullet W_2} (X,D)$ in $\mathcal{C}$, then there are edges $(X'',D'') \xrightarrow{W_2} (X',D') \xrightarrow{W_1} (X,D)$.*

Proof. Consider any $(X,D) \in \mathcal{C}$ and $\sigma \in \overline{D}(W_1 \bullet W_2)$ witnessing the given edge. Consider also the morphism $u: W_1 \to (W_1 \bullet W_2)$ given by (W_2, id), and let $\sigma_1 = \sigma \cdot u \in \overline{D}(W_1)$. Let $D_1 = (D \cdot W_1)_{|\psi_{W_1}(\sigma_1)}$. We have $\sigma \in \{\sigma' \in \overline{D}(W_1 \bullet W_2) \mid \sigma' \cdot u = \sigma_1\}$, hence $\sigma \in \overline{D_1}(W_2)$. Furthermore,

$$(D_1 \cdot W_2)_{|\psi_{W_2}(\sigma)} \cong (D \cdot (W_1 \bullet W_2))_{|\psi_{W_1 \bullet W_2}(\sigma)} \cong D'',$$

so we have two edges

$$(X,D) \xleftarrow{(W_1,\sigma_1)} (X',D_1) \xleftarrow{(W_2,\sigma)} (X'',D'')$$

as desired. $\square$

As an easy consequence, for any closed-world trace W and edge e of $\mathcal{C}$ over W, any decomposition of W into actions induces a corresponding decomposition of e. Formally:

Notation 5.20. By default, composition in $\mathbb{D}_v^{\mathbb{W}}$ associates to the right, i.e., $W \circ W' \circ W''$ denotes $W \circ (W' \circ W'')$.

Corollary 5.21. *For any path p, say*

$$X = X_0 \xleftarrow{M_1} X_1 \xleftarrow{M_2} \dots X_n = X', \tag{5.2}$$

in $\mathbb{W}$ and edge $(X',D') \xrightarrow{W} (X,D)$ in $\mathcal{C}$ over its right-associated, n-ary composition $W = (M_1 \bullet (\dots \bullet M_n))$, there is a path $e = (e_1,\dots,e_n)$ in $\mathcal{C}$ such that $(\mathsf{p}^{\mathcal{C}})^{\star}_{(X,D),(X',D')}(e) = p$.

Proof. By induction on n. $\square$

Lemma 5.22. *If $W: X' \to X$ is a closed-world action (i.e., has length 1), then for all D and D' both fibres of $\mathcal{C}((X',D'),(X,D))$ and $\mathcal{S}((X',D'),(X,D))$ over W are equal.*

Proof. By construction. $\square$

Corollary 5.23. *For all closed-world paths as in* (5.2)*, and* $(X,D),(X',D') \in \mathcal{S}$*, we have*

$$((\mathsf{p}^{\mathcal{S}})^{\star}_{(X',D'),(X,D)})^{-1}(p) \cong (\mathsf{p}^{\mathcal{C}}_{(X',D'),(X,D)})^{-1}(P),$$

for any special isomorphism $P \cong (M_1 \bullet (\ldots \bullet M_n))$.

Proof. Consider any special isomorphism $\alpha: P \Rrightarrow (M_1 \bullet (\ldots \bullet M_n))$. We have

$$\begin{array}{ll} ((\mathsf{p}^{\mathcal{S}})^{\star}_{(X',D'),(X,D)})^{-1}(p) & \\ \cong ((\mathsf{p}^{\mathcal{C}})^{\star}_{(X',D'),(X,D)})^{-1}(p) & \text{(by Lemma 5.22)} \\ \cong (\mathsf{p}^{\mathcal{C}}_{(X',D'),(X,D)})^{-1}(M_1 \bullet (\ldots \bullet M_n)) & \text{(by Lemma 5.14 and Corollary 5.21)} \\ \cong (\mathsf{p}^{\mathcal{C}}_{(X',D'),(X,D)})^{-1}(P) & \text{(by Lemma 5.18).} \end{array}$$

□

As a corollary, we get that the identity relation on objects is a strong bisimulation between $\mathsf{fc}(\mathcal{S})$ and $\mathcal{C}$:

Corollary 5.24. *For all* $w \in \Sigma^{\star}(\star,\star)$ *and* $(X,D),(X',D') \in \mathcal{S}$*, we have* $(X,D) \stackrel{w}{\Longleftarrow} (X',D')$ *in* $\mathcal{S}$ *iff* $(X,D) \xleftarrow{\widetilde{w}} (X',D')$ *in* $\mathcal{C}$.

The last statement is slightly subtle, in that $(X,D) \xleftarrow{\widetilde{w}} (X',D')$ denotes a single edge in $\mathcal{C}$, lying over the composite $\widetilde{w}$ in $\mathsf{fc}(\Sigma)$.

Proof. If $(X,D) \stackrel{w}{\Longleftarrow} (X',D')$ in $\mathcal{S}$, then there exists $p \in \mathbb{W}^{\star}$ such that $\mathsf{fc}(\mathsf{p}^{\mathbb{W}})(p) = \widetilde{w}$ and there is a path $e: (X',D') \to (X,D)$ over p in $\mathcal{S}$. Let $W: X' \multimap X$ denote the composition of p. By Corollary 5.23, we get an edge $(X',D') \to (X,D)$ over W in $\mathcal{C}$. So since $\mathsf{p}^{\mathbb{W}}(W) = \widetilde{w}$, this gives us the expected transition.

Conversely, if $(X,D) \xleftarrow{\widetilde{w}} (X',D')$ in $\mathcal{C}$, then let $W: X' \multimap X$ denote the corresponding edge in $\mathbb{D}^{\mathbb{W}}_v$. In particular, we have $\mathsf{p}^{\mathbb{W}}(W) = \widetilde{w}$. Decomposing W as some path p in $\mathbb{W}$, we obtain by Corollary 5.23 a transition sequence $(X,D) \xLeftarrow{(\mathsf{p}^{\mathbb{W}})^{\star}(p)} (X',D')$ in $\mathcal{S}$. But $\widetilde{(\mathsf{p}^{\mathbb{W}})^{\star}(p)} = \mathsf{p}^{\mathbb{W}}(W) = \widetilde{w}$, as desired. □

As promised, we obtain:

Corollary 5.25. *We have* $\bot^{\mathcal{S}} = \bot^{\mathcal{C}}$.

Finally, here is the long awaited

Lemma 5.26. *We have* $D \in \perp\!\!\!\perp_X$ *iff* $(X,D) \in \bot^{\mathcal{C}}$.

Proof. Assume $D \in \perp\!\!\!\perp_X$, and consider any $(X,D) \leftarrow (X',D')$. The latter is witnessed by some unsuccessful, closed-world trace $W: X' \multimap X$ such that $D' \cong (D \cdot W)_{|\psi_W(\sigma)}$ for a certain $\sigma \in \overline{D}(W)$ and some isomorphism $h: (D \cdot W)_{|\psi_W(\sigma)} \Rrightarrow D'$.

By hypothesis, σ admits an extension $\sigma' \in \overline{D}(W \bullet W')$ for some successful $W': X'' \multimap X'$. Letting $D'' = (D \cdot W \bullet W')_{|\psi_{W \bullet W'}(\sigma')}$, we have

$$D'' \cong (((D \cdot W)_{|\psi_W(\sigma)}) \cdot W')_{|\psi_{W'}(\sigma')} \cong (D' \cdot W')_{|\psi_{W'}(\overline{h}_{W'}(\sigma'))},$$

and hence $(X',D') \xleftarrow{\heartsuit^n} (X'',D'')$ for some $n > 0$. This shows that $(X,D) \in \bot^{\mathcal{C}}$.

Conversely, assume $(X,D) \in \bot^{\mathcal{C}}$ and consider any unsuccessful, closed-world trace $W: X' \multimap X$ and state $\sigma \in \overline{D}(W)$. Letting $D' = (D \cdot W)_{|\psi_W(\sigma)}$, we have $(X,D) \leftarrow (X',D')$.

By hypothesis, we find some transition $(X', D') \xleftarrow{\heartsuit} (X'', D'')$, witnessed by some successful $W': X'' \rightarrow\!\!\!\!\bullet\!\!\rightarrow X'$. Hence, $D'' \cong (D' \cdot W')_{|\psi_{W'}(\sigma')}$ for a certain $\sigma' \in \overline{D'}(W')$. By definition of D', σ' is a state in $\overline{D}(W \bullet W')$ such that $\sigma' \cdot u = \sigma$, where $u: W \to (W \bullet W')$ is (W', id). This gives the desired successful extension of σ, which shows that $D \in \perp\!\!\!\perp_X$. ☐

5.2. **A further labelled transition system for behaviours.** In the previous section, we have characterised semantic fair testing equivalence using the graph with testing $\mathcal{S}$. We now define a further graph with testing, $\mathcal{M}$, which will help us to bridge the gap between behaviours and π-calculus configurations. Indeed, we define a surjective morphism $\mathsf{m}: \mathcal{S} \to \mathcal{M}$ over Σ, and we then prove that m is 'fully abstract' for fair testing equivalence.

Recall from Definition 2.15 that $-^{\odot}$ denotes the finite multiset monad on sets.

Definition 5.27. Let the set $\mathcal{M}_0$ of *mixed behaviours* be

$$\sum_{\gamma \in \mathcal{P}_f(\mathbb{N})} \Big(\sum_{n \in \mathbb{N}} \mathsf{D}_{[n]} \times \gamma^n \Big)^{\odot}.$$

Notation 5.28. Similarly to the notation for configurations, i.e., we denote

$$(\gamma, [(n_1, D_1, \sigma_1), \ldots, (n_p, D_p, \sigma_p)])$$

by

$$\langle \gamma \| D_1[\sigma_1], \ldots, D_p[\sigma_p] \rangle.$$

Let $\mathsf{m}: \mathrm{ob}(\mathcal{S}) \to \mathcal{M}_0$ map any (X, D) to the mixed behaviour

$$\langle X(\star) \| [(D \cdot x)[\sigma_x] \mid (n, x) \in \mathrm{Agents}(X)] \rangle,$$

where σ_x is the map $n \xrightarrow{[\ulcorner x \cdot s_i \urcorner]_{i \in n}} X(\star)$.

Proposition 5.29. *Consider any $x, x', y, y' \in \mathrm{ob}(\mathcal{S})$ such that $\mathsf{m}(x) = \mathsf{m}(x')$ and $\mathsf{m}(y) = \mathsf{m}(y')$. Then we have $\mathcal{S}(y, x) \cong \mathcal{S}(y', x')$ over Σ.*

Proof. Let $x = (X, D)$, $x' = (X', D')$, $y = (Y, E)$, $y' = (Y', E')$. Under the given hypotheses, we may construct isomorphisms $h: X \xrightarrow{\sim} X'$ and $k: Y \xrightarrow{\sim} Y'$ such that $D' \cdot h = D$ and $E' \cdot k = E$. It is then easy to see that $x \xleftarrow{W} y$ iff $x' \xleftarrow{W'} y'$, for all closed-world actions W, with the obvious W' and isomorphism α in

$$\begin{tikzcd} Y \arrow[r, "k"] \arrow[d, "W"'] \arrow[dr, phantom, "\xRightarrow{\alpha}"] & Y' \arrow[d, "W'"] \\ X \arrow[r, "h"'] & X', \end{tikzcd}$$

which entails the result. ☐

Proposition 5.30. *The map* $\mathsf{m}\colon \mathcal{S} \to \mathcal{M}$ *is surjective. Let* a *denote any section of* m.

Recalling Notation 5.17, we put:

Definition 5.31. Let $\mathcal{M}$ have $\mathcal{M}_0$ as vertex set, with

$$\mathcal{M}(M', M) = \mathrm{Im}(\mathsf{p}^{\mathcal{S}}_{\mathsf{a}(M'),\mathsf{a}(M)}).$$

Thus, we take as edges $M' \to M$ the set of edges σ in Σ such that $\mathsf{a}(M) \,{}_{\Sigma}\!\xleftarrow{\sigma} \mathsf{a}(M')$. Otherwise said:

Proposition 5.32. *We have*

$$M \,{}_{\Sigma}\!\xleftarrow{\sigma} M' \text{ iff } \mathsf{a}(M) \,{}_{\Sigma}\!\xleftarrow{\sigma} \mathsf{a}(M')$$

and

$$\mathsf{m}(X, D) \,{}_{\Sigma}\!\xleftarrow{\sigma} \mathsf{m}(X', D') \text{ iff } (X, D) \,{}_{\Sigma}\!\xleftarrow{\sigma} (X', D').$$

This directly makes $\mathcal{M}$ into a graph over Σ, which by Proposition 5.29 does not depend on the choice of a.

Definition 5.33. Let $\mathsf{p}^{\mathcal{M}}\colon \mathcal{M} \to \Sigma$ denote the projection.

Recalling Notation 2.17, we have:

Lemma 5.34. *Edges of* $\mathcal{M}$ *may be inductively defined by the rules in Figure 8.*

$$\frac{i \in |D \cdot \pi^l_n| \qquad j \in |D \cdot \pi^r_n|}{\langle \gamma \| D[\sigma] \rangle \xleftarrow{id} \langle \gamma \| (D \cdot \pi^l_n)_{|i}[\sigma], (D \cdot \pi^r_n)_{|j}[\sigma] \rangle} \qquad \frac{i \in |D \cdot \tau_n|}{\langle \gamma \| D[\sigma] \rangle \xleftarrow{id} \langle \gamma \| (D \cdot \tau_n)_{|i}[\sigma] \rangle}$$

$$\frac{i \in |D \cdot \heartsuit_n|}{\langle \gamma \| D[\sigma] \rangle \xleftarrow{\heartsuit} \langle \gamma \| (D \cdot \heartsuit_n)_{|i}[\sigma] \rangle} \qquad \frac{i \in |D \cdot \nu_n| \qquad a \notin \gamma}{\langle \gamma \| D[\sigma] \rangle \xleftarrow{id} \langle \gamma, a \| (D \cdot \nu_n)_{|i}[n+1 \xrightarrow{\sigma + \ulcorner a \urcorner} \gamma, a] \rangle}$$

$$\frac{i \in |D_1 \cdot \iota_{n_1,a_1}| \qquad j \in |D_2 \cdot o_{n_2,a_2,b_2}| \qquad \sigma_1(a_1) = \sigma_2(a_2)}{\langle \gamma \| D_1[\sigma_1], D_2[\sigma_2] \rangle \xleftarrow{id} \langle \gamma \| (D_1 \cdot \iota_{n_1,a_1})_{|i}[n_1 + 1 \xrightarrow{[\sigma_1, \ulcorner \sigma_2(b_2) \urcorner]} \gamma], (D_2 \cdot o_{n_2,a_2,b_2})_{|j}[\sigma_2] \rangle}$$

$$\frac{\langle \gamma_1 \| S_1 \rangle \xleftarrow{\alpha} \langle \gamma_2 \| S_2 \rangle}{\langle \gamma_1 \| S \cup S_1 \rangle \xleftarrow{\alpha} \langle \gamma_2 \| S[\gamma_1 \subseteq \gamma_2] \cup S_2 \rangle} \qquad \frac{}{\langle \gamma \| S \rangle \xleftarrow{id} \langle \gamma \| S \rangle}$$

Figure 8: Inductive characterisation of transitions in $\mathcal{M}$

Proof. An easy case analysis. Let $\leftharpoondown$ denote transitions as in Figure 8. E.g., consider M and M' and let $\mathsf{a}(M) = (X, D)$ and $\mathsf{a}(M') = (X', D')$. If $(X, D) \,{}_{\Sigma}\!\xleftarrow{\heartsuit} (X', D')$, then we have an iso $h\colon X' \Rrightarrow X$, and there is some agent (n_0, x'_0) in X' and i such that $D' \cdot x'_0 = (D \cdot h(x_0) \cdot \heartsuit_{n_0})_{|i}$. Furthermore, for all $(n, x') \neq (n_0, x'_0)$ in $\mathrm{Agents}(X')$, we have $D' \cdot x' = D \cdot h(x')$. So, letting $x = h(x')$ for all such x', we do have

$$M = \langle \gamma \| (D \cdot x_0)[\sigma_{x_0}] :: [(D \cdot x)[\sigma_x] \mid x \neq x_0] \rangle \overset{\heartsuit}{\leftharpoondown} \langle \gamma \| (D' \cdot x'_0)[\sigma_{x'_0}] :: [(D \cdot x)[\sigma_x] \mid x \neq x_0] \rangle = M'.$$

∎

Mimicking Definition 2.18, we put:

Definition 5.35. For any $\langle \gamma \| S \rangle, \langle \gamma' \| S' \rangle \in \mathcal{M}$, let $\langle \gamma \| S \rangle @ \langle \gamma' \| S' \rangle$ denote $\langle \gamma \| S \cup S' \rangle$ if $\gamma = \gamma'$ and be undefined otherwise. Let furthermore $\epsilon_\gamma = \langle \gamma \| \rangle$.

By Lemma 2.5, we have:

Proposition 5.36. *The morphism* $\mathsf{fc}(\mathsf{p}^{\mathcal{M}}) : \mathsf{fc}(\mathcal{M}) \to \mathsf{fc}(\Sigma)$*, with* @ *as testing relation, forms a free graph with testing.*

Proposition 5.37. *For all* $x, y \in \mathcal{S}$*,* $x \sim_f y$ *iff* $\mathsf{m}(x) \sim_f \mathsf{m}(y)$*. Conversely, for all* $M, N \in \mathcal{M}$*,* $M \sim_f N$ *iff* $\mathsf{a}(M) \sim_f \mathsf{a}(N)$*.*

Proof. The graph of m defines a weakly fair relation $\mathrm{ob}(\mathsf{fc}(\mathcal{S})) \longrightarrow\!\!\!\!\mapsto \mathrm{ob}(\mathsf{fc}(\mathcal{M}))$. Indeed, the first two requirements are trivially satisfied. The third one follows from the fact that any surjective-on-objects, full morphism over $\mathsf{fc}(\Sigma)$ is a bisimulation. Finally, in the situation of Definition 5.7, we obviously have $\mathsf{m}(Z, D) = \mathsf{m}(X_1, D_1) @ \mathsf{m}(X_2, D_2)$ as desired. $\square$

5.3. **Interpretation of π and intensional full abstraction.** We at last prove our main result. We first define a relation $\bowtie : \mathrm{Conf} \longrightarrow\!\!\!\!\mapsto \mathcal{M}$ over Σ which

- relates any process to its translation, and
- is surjective, i.e., relates any mixed behaviour to some process.

We will then show that this relation is a weak bisimulation over Σ. We thus need maps from processes to behaviours and back.

To start with, we define a family of maps in

$$\prod_{\gamma \in \mathcal{P}_f(\mathbb{N})} \mathsf{D}_{|\gamma|}^{Pi_\gamma \times \mathrm{Bij}(\gamma, |\gamma|)},$$

where $\mathrm{Bij}(A, B)$ denotes the set of all bijections $A \xrightarrow{\sim} B$. For any such γ, P, and h in the domain, we denote the result by $[\![P]\!]_h$, or $[\![P]\!]_{\gamma,h}$ when needed. Letting $n = |\gamma|$, it is coinductively defined by

$$\begin{array}{rcl}
[\![\sum_i P_i]\!]_h & = & \boxplus_i [\![P_i]\!]_h \\
[\![P \mid Q]\!]_h & = & \left\langle \begin{array}{ccc} \pi_n^l & \mapsto & [\![P]\!]_h \\ \pi_n^r & \mapsto & [\![Q]\!]_h \end{array} \right\rangle \\
[\![\nu b.P]\!]_h & = & \left\langle \begin{array}{ccc} \nu_n & \mapsto & [\![P]\!]_{h'} \end{array} \right\rangle \\
[\![\heartsuit.P]\!]_h & = & \left\langle \begin{array}{ccc} \heartsuit_n & \mapsto & [\![P]\!]_h \end{array} \right\rangle \\
[\![\tau.P]\!]_h & = & \left\langle \begin{array}{ccc} \tau_n & \mapsto & [\![P]\!]_h \end{array} \right\rangle \\
[\![a(b).P]\!]_h & = & \left\langle \begin{array}{ccc} \iota_{n,h(a)} & \mapsto & [\![P]\!]_{h'} \end{array} \right\rangle \\
[\![\bar{a}\langle b \rangle.P]\!]_h & = & \left\langle \begin{array}{ccc} o_{n,h(a),h(b)} & \mapsto & [\![P]\!]_h \end{array} \right\rangle
\end{array}$$

where

- in any list $\langle b_1 \mapsto B_1, \ldots, b_m \mapsto B_m \rangle$, all unmentioned basic actions are meant to be mapped to the empty behaviour;
- the sum $\boxplus_i D_i$ of definite behaviours D_i is the definite behaviour determined by $(\boxplus_i D_i) \cdot b = \sum_i (D_i \cdot b)$, for all basic actions $b : [n'] \longrightarrow\!\!\!\!\bullet\, [n]$;
- and $h' : \gamma, b \xrightarrow{\sim} n + 1$ maps any $a \in \gamma$ to $h(a)$, and b to $n + 1$.

In the other direction, we coinductively define $\zeta \in \prod_n Pi_n^{\mathsf{D}[n]}$ by:

$$\zeta(n \vdash_{\mathsf{D}} D) \quad = \quad \left(\begin{array}{ll} & \sum_{i \in |D \cdot \pi_n^l|, j \in |D \cdot \pi_n^r|} \tau.(\zeta(n \vdash_{\mathsf{D}} (D \cdot \pi_n^l)_{|i}) \mid \zeta(n \vdash_{\mathsf{D}} (D \cdot \pi_n^r)_{|j})) \\ + & \sum_{i \in |D \cdot \tau_n|} \tau.\zeta(n \vdash_{\mathsf{D}} (D \cdot \tau_n)_{|i}) \\ + & \sum_{i \in |D \cdot \heartsuit_n|} \heartsuit.\zeta(n \vdash_{\mathsf{D}} (D \cdot \heartsuit_n)_{|i}) \\ + & \sum_{i \in |D \cdot \nu_n|} \nu(n+1).\zeta(n+1 \vdash_{\mathsf{D}} (D \cdot \nu_n)_{|i}) \\ + & \sum_{a \in n, i \in |D \cdot \iota_{n,a}|} a(n+1).\zeta(n+1 \vdash_{\mathsf{D}} (D \cdot \iota_{n,a})_{|i}) \\ + & \sum_{a,b \in n, i \in |D \cdot o_{n,a,b}|} \bar{a}\langle b \rangle.\zeta(n \vdash_{\mathsf{D}} (D \cdot o_{n,a,b})_{|i}) \end{array}\right).$$

We first observe that for any D and $i \in |D \cdot \pi_n^l|$ and $j \in |D \cdot \pi_n^r|$,

- on the one hand D has a silent transition to $\partial_{i,j} D$, the behaviour on $[n] \mid [n]$ such that $(\partial_{i,j} D) \cdot x_1 = (D \cdot \pi_n^l)_{|i}$ and $(\partial_{i,j} D) \cdot x_2 = (D \cdot \pi_n^r)_{|j}$ (where x_1 and x_2 denote the two players of $[n] \mid [n]$);
- on the other hand $\zeta(D)$ has a silent transition to
$$\zeta'_{i,j}(D) = (\zeta((D \cdot \pi_n^l)_{|i}) \mid \zeta((D \cdot \pi_n^r)_{|j})),$$
which then has a further silent transition to the two-process configuration consisting of $\zeta((D \cdot \pi_n^l)_{|i})$ and $\zeta((D \cdot \pi_n^r)_{|j})$.

Thus, when we will try below to relate D and $\zeta(D)$, the transition $\zeta(D) \xleftarrow{id} \zeta'_{i,j}(D)$ has to be matched by the former transition $D \xleftarrow{id} \partial_{i,j} D$. So our relation $\bowtie$ should somehow include pairs $(\zeta'_{i,j}(D), \partial_{i,j} D)$.

Before moving on to the definition, we extend $[\![-]\!]$ and ζ to maps

$$\begin{tikzcd} \mathrm{Conf} \arrow[r, bend left, "\mathsf{T}"] & \mathcal{M}. \arrow[l, bend left, "\mathsf{Z}"] \end{tikzcd}$$

First, for any γ, there is a canonical bijection $h_\gamma \colon \gamma \xrightarrow{\sim} |\gamma|$ mapping each $a \in \gamma$ to its position in the ordering induced by the one on natural numbers.

Definition 5.38. Let $\mathsf{T} \colon \mathrm{ob}(\mathrm{Conf}) \to \mathrm{ob}(\mathcal{M})$ be $\sum_\gamma (\mathsf{T}^0_\gamma)^{\odot}$, where

$$\mathsf{T}^0_\gamma(P) = [\![P]\!]_{h_\gamma}[h_\gamma^{-1}].$$

Conversely:

Definition 5.39. Let $\mathsf{Z} \colon \mathrm{ob}(\mathcal{M}) \to \mathrm{ob}(\mathrm{Conf})$ to be $\sum_\gamma (\mathsf{Z}^0_\gamma)^{\odot}$, where

$$\mathsf{Z}^0_\gamma(D[\sigma]) = \zeta(D)[\sigma].$$

We now come to our relation $\bowtie$, which by construction contains the graphs of T and Z:

Definition 5.40. Let the relation $\bowtie \colon \mathrm{Conf} \mathrel{+\!\!\!\to} \mathcal{M}$ over Σ be defined inductively by the rules in Figure 9.

(In the last rule, ϵ_γ is understood in Conf on the left, and in $\mathcal{M}$ on the right.)

Lemma 5.41. *We have $C \bowtie \mathsf{T}(C)$ for all C and $\mathsf{Z}(M) \bowtie M$ for all M, and so $\bowtie$ is total and surjective.*

Proof. By construction. □

$$\frac{\gamma' \vdash P \qquad h{:}\,\gamma' \mathrel{\dot{\Rightarrow}} n \qquad \sigma{:}\,n \to \gamma}{P[\sigma \circ h] \bowtie [\![P]\!]_h[\sigma]} \qquad \frac{n \vdash_{\mathsf{D}} D \qquad \sigma{:}\,n \to \gamma}{\zeta(D)[\sigma] \bowtie D[\sigma]}$$

$$\frac{n \vdash_{\mathsf{D}} D^1, D^2 \qquad \sigma{:}\,n \to \gamma}{(\zeta(D^1) \mid \zeta(D^2))[\sigma] \bowtie D^1[\sigma], D^2[\sigma]} \qquad \frac{C \bowtie M \qquad D \bowtie N}{C@D \bowtie M@N} \qquad \frac{}{\epsilon_\gamma \bowtie \epsilon_\gamma} \cdot$$

Figure 9: The relation $\bowtie$

Lemma 5.42. *The relation $\bowtie$ is a weak bisimulation over Σ.*

Proof. First, we observe that $\bowtie$ may equivalently be defined by first letting $\bowtie_0$ be generated by all rules except the last two, and then adding the rule

$$\frac{C_1 \bowtie_0 M_1 \qquad \ldots \qquad C_n \bowtie_0 M_n}{C_1@\ldots@C_n \bowtie M_1@\ldots@M_n}\ (n \in \mathbb{N}) \cdot$$

The advantage of this presentation is that proofs of $C \bowtie M$ all have depth at most 1.

The rest is then a tedious case analysis, which we defer to Appendix A to avoid disrupting the flow of the paper. In summary, we easily observe that the 'forwards' clause of weak bisimulation is satisfied, the only subtlety being that heating $(\zeta(D^1) \mid \zeta(D^2))[\sigma]$ should be matched by the identity edge on the corresponding behaviour. Furthermore, the 'backwards' clause is also easily satisfied, the only subtlety being that if the considered behaviour performs a transition involving one or several $[D^1[\sigma], D^2[\sigma]]$'s, related on the left to $(\zeta(D^1) | \zeta(D^2))[\sigma]$'s, then all of the latter first have to heat to $[\zeta(D^1)[\sigma], \zeta(D^2)[\sigma]]$, and only then perform the matching transition. □

We now come to one of our main results:

Theorem 5.43. *For all C_1, C_2, M_1, M_2, if $C_1 \bowtie M_1$ and $C_2 \bowtie M_2$, then $C_1 \sim_f C_2$ iff $M_1 \sim_f M_2$.*

Proof. The relation $\bowtie$ is weakly fair, the only difficult points being proved by Lemmas 5.41 and 5.42 above. We thus conclude by Corollary 2.12. □

Considering processes as singleton configurations, we get a notion of fair testing equivalence for the former, and a translation map $[\![\gamma \vdash P]\!] = \mathsf{T}\langle \gamma \| P \rangle$.

Corollary 5.44. *For all processes P_1 and P_2 over γ, $P_1 \sim_f P_2$ iff $[\![P_1]\!] \sim_f [\![P_2]\!]$. Furthermore, for all behaviours $B \in \mathsf{B}_n$, there exists $P \in Pi_n$ such that $[\![P]\!] \sim_f B$.*

Proof. The first point is a trivial consequence of the theorem. For the second, we observe that any $B = \sum_i D_i$ over $[n]$ is fair testing equivalent to the definite behaviour

$$\langle \tau_n \mapsto \sum_i D_i \rangle,$$

except if $B = \varnothing$, in which case $B \sim_f \langle \heartsuit_n \mapsto \langle\rangle \rangle$. Thus, we may restrict attention to definite behaviours. But then $D \sim_f [\![\zeta(D)]\!]$ because for all M over n, we have

$D@M \in \bot$ iff
$\mathsf{Z}(D)@\mathsf{Z}(M) \in \bot$ (because $D@M \bowtie \mathsf{Z}(D)@\mathsf{Z}(M)$ and by Lemma 5.42) iff
$\mathsf{T}(\mathsf{Z}(D))@M \in \bot$ (because $\mathsf{Z}(D)@\mathsf{Z}(M) \bowtie \mathsf{T}(\mathsf{Z}(D))@M$ and by Lemma 5.42), i.e.,
$[\![\zeta(D)]\!]@M \in \bot$,

as desired. □

5.4. **Generalisation.** We now show that our main results generalise beyond fair testing equivalence. Indeed, let us put:

Definition 5.45. A *pole* is a property of states over $\mathsf{fc}(\Sigma)$ which is stable under strong bisimilarity.

There is a slight size issue in this definition, as it quantifies over elements of all graphs over $\mathsf{fc}(\Sigma)$. The reader may understand this using whatever fix she prefers, e.g., using a universe or some modal logic.

Example 5.46. Consider any $x \in G$ over $\mathsf{fc}(\Sigma)$. We have

- x is in the pole for fair testing equivalence iff for all $x \leftarrow x'$ there exists $x' \overset{\heartsuit}{\leftarrow} x''$;
- x is in the pole for may testing equivalence iff there exists $x \overset{\heartsuit}{\leftarrow} x'$.

Must testing equivalence is less easy to capture, for reasons explained in [36]. Here is an exotic, yet perhaps relevant pole: x is in it iff for all finite, not-necessarily silent transition sequences $x \leftarrow^{\star} x'$, there exists $x' \overset{\heartsuit}{\leftarrow} x''$. In other words, x never loses the ability to tick. The induced equivalence is clearly finer than fair testing equivalence, but we leave open the question of whether it is strictly so.

Definition 5.47. For any such pole $\perp$, let $\sim_{\perp}$ denote the testing equivalence induced by replacing $\perp^{G}$ by $\perp$ in the definition of fair testing equivalence (Definition 2.3).

Semantic testing equivalence may then be taken to be testing equivalence on $\mathcal{C}$ (Definition 5.13), and we get the exact analogues of Theorem 5.43 and Corollary 5.44 (without changing the model in any way).

6. Conclusion and future work

We have described our playground for π and the induced sheaf model, which we have proved intensionally fully abstract for a wide range of testing equivalences.

Regarding future work: our proof that traces form a playground uses a new technique based on factorisation systems. We are currently designing a general setting where this technique applies, hopefully encompassing some aspects of graph rewriting, and in particular Petri net unfolding [3]. We also consider applying our notion of trace to error diagnostics [27] or efficient machine representation of reversible π-calculus processes [16]. Longer-term directions include applying the approach to more complex calculi, e.g., calculi with passivation [45] or functional calculi, and eventually consider some full-fledged functional language with concurrency primitives. Finally, deriving the complex notion of trace evoked in Section 1.5 from the one exposed here is akin to deriving LTSs from reduction rules [43, 61]. Since the issue still seems easier on traces than on a full operational semantics specification, this might be worthwile to investigate further. In the same vein, the emphasis we put on traces suggests that we might be able to deduce properties of type systems (soundness, progress, etc) or compilers (correctness) from corresponding properties on traces.

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Appendix A. Proof of Lemma 5.42

One wants to check two properties:

(LH) for all transitions $C' \xrightarrow{a}_A C$ with $C \bowtie M$, there exists $M' \xrightarrow{a}_A M$ with $C' \bowtie M'$;

(RH) for all transitions $M \; {}_A\!\xleftarrow{a} M'$ with $C \bowtie M$, there exists $C \; {}_A\!\xLeftarrow{a} C'$ with $C' \bowtie M'$.

The attentive reader will have noticed that (LH) imposes y to answer with a single transition. This means we actually prove that $\bowtie$ is an *expansion* [64, Chapter 6]. Any expansion being in particular a weak bisimulation, this suffices.

Notation A.1. We sometimes cast processes P (resp. pairs $D[\sigma]$) over γ into configurations $\langle\gamma\|P\rangle$ (resp. mixed behaviours $\langle\gamma\|D[\sigma]\rangle$). We proceed similarly for multisets of processes.

We start by proving (LH) for all cases, before proving that (RH) holds as well.

Synchro, (LH). We begin by the case of a synchronisation, i.e., when one has a transition

$$C = \langle\gamma\|a(b).P +_{k_1} R_1, \bar{a}\langle c\rangle.Q +_{k_2} R_2\rangle@C_0 \xleftarrow{id} \langle\gamma\|P[b \mapsto c], Q\rangle@C_0 = C'.$$

We want to show that there exists a transition $M \xleftarrow{id} M'$ with $C' \bowtie M'$.

We write $P_1 = a(b).P +_{k_1} R_1$ and $P_2 = \bar{a}\langle c\rangle.Q +_{k_2} R_2$. Neither of them are of the form $(\cdot\,|\,\cdot)$ so they can only be related to mixed behaviours using the first two rules. Therefore, four sub-cases should be considered, as detailed in Figure 10. If we are in case i_1 for P_1 and i_2 for P_2, then we have two mixed behaviours $D_1[\sigma_1]$ and $D_2[\sigma_2]$ such that $n_i \vdash_{\mathsf{D}} D_i$, $\sigma_i \colon n_i \to \gamma$, and $P_i \bowtie D_i[\sigma_i]$ for $i = 1, 2$, plus $M = \langle\gamma\|D_1[\sigma_1], D_2[\sigma_2]\rangle@M_0$ with $C_0 \bowtie M_0$.

- **Case 1 for both P_1 and P_2.** We have $M = [\![P_1']\!]_{h_1}[\sigma_1]@[\![P_2']\!]_{h_2}[\sigma_2]@M_0$, and there is a transition
$$M \xleftarrow{id} [\![P']\!]_{h_1'}[\sigma_1']@[\![Q']\!]_{h_2}[\sigma_2]@M_0,$$
where h_1' is $\gamma_1', b' \xrightarrow{h_1+!} n_1 + 1$ and σ_1' is $n_1 + 1 \xrightarrow{[\sigma_1, c]} \gamma$.
Since $\sigma_1' \circ h_1'$ equals $\gamma_1' \xrightarrow{h_1+!} n_1 + 1 \xrightarrow{\sigma_1 + \ulcorner b \urcorner} (\gamma, b) \xrightarrow{b \mapsto c} \gamma$, we have that $P'[\sigma_1' \circ h_1'] = P'[(\sigma_1 + \ulcorner b \urcorner) \circ (h_1 + !)][b \mapsto c] = P[b \mapsto c]$, and therefore $P[b \mapsto c] \bowtie [\![P']\!]_{h_1'}[\sigma_1']$. Moreover, it is clear that $Q = Q'[\sigma_2 \circ h_2] \bowtie [\![Q']\!]_{h_2}[\sigma_2]$, and finally $[P[b \mapsto c], Q]@C_0 \bowtie M'$.
- **Case 1 for P_1, Case 2 for P_2.** We have a transition
$$M \xleftarrow{id} [\![P']\!]_{h_1'}[\sigma_1']@(D_2 \cdot o_{n_2,a_2,c_2})_{|j}[\sigma_2]@M_0 = M',$$
where h_1' is $\gamma_1', b' \xrightarrow{h_1+!} n_1 + 1$ and σ_1' is $n_1 + 1 \xrightarrow{[\sigma_1, c]} \gamma$.
As is the previous case, one can check that $P[b \mapsto c] \bowtie [\![P']\!]_{h_1'}[\sigma_1']$. Moreover, $Q \bowtie (D_2 \cdot o_{n_2,a_2,c_2})_{|j}[\sigma_2]$ and therefore $[P[b \mapsto c], Q]@C_0 \bowtie M'$.
- **Case 2 for P_1, Case 1 for P_2.** We have a transition
$$M \xleftarrow{id} (D_1 \cdot \iota_{n_1,a_1})_{|i}[n_1 + 1 \xrightarrow{[\sigma_1, \ulcorner c \urcorner]} \gamma]@[\![Q']\!]_{h_2'}[\sigma_2']@M_0 = M'.$$
As in the first case, we have $Q = Q'[\sigma_2 \circ h_2] \bowtie [\![Q']\!]_{h_2}[\sigma_2]$. Furthermore, since
$$\begin{aligned}&\zeta((D_1 \cdot \iota_{n_1,a_1})_{|i})[n_1 + 1 \xrightarrow{[\sigma_1, \ulcorner c \urcorner]} \gamma] \\ &\quad = \zeta((D_1 \cdot \iota_{n_1,a_1})_{|i})[n_1 + 1 \xrightarrow{\sigma_1 + \ulcorner b \urcorner} \gamma, b][b \mapsto c] \\ &\quad = P[b \mapsto c],\end{aligned}$$
we have that $[P[b \mapsto c], Q]@C_0 \bowtie M'$.

	Case 1	Case 2
P_1	There exist $\gamma'_1 \vdash P'_1 = a'_1(b').P' +_{k_1} R'_1$ and $h_1 : \gamma'_1 \Rrightarrow n_1$ such that $P_1 = P'_1[\sigma_1 \circ h_1] \quad \sigma_1(h_1(a'_1)) = a$ $P = P'[\gamma'_1, b' \xrightarrow{h_1+!} n_1 + 1 \xrightarrow{\sigma_1 + \ulcorner b \urcorner} \gamma, b]$ $D_1 = [\![P'_1]\!]_{h_1} \quad R_1 = R'_1[\sigma_1 \circ h_1].$	There exist $n_1 \Vdash_{\mathsf{D}} D_1$, $a_1 \in n_1$, and $i \in \lvert D_1 \cdot \iota_{n_1,a_1} \rvert$ such that $\sigma_1(a_1) = a \quad P_1 = \zeta(D_1)[\sigma_1]$ $P = \zeta((D_1 \cdot \iota_{n_1,a_1})_{\lvert i})[n_1 + 1 \xrightarrow{\sigma_1 + \ulcorner b \urcorner} \gamma, b].$
P_2	There exist $\gamma'_2 \vdash P'_2 = \bar{a'_2}\langle c' \rangle.Q' +_{k_2} R'_2$ and $h_2 : \gamma'_2 \Rrightarrow n_2$ such that $P_2 = P'_2[\sigma_2 \circ h_2]$ $Q = Q'[\gamma'_2, b \xrightarrow{h_2+1} n_2 + 1 \xrightarrow{\sigma_2 + \ulcorner b \urcorner} \gamma, b]$ $\sigma_2(h_2(a'_2)) = a \quad \sigma_2(h_2(c'_2)) = c$ $D_2 = [\![P'_2]\!]_{h_2} \quad R_2 = R'_2[\sigma_2 \circ h_2].$	There exist $n_2 \Vdash_{\mathsf{D}} D_2$, $a_2, c_2 \in n_2$, and $j \in \lvert D_2 \cdot o_{n_2,a_2,c_2} \rvert$ such that $\sigma_2(a_2) = a \quad \sigma_2(c_2) = c$ $P_2 = \zeta(D_2)[\sigma_2]$ $Q = \zeta((D_2 \cdot o_{n_2,a_2,c_2})_{\lvert j})[\sigma_2].$

Figure 10: Synchro, (LH) cases

- **Case 2 for both P_1 and P_2.** We have a transition
$$M \xleftarrow{id} (D_1 \cdot \iota_{n_1,a_1})_{\lvert i}[n_1 + 1 \xrightarrow{[\sigma_1, \ulcorner c \urcorner]} \gamma]@(D_2 \cdot o_{n_2,a_2,c_2})_{\lvert j}[\sigma_2]@M_0 = M'.$$
As in the previous cases, one can show that $[P[b \mapsto c], Q]@C_0 \bowtie M'$.

Heating, (LH). We now consider the case of heating, i.e., when one has a transition
$$C = (P \mid Q)@C_0 \xleftarrow{\tau} [P, Q]@C_0 = C'.$$
We want to show that there exists a transition $M \xleftarrow{\tau} M'$ with $[P, Q]@C_0 \bowtie M'$.

We now have to consider a few cases, depending on which rule is applied for $P \mid Q$ in the proof of $(P \mid Q)@C_0 \bowtie M$. Notice that $P \mid Q$ cannot be of the form $\zeta(D)[\sigma]$. We are therefore left with two cases, depending on whether the first or third rule is applied.

- If the first rule is applied, we find M_0, $\gamma' \vdash P' \mid Q'$, $h : \gamma' \Rrightarrow n$, and $\sigma : n \to \gamma$ such that $P \mid Q = (P' \mid Q')[\sigma \circ h]$ and $M = [\![P' \mid Q']\!]_h[\sigma]@M_0$, with $C_0 \bowtie M_0$. Letting $D = [\![P' \mid Q']\!]_h$, we notice that
$$D = \langle \pi^l_n \mapsto [\![P']\!]_h, \pi^r_n \mapsto [\![Q']\!]_h \rangle.$$
Thus, there is a transition
$$M \xleftarrow{\tau} M' = [[\![P']\!]_h[\sigma], [\![Q']\!]_h[\sigma]]@M_0$$
with $[P, Q]@C_0 \bowtie M'$, as desired.
- If the third rule is applied, we find M_0, $n \Vdash_{\mathsf{D}} D^1, D^2$ and $\sigma : n \to \gamma$ such that
$$P \mid Q = (\zeta(D^1) \mid \zeta(D^2))[\sigma]$$

and $M = [D^1[\sigma], D^2[\sigma]]@M_0$. We notice that $[P,Q] = [\zeta(D^1)[\sigma], \zeta(D^2)[\sigma]]$, so $[P,Q] \bowtie [D^1[\sigma], D^2[\sigma]]$, hence $[P,Q]@C_0 \bowtie M$. The identity transition $M \xleftarrow{\tau} M$ thus fits our needs.

Nu, (LH). We now consider the case of a ν rule, i.e., when one has a transition

$$C = \langle\gamma \| \nu a.P +_k R\rangle@C_0 \xleftarrow{\tau} \langle\gamma, a \| P\rangle@C_0[\gamma \subset \gamma, a] = C'.$$

We want to show that there exists a transition $M \xleftarrow{\tau} M'$ with $\langle\gamma, a\|P\rangle@C_0[\gamma \subset \gamma, a] \bowtie M'$. We notice that $\nu a.P +_k R$ cannot be obtained from the third rule. We thus consider two cases corresponding to the first and second rules.

- If the first rule is applied, there exist M_0, $\gamma' \vdash \nu a.P' +_k R'$, $h{:}\gamma' \doteqdot n$, and $\sigma{:}n \to \gamma$ such that $M = [\![\nu a.P' +_k R']\!]_h[\sigma]@M_0$ and
$$\nu a.P +_k R = (\nu a.P' +_k R')[\sigma \circ h].$$
We write $D = [\![\nu a.P' +_k R']\!]_h$ as $\langle \nu_n \mapsto [\![P']\!]_{h'}\rangle \boxplus [\![R']\!]_h$, where h' is $\gamma', a \xrightarrow{h+1} n+1$. Thus, there is a transition
$$M \xleftarrow{\tau} M' = \langle\gamma, a \| [\![P']\!]_{h'}[n+1 \xrightarrow{\sigma + \ulcorner a \urcorner} \gamma, a]\rangle@M_0[\gamma \subset \gamma, a]$$
and, modulo the fact that $C \bowtie M$ implies $C[\sigma] \bowtie M[\sigma]$, we have $P@C_0[\gamma \subset \gamma, a] \bowtie [\![P']\!]_{h'}[\sigma + \ulcorner a \urcorner]@M_0[\gamma \subset \gamma, a]$ since $P = P'[\gamma', a \xrightarrow{h'} n+1 \xrightarrow{\sigma+1} \gamma, a]$, as desired.
- If the second rule is applied, there exist M_0, $n \vdash_{\mathsf{D}} D$, and $\sigma{:}n \to \gamma$ such that $M = \zeta(D)[\sigma]@M_0$ and
$$\nu a.P +_k R = \zeta(D)[\sigma].$$
Thus, there exists $i \in |D \cdot \nu_n|$ such that $\zeta((D \cdot \nu_n)_{|i})[n+1 \xrightarrow{\sigma + \ulcorner a \urcorner} \gamma, a] = P$. There is thus a transition
$$M \xleftarrow{\tau} M' = \langle\gamma, a \| (D \cdot \nu_n)_{|i}[\sigma + \ulcorner a \urcorner]\rangle@M_0[\gamma \subset \gamma, a]$$
with $P@C_0[\gamma \subset \gamma, a] \bowtie M'$, as desired.

Tick and Tau, (LH). We now consider the cases $\heartsuit$ and τ, i.e., when one has a transition

$$C = (\xi.P +_k R)@C_0 \xleftarrow{\xi} P@C_0,$$

where $\xi \in \{\heartsuit, \tau\}$. We want to show that there exists a transition $M \xleftarrow{\xi} M'$ with $P@C_0 \bowtie M'$.

Once again, the third rule could not have been applied, and we are left with two cases corresponding to the first and second rules.

- If the first rule is applied, then we find M_0, $\gamma' \vdash \xi.P' +_k R'$, $h{:}\gamma' \doteqdot n$, and $\sigma{:}n \to \gamma$ such that $M = [\![\xi.P' +_k R']\!]_h[\sigma]@M_0$, $C_0 \bowtie M_0$, and
$$\xi.P +_k R = (\xi.P' +_k R')[\sigma \circ h].$$
We write $D = [\![\xi.P' +_k R']\!]_h$ as $\langle \xi_n \mapsto [\![P']\!]_h\rangle \boxplus [\![R']\!]_h$. Thus, there is a transition
$$M = D[\sigma]@M_0 \xleftarrow{\xi} [\![P']\!]_h[\sigma]@M_0 = M',$$
with $P = P'[\sigma \circ h]$ and thus $P@C_0 \bowtie M'$, as desired.

- If the second rule is applied, then we find M_0, $n \vdash_{\mathsf{D}} D$, and $\sigma: n \to \gamma$ such that $C_0 \bowtie M_0$, $M = D[\sigma]@M_0$, and
$$\xi.P +_k R = \zeta(D)[\sigma].$$
Then, there exists $i \in |D \cdot \xi_n|$ such that $\zeta((D \cdot \xi_n)_{|i})[\sigma] = P$. Hence, there is a transition
$$M \xleftarrow{\xi} (D \cdot \xi_n)_{|i}[\sigma]@M_0 = M'$$
with $P@C_0 \bowtie M'$, as desired.

We have thus proved that (LH) holds. We now proceed with the case analysis for (RH).

Synchro, (RH). We start, once again, with the case of the synchronisation, i.e., when M has the shape $\langle \gamma \| D_1[\sigma_1], D_2[\sigma_2] \rangle @M_0$ and we consider a silent transition to
$$M' = \langle \gamma \| (D_1 \cdot \iota_{n_1,a_1})_{|i}[n_1 + 1 \xrightarrow{[\sigma_1, \ulcorner \sigma_2(b_2) \urcorner]} \gamma], (D_2 \cdot o_{n_2,a_2,b_2})_{|j}[\sigma_2] \rangle @M_0,$$
where $n_i \vdash_{\mathsf{D}} D_i$ and $\sigma_i: n_i \to \gamma$ for $i = 1, 2$, $\sigma_1(a_1) = a = \sigma_2(a_2)$, and $\sigma_2(c_2) = c$. We want to show that there exists a transition $C \xleftarrow{id} C'$ with $C' \bowtie M'$. There are exactly 10 cases here. Firstly, we have the case where the third rule is applied to $D_1[\sigma_1], D_2[\sigma_2]$. Otherwise, one could have used each of the three rules for $\bowtie_0$ for each of D_1 and D_2, yielding nine cases. We start with the first case, and then treat the nine others.

- If the third rule is applied on $D_1[\sigma_1], D_2[\sigma_2]$ (hence $n_1 = n_2 = n$ and $\sigma_1 = \sigma_2 = \sigma$), then we find C_0 such that $C_0 \bowtie M_0$ and $C = (\zeta(D_1) \mid \zeta(D_2))[\sigma]@C_0$. Then, we have a transition
$$(\zeta(D_1) \mid \zeta(D_2))[\sigma]@C_0 \xleftarrow{id} [\zeta(D_1)[\sigma_1], \zeta(D_2)[\sigma_2]]@C_0.$$
Since the latter configuration is again related to M, this reduces to the case where the second rule is applied for both $D_1[\sigma_1]$ and $D_2[\sigma_2]$.
- If the third rule is applied for $D_1[\sigma_1]$ and any of the three rules is applied for $D_2[\sigma_2]$, then we find P, M_1, M_2, C_0, and $n_1 \vdash_{\mathsf{D}} D_3$ such that M_1 has length 1 if the third rule is also used for D_2 and 0 otherwise, $M_0 = D_3[\sigma_3]@M_1@M_2$, $P \bowtie D_2[\sigma_2]@M_1$, $C_0 \bowtie M_2$, and $C = (\zeta(D_1) \mid \zeta(D_3))[\sigma_1]@P@C_0$.
  Thus, there is a transition
$$(\zeta(D_1) \mid \zeta(D_3))[\sigma_1]@P@C_0 \xleftarrow{id} [\zeta(D_1)[\sigma_1], \zeta(D_3)[\sigma_1]]@P@C_0$$
which reduces this case to the one where the second rule is applied to D_1.
- If the third rule is applied for $D_2[\sigma_2]$ and any of the first two rules is applied for D_1, then we find M_1, P, C_0, and $n_2 \vdash_{\mathsf{D}} D_3$ such that $M_0 = D_3[\sigma_3]@M_1$, $P \bowtie D_1[\sigma_1]$, $C_0 \bowtie M_1$, and $C = (\zeta(D_2) \mid \zeta(D_3))[\sigma_2]@P@C_0$. Again, we are reduced to the case where the second rule is applied for D_2, using the transition
$$(\zeta(D_2) \mid \zeta(D_3))[\sigma_2]@P@C_0 \xleftarrow{id} [\zeta(D_2)[\sigma_2], \zeta(D_3)[\sigma_2]]@P@C_0.$$

In the remaining cases, we have $C = P_1^0@P_2^0@C_0$ with $P_i^0 \bowtie D_i[\sigma_i]$ for $i = 1, 2$, and $C_0 \bowtie M_0$. We thus have four cases, described in Figure 11 just as we did in Figure 10.

- If the second rule is applied for both D_1 and D_2, then there is a transition
$$C \xleftarrow{id} C' = \zeta((D_1 \cdot \iota_{n_1,a_1})_{|i})[[\sigma_1, \ulcorner c \urcorner]]@\zeta((D_2 \cdot o_{n_2,a_2,c_2})_{|j})[\sigma_2]@M_0$$
with $C' \bowtie M'$ as desired.

	Case 1	Case 2
D_1	There exist $\gamma_1' \vdash P_1 = a_1'(b).P_1' +_{k_1} R_1$ and $h_1 \colon \gamma_1' \Rrightarrow n_1$ such that $P_1^0 = P_1[\sigma_1 \circ h_1] \qquad h_1(a_1') = a_1$ $D_1 = [\![P_1]\!]_{h_1} \qquad i \in \lvert D_1 \cdot \iota_{n_1,a_1} \rvert$ $(D_1 \cdot \iota_{n_1,a_1})_{\lvert i} = [\![P_1']\!]_{h_1'}$ where h_1' is $\gamma_1', b \xrightarrow{h_1+1} n_1 + 1$.	$P_1^0 = \zeta(D_1)[\sigma_1]$.
P_2	There exist $\gamma_2' \vdash P_2 = a_2'\langle c'\rangle.P_2' +_{k_2} R_2$ and $h_2 \colon \gamma_2' \Rrightarrow n_2$ such that $P_2^0 = P_2[\sigma_2 \circ h_2] \qquad h_2(a_2') = a_2$ $h_2(c_2') = c_2 \qquad D_2 = [\![P_2]\!]_{h_2}$ $j \in \lvert D_2 \cdot o_{n_2,a_2,c_2} \rvert$ $(D_2 \cdot o_{n_2,a_2,c_2})_{\lvert j} = [\![P_2']\!]_{h_2}$.	$P_2^0 = \zeta(D_2)[\sigma_2]$.

Figure 11: Synchro, (RH) cases

- If we apply the second rule for D_1 and the first rule for D_2, then there is a transition
$$C \xleftarrow{id} C' = \zeta((D_1 \cdot \iota_{n_1,a_1})_{\lvert i})[[\sigma_1, \ulcorner c \urcorner]]@P_2'[\sigma_2 \circ h_2]@C_0$$
with $C' \bowtie M'$ as desired.
- If the first rule is applied for both D_1 and D_2, then we have a synchronisation between $P_1[\sigma_1 \circ h_1]$ and $P_2[\sigma_2 \circ h_2]$. In order to determine the result of this synchronisation we need to choose a representative for $P_1[\sigma_1 \circ h_1]$, i.e., pick a channel for what b becomes after substitution. A reasonable choice here is $\gamma + 1$, we choose
$$P_1[\sigma_1 \circ h_1] = a(\gamma + 1).(P_1'[(\sigma_1 + 1) \circ h_1']) +_{k_1} R_1[\sigma_1 \circ h_1].$$
We thus have a transition to
$$C' = P_1'[(\sigma_1 + 1) \circ h_1'][\gamma + 1 \mapsto c]@P_2'[\sigma_2 \circ h_2]@C_0.$$
But the diagram
$$\begin{tikzcd} n_1 + 1 \arrow[r, "\sigma_1+1"] \arrow[dr, "{[\sigma_1, \ulcorner c \urcorner]}"'] & \gamma + 1 \arrow[d, "{[\gamma+1 \mapsto c]}"] \\ & \gamma \end{tikzcd}$$
commutes, so $C' = P_1'[[\sigma_1, \ulcorner c \urcorner] \circ h_1']@P_2'[\sigma_2 \circ h_2]@C_0$. Finally, we also know:
$$\begin{aligned} M' &= ((D_1 \cdot \iota_{n_1,a_1})_{\lvert i})[[\sigma_1, \ulcorner c \urcorner]]@(D_2 \cdot o_{n_2,a_2,c_2})_{\lvert j}[\sigma_2]@M_0 \\ &= [\![P_1']\!]_{h_1'}[[\sigma_1, \ulcorner c \urcorner]]@[\![P_2']\!]_{h_2}[\sigma_2]@M_0, \end{aligned}$$
which entails $C' \bowtie M'$ as desired.

- If we apply the first rule for D_1 and the second rule for D_2, then, choosing the same representative as before for $P_1[\sigma_1 \circ h_1]$, there is a transition
$$C \xleftarrow{id} P_1'[(\sigma_1+1)\circ h_1'][\gamma+1 \mapsto c]@\zeta((D_2 \cdot o_{n_2,a_2,c_2})_{|j})[\sigma_2]@C_0 = C'$$
satisfying $C' \bowtie M'$ (for the same reason as in the last case).

Fork, (RH). We consider the case of a forking action, i.e., $M = \langle \gamma \| D[\sigma] \rangle @M_0$ with $n \vdash_{\mathsf{D}} D$ and $\sigma: n \to \gamma$, and we have a transition
$$\langle \gamma \| D[\sigma] \rangle @M_0 \xleftarrow{id} \langle \gamma \| (D \cdot \pi_n^l)_{|i}[\sigma], (D \cdot \pi_n^r)_{|j}[\sigma] \rangle @M_0$$
for some i and j. We want to show that there exists a transition $C \xleftarrow{id} C'$ with $C' \bowtie M'$. We proceed by case analysis on the rule applied for $D[\sigma]$ in the proof of $C \bowtie M$.

- If the first rule is applied, then we find $C_0 \bowtie M_0$, $\gamma' \vdash P = P_1 \,|\, P_2$, and $h: \gamma' \rightleftharpoons n$ such that $D = [\![P]\!]_h$, $C = P[\sigma \circ h]@C_0$, $(D \cdot \pi_n^l)_{|i} = [\![P_1]\!]_h$, and $(D \cdot \pi_n^r)_{|j} = [\![P_2]\!]_h$.
  Thus, there is a transition
$$P[\sigma \circ h]@C_0 \xleftarrow{id} P_1[\sigma \circ h]@P_2[\sigma \circ h]@C_0 = C'$$
with $C' \bowtie M'$ as desired.
- If the second rule is applied, then we find $C_0 \bowtie M_0$ such that $C = \zeta(D)[\sigma]@C_0$. Thus, $\zeta(D)[\sigma]$ has the shape
$$\tau.(\zeta((D \cdot \pi_n^l)_{|i} \,|\, \zeta((D \cdot \pi_n^r)_{|j})))[\sigma] +_k R \tag{A.1}$$
  so we have
$$\zeta(D)[\sigma]@C_0 \quad \xleftarrow{id} \quad (\zeta((D \cdot \pi_n^l)_{|i}) \,|\, \zeta((D \cdot \pi_n^r)_{|j}))[\sigma]@C_0 = C'$$
with $C' \bowtie M'$ (using the third rule) as desired.
- If the third rule is applied, then we find $n \vdash_{\mathsf{D}} D'$, M_1, and C_0 such that $C_0 \bowtie M_1$, $C = (\zeta(D) \,|\, \zeta(D'))[\sigma]@C_0$, and $M_0 = D'[\sigma]@M_1$. But then, as in the previous case, $\zeta(D)[\sigma]$ has the shape (A.1) and we have transitions:
$$\begin{array}{rcl} (\zeta(D) \,|\, \zeta(D'))[\sigma]@C_0 & \xleftarrow{id} & \zeta(D)[\sigma]@\zeta(D')[\sigma]@C_0 \\ & \xleftarrow{id}^{2} & \zeta((D \cdot \pi_n^l)_{|i})[\sigma]@\zeta((D \cdot \pi_n^r)_{|j})[\sigma]@\zeta(D')[\sigma]@C_0 \\ & = & C' \end{array}$$
with $C' \bowtie M'$ as desired.

Nu, (RH). We consider the case of a ν rule, i.e., one has a transition
$$C = \langle \gamma \| D[\sigma] \rangle @M_0 \xleftarrow{id} \langle \gamma \| (D \cdot \nu_n)_{|i}[n+1 \xrightarrow{\sigma + \ulcorner a \urcorner} \gamma, a] \rangle @M_0[\gamma \subset \gamma, a]$$
with $n \vdash_{\mathsf{D}} D$ and $\sigma: n \to \gamma$. We want to show that there exists a transition $C \xleftarrow{id} C'$ with $C' \bowtie M'$. We again proceed by case analysis on the rule applied for $D[\sigma]$.

- If the first rule is applied, then we find $C_0 \bowtie M_0$, $\gamma' \vdash P = \nu a.P' +_k R$, and $h: \gamma' \rightleftharpoons n$ such that $C = P@C_0$, $D = [\![P]\!]_h$ and $(D \cdot \nu_n)_{|i} = [\![P']\!]_{h+1}$. There are thus transitions
$$C = P[\sigma \circ h]@C_0 \xleftarrow{id} P'[\gamma', a \xrightarrow{h+1} n+1 \xrightarrow{\sigma + \ulcorner a \urcorner} \gamma, a]@C_0[\gamma \subset \gamma, a] = C'$$
with $C' \bowtie M'$ as desired.

- If the second rule is applied, then we find $C_0 \bowtie M_0$ such that $C = \zeta(D)[\sigma]@C_0$, and there is a transition
$$\zeta(D)[\sigma]@C_0 \xleftarrow{id} \zeta((D \cdot \nu_n)_{|i})[n+1 \xrightarrow{\sigma + \ulcorner a \urcorner} \gamma, a]@C_0[\gamma \subset \gamma, a] = C'$$
with $C' \bowtie M'$ as desired.
- If the third rule is applied, then we find M_1, C_0, and $n \vdash_{\mathrm{D}} D'$ such that $M_0 = D'[\sigma]@M_1$, $C_0 \bowtie M_1$, and $C = (\zeta(D) \mid \zeta(D'))[\sigma]@C_0$. But then we have
$$\begin{aligned}(\zeta(D) \mid \zeta(D'))[\sigma]@C_0 &\xleftarrow{id} \zeta(D)[\sigma]@\zeta(D')[\sigma]@C_0 \\ &\xleftarrow{id} \zeta((D \cdot \nu_n)_{|i})[n+1 \xrightarrow{\sigma + \ulcorner a \urcorner} \gamma, a]@\zeta(D')[\sigma][\gamma \subset \gamma, a]@C_0[\gamma \subset \gamma, a] = C'\end{aligned}$$
and $C' \bowtie M'$ as desired.

Tick and Tau, (RH). We now consider the cases $\heartsuit$ and τ, i.e., when one has a transition
$$\langle \gamma \| D[\sigma] \rangle @M_0 \xleftarrow{\xi} \langle \gamma \| (D \cdot \xi_n)_{|i}[\sigma] \rangle @M_0$$
where $\xi \in \{\heartsuit, \tau\}$, with $n \vdash_{\mathrm{D}} D$ and $\sigma : n \to \gamma$. We want to show that there exists a transition $C \xleftarrow{\xi} C'$ with $C' \bowtie M'$. Again, we proceed by case analysis on the rule applied for $D[\sigma]$.

- If the first rule is applied, we find $C_0 \bowtie M_0$, $\gamma' \vdash P = \xi.P' +_k R$, and $h : \gamma' \rightleftharpoons n$ such that $C = P[\sigma \circ h]@C_0$, $D = [\![P]\!]_h[\sigma]$, and $(D \cdot \xi_n)_{|i} = [\![P']\!]_h$. But then we have
$$P[\sigma \circ h]@C_0 \xleftarrow{\xi} P'[\sigma \circ h]@C_0 \bowtie [\![P']\!]_h[\sigma]@M_0,$$
as expected.
- If the second rule is applied, then $C = \zeta(D)[\sigma]@C_0$ for some $C_0 \bowtie M_0$ and we have
$$\zeta(D)[\sigma]@C_0 \xleftarrow{\xi} \zeta((D \cdot \xi_n)_{|i})[\sigma]@C_0 \bowtie (D \cdot \xi_n)_{|i}[\sigma]@M_0,$$
as desired.
- Finally, if the third rule is applied, then we find C_0, M_1, and $n \vdash_{\mathrm{D}} D'$ such that $C = (\zeta(D) \mid \zeta(D'))[\sigma]@C_0$, $M_0 = D'[\sigma]@M_1$, and $C_0 \bowtie M_1$. But then, we have
$$(\zeta(D) \mid \zeta(D'))[\sigma]@C_0 \xleftarrow{id} \zeta(D)[\sigma]@\zeta(D')[\sigma]@C_0 \xleftarrow{\xi} \zeta((D \cdot \xi_n)_{|i})[\sigma]@\zeta(D')[\sigma]@C_0 = C'$$
with $C' \bowtie M'$ as desired.