Presheaves for Processes and Unfoldings *

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1 Introduction

Baldan et al. [?] give a general framework for rewriting systems based on adhesive categories. They give a definition of *process* in their setting, which is a sequence of rewriting steps considered up to permutation of independent steps, and they build the *unfolding* of any object. Here, we propose a different approach to model rewriting, processes, and unfoldings, which is based on a formalism inherited from *playgrounds* [?]. The main contribution of our work is a direct, combinatorial description of processes and unfoldings in terms of presheaves. Just like in Bladan et al.'s work, the unfolding of an object has a universal property (slightly different from theirs, though very close in spirit). This framework can adequately model executions of Petri nets, term graphs, or interaction nets, as well as processes in simple concurrent languages such as CCS [?], the π-calculus, or the join-calculus. It can also model some form of graph rewriting, which has yet to be compared to existing frameworks, such as SPO and DPO.

The main difficulty in Baldan et al.'s work is that a process is more than the sequence of its successive positions: it also records all the different occurrences of the rules that were applied. In our framework, this is dealt with by describing the whole process as a single presheaf that describes a combinatorial object of higher dimension. Given a signature Σ that describes a set of objects (which are the presheaves over a category $\mathbb{C}$) and rewriting rules on these objects (given as a set $\mathcal{R}$ of spans of such objects), we build a bicategory $\mathbb{B}$ that models rewriting on this set of objects according to the given rewriting rules. This bicategory will have as objects the set of objects described by Σ, and as morphisms from an object Y to an object X all the processes (also called *execution traces* or *rewriting traces*) that correspond to rewriting X into Y according to the rules given by Σ. Processes are defined as particular cospans in $\widehat{\mathbb{C}[\mathcal{R}]}$ (the category of presheaves over some category $\mathbb{C}[\mathcal{R}]$ constructed from $\mathbb{C}$ and $\mathcal{R}$). Such a cospan $Y \to U \leftarrow X$ describes an execution trace whose starting position is X and whose final position is Y. We build $\mathbb{C}[\mathcal{R}]$ from $\mathbb{C}$ by adding an object r for each rewriting rule in $\mathcal{R}$ and some meaningful morphisms from objects of $\mathbb{C}$ to r. We then define *seeds*, which are the local shape of a rewriting step, by equipping each representable presheaf y_r with its initial and final positions X and Y (as given by the rewriting rule), thus forming a cospan $Y \to \mathsf{y}_r \leftarrow X$. Rewriting steps are then defined by embedding seeds into larger positions (thus allowing rewriting steps to occur in context). Finally, rewriting traces are defined to be compositions of rewriting steps in the bicategory of cospans in $\widehat{\mathbb{C}[\mathcal{R}]}$. We then derive the unfolding U_X of any object X as the colimit of all the rewriting traces whose starting position is X.

Under some assumptions on the category $\mathbb{C}$, the presheaves over $\mathbb{C}[\mathcal{R}]$ have a graphical notation that is both close in spirit to string diagrams, and also inspired by that of graphs (but with multiple types of edges and higher-dimensional edges). For this reason, we usually call them *higher-dimensional* graphs or string diagrams (though none of these terms are actually very accurate).

* The authors acknowledge support from French ANR projets blanc Récré ANR-11-BS02-0010.

6th Conference on Algebra and Coalgebra in Computer Science.
Editors: Larry Moss and Pawel Sobociński; pp. 1–9
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

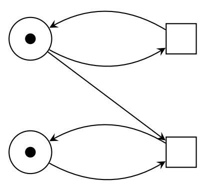


Figure 1 Our running example.

2 From Signatures to Bicategories and Unfoldings

Definition 1 (Signature). A signature Σ is given by:

- a "base category" $\mathbb{C}$ that describes positions (more precisely, positions are presheaves over $\mathbb{C}$),
- a finite set $\mathcal{R}$ of spans $Y \xleftarrow{w} I \xrightarrow{u} X$ in the category $\widehat{\mathbb{C}}$ of presheaves over $\mathbb{C}$, called "rules", which describe the possibility to rewrite X into Y,
- and a "starting position" S in $\widehat{\mathbb{C}}$.

Example 2. We give here two motivating examples our techniques apply to.

- In the case of Petri nets, the Petri net $\langle P, T, \delta, M\rangle$, where P and T are the sets of its places and transitions, $\delta : T \to (\mathbb{N}^P) \times (\mathbb{N}^P)$ its transition function (where the first component encodes consumed tokens and the second one created tokens), and $M : P \to \mathbb{N}$ its initial marking, is modeled by:
  - the base category $\mathbb{C} = P$, where P is viewed as the discrete category with $|P|$ elements and only identity morphisms (note that presheaves on $\mathbb{C}$ indeed model markings of the Petri net),
  - if $\delta = \langle \delta_{\text{cons}}, \delta_{\text{crea}}\rangle$, $\mathcal{R}$ contains a rule $Y_t \leftarrow \varnothing \to X_t$ for each transition $t \in T$, where Y_t and X_t are resp. the multisets $\delta_{\text{crea}}(t)$ and $\delta_{\text{cons}}(t)$, seen as presheaves,
  - the starting position is $S = M$, seen as a presheaf.
- In the case of graph rewriting, rewriting a graph G according to a given set of rewriting rules $\mathcal{R}_{\mathsf{Gph}}$ given as spans $r = L_r \leftarrow I_r \to R_r$, describing the possibility to rewrite L_r into R_r, is modeled by:
  - the base category is the category with two objects e, v and two non-trivial arrows $s, t : v \to e$ (again, note that presheaves over $\mathbb{C}$ are indeed graphs),
  - $\mathcal{R}$ contains a rule $R_r \leftarrow I_r \to L_r$ for each $r \in \mathcal{R}_{\mathsf{Gph}}$, where L_r and R_r are seen as presheaves over $\mathbb{C}$,
  - the starting position is $S = G$, where G is seen as a presheaf over $\mathbb{C}$.

Remark. In the case of Petri nets, we could also model "read arcs" by changing the middle component of the span (and therefore Y_t and X_t as well) to include the place the transition reads from.

Example 3. We further specialize the example of Petri net rewriting to the particular Petri net and marking shown in Figure 1. In our setting, this is modeled by:

- the set $\{a, b\}$ as its base category, where a represents the top-left place and b the bottom-left one,

- rules $a \leftarrow \varnothing \rightarrow a$ and $b \leftarrow \varnothing \rightarrow a+b$ [1], which represent the top-right and bottom-right transitions respectively,
- the starting position is $a+b$.

We will also use a more graphical notation for positions, i.e., presheaves over the set $\{a,b\}$. We could of course, as it is usually done for Petri nets, represent a position X by drawing a circle for each place and drawing as many bullets inside the circle that corresponds to a place p as there are elements in $X(p)$. However, this will prove quite impractical in our case, so we use a different representation. We will represent tokens in the place a by bullets and tokens in the place b by circles. For example, $\bullet \quad \bullet \quad \circ$ represents a marking with two tokens in a and one token in b.

For each rule $r \in \mathcal{R}$, we define a new category $\mathbb{C}_r$ that has the same objects as $\mathbb{C}$, plus an additional object r and morphisms from some objects of $\mathbb{C}$ to that new object.

▸ **Definition 4.** For any presheaf U, $\mathbb{C}[U]$ is the following cocomma category:

$$\begin{tikzcd} \int U \arrow[r, "!"] \arrow[d, "p_U"'] \arrow[dr, phantom, "\xRightarrow{\lambda}"] & 1 \arrow[d, "\ulcorner u \urcorner"] \\ \mathbb{C} \arrow[r, "i_U"'] & \mathbb{C}[U], \end{tikzcd}$$

where $\int U$ is U's category of elements.

▸ Remark. In more simple terms, $\mathbb{C}[U]$ has the same objects as $\mathbb{C}$, plus an additional object, say u. Between two objects of $\mathbb{C}$, $\mathbb{C}[U]$ has the same morphisms as $\mathbb{C}$, and the only morphism from u to itself is the identity morphism. For c an object of $\mathbb{C}$, there is a morphism $\bar{f} : c \to u$ for each morphism $f : c \to U$ (here again, we identify c and y_c), and there are no morphisms from u to c. Composition is defined in the obvious way.

For each presheaf X over $\mathbb{C}$, there is a presheaf $\overline{X} = \mathrm{Lan}_{i_U^{op}}(X)$ over $\mathbb{C}[U]$. In simple terms, for each object c of $\mathbb{C}$, $\overline{X}(c)$ is isomorphic to $X(c)$, and $\overline{X}(u)$ is empty. Therefore, we will often denote X instead of $\overline{X}$ when it's obvious that the presheaf is over $\mathbb{C}[U]$.

▸ **Proposition 5.** *Each presheaf $\overline{X}$ is the colimit of $\int X \xrightarrow{p_X} \mathbb{C} \xrightarrow{i_U} \mathbb{C}[U] \xrightarrow{\mathsf{y}} \widehat{\mathbb{C}[U]}$*

Proof. We consider the square

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\mathsf{y}"] \arrow[d, "i_U"'] \arrow[dr, phantom, "\Longrightarrow"] & \widehat{\mathbb{C}} \arrow[d] \\ \mathbb{C}[U] \arrow[r, "\mathsf{y}"'] & \widehat{\mathbb{C}[U]}. \end{tikzcd}$$

There are two possible ways to fill the morphism on the right and the natural transformation: either with $\mathrm{Lan}_{\mathsf{y}}(\mathsf{y} i_U)$ or with $\mathrm{Lan}_{i_U^{op}}(-)$, but the two possible fillings are actually isomorphic:

$$\mathrm{Lan}_{i_U^{op}}(X)(d) = \int^{c \in \mathbb{C}} \mathbb{C}[U](i_U^{op} c, d) \times X(c) \cong \int^{c \in \mathbb{C}} X(c) \times \mathbb{C}[U](i_U c, d)$$
$$\mathrm{Lan}_{\mathsf{y}}(\mathsf{y} i_U)(X)(d) = \left(\int^{c \in \mathbb{C}} \widehat{\mathbb{C}}(\mathsf{y} c, X) \times \mathsf{y} i_U c \right)(d) \cong \int^{c \in \mathbb{C}} X(c) \times \mathbb{C}[U](i_U c, d).$$

Now, since

[1] Here, a and b are actually y_a and y_b, the representable presheaves on a and b, i.e., $\mathbb{C}(-,a)$ and $\mathbb{C}(-,b)$. We will often identify an object of a category and its representable presheaf and let the context disambiguate.

$$\begin{tikzcd}
\int X \arrow[r, "!"] \arrow[d, "p_X"'] \arrow[dr, phantom, "\Longrightarrow"] & 1 \arrow[d, "\ulcorner X \urcorner"] \\
\mathbb{C} \arrow[r, "\mathsf{y}"'] & \widehat{\mathbb{C}}
\end{tikzcd}$$

is a comma square and Kan extensions are pointwise in Cat, the composition of any left Kan extension $\mathrm{Lan}_{\mathsf{y}} F : \widehat{\mathbb{C}} \to \mathbb{C}'$ with $\ulcorner X \urcorner$ is again a left Kan extension. In particular, since $\mathrm{Lan}_{i_U^{op}}(-)$ is isomorphic to $\mathrm{Lan}_{\mathsf{y}}(\mathsf{y} i_U)$, which is a left Kan extension, it is also a left Kan extension, and therefore, so is $\mathrm{Lan}_{i_U^{op}}(-) \circ \ulcorner X \urcorner = \ulcorner \mathrm{Lan}_{i_U^{op}}(X) \urcorner = \ulcorner \overline{X} \urcorner$, so $\overline{X}$ is indeed the desired colimit. ◂

In particular, $\overline{U}$ is the colimit of $\int U \xrightarrow{p_U} \mathbb{C} \xrightarrow{i_U} \mathbb{C}[U] \xrightarrow{\mathsf{y}} \widehat{\mathbb{C}[U]}$. Moreover, we have the following natural transformation:

$$\begin{tikzcd}
\int U \arrow[r, "!"] \arrow[d, "p_U"'] \arrow[dr, phantom, "\Longrightarrow"] & 1 \arrow[d, "\ulcorner u \urcorner"] \arrow[dr, "\ulcorner u \urcorner"] & \\
\mathbb{C} \arrow[r, "i_U"'] & \mathbb{C}[U] \arrow[r, "\mathsf{y}"'] & \widehat{\mathbb{C}[U]},
\end{tikzcd}$$

so, by universal property of the colimit, there is a unique morphism from $\overline{U}$ to u such that

$$\begin{tikzcd}
\int U \arrow[r, "!"] \arrow[d, "i_U p_U"'] \arrow[dr, phantom, "\Longrightarrow"] & 1 \arrow[r, equal] \arrow[d, "\ulcorner \overline{U} \urcorner"] \arrow[dr, phantom, "\Longrightarrow"] & 1 \arrow[d, "\ulcorner u \urcorner"] \\
\mathbb{C}[U] \arrow[r, "\mathsf{y}"'] & \widehat{\mathbb{C}[U]} \arrow[r, equal] & \widehat{\mathbb{C}[U]}
\end{tikzcd}$$

gives back the natural transformation above.

▸ **Definition 6.** We call $\partial u : U \to u$ be the morphism constructed above.

▸ **Definition 7.** $\mathbb{C}_r$ is $\mathbb{C}[M_r]$, where M_r is defined as the following pushout:

$$\begin{tikzcd}
I_r \arrow[r, "w_r"] \arrow[d, "u_r"'] & Y_r \arrow[d] \\
X_r \arrow[r] & M_r. \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

▸ Remark. $\mathbb{C}_r$ is defined as any $\mathbb{C}[U]$, and, if we call r its new object, then the morphisms from c to r are as defined below:

- for each $f : \mathsf{y}_c \to X_r$ in $\widehat{\mathbb{C}}$, add a morphism $t_f : c \to r$ in $\mathbb{C}_r$,
- for each $f : \mathsf{y}_c \to Y_r$ in $\widehat{\mathbb{C}}$, add a morphism $s_f : c \to r$ in $\mathbb{C}_r$,
- for each $f : \mathsf{y}_c \to I_r$ in $\widehat{\mathbb{C}}$, add a morphism $\tilde{f} : c \to r$ in $\mathbb{C}_r$,
- quotient these morphisms by $\tilde{f} = t_{u_r f}$ and $\tilde{f} = s_{w_r f}$.

▸ Remark. In fact, we have only added morphisms t_f and s_f, since each morphism $\tilde{f}$ will be identified with $t_{u_r f}$ and $s_{w_r f}$.

▸ **Definition 8.** In $\widehat{\mathbb{C}_r}$, we call $Y_r \xrightarrow{s_r} r \xleftarrow{t_r} X_r$ the cospan obtained by composing $Y_r \to M_r \leftarrow X_r$ with ∂r.

▸ **Proposition 9.** *$Y_r \xrightarrow{s_r} r \xleftarrow{t_r} X_r$ is such that*

$$\begin{tikzcd}
I_r \arrow[r, "w_r"] \arrow[d, "u_r"'] & Y_r \arrow[d, "s_r"] \\
X_r \arrow[r, "t_r"'] & r
\end{tikzcd} \tag{1}$$

commutes.

Proof. It is the composition of a commutative square with ∂r. ◂

We finally define $\mathbb{C}[\mathcal{R}]$, the base category our pseudo double category will be built on.

▸ **Definition 10.** For a signature $\Sigma = (\mathbb{C}, \mathcal{R}, S)$, we define the base category $\mathbb{C}[\mathcal{R}]$ as the wide pushout of $\mathbb{C} \xrightarrow{i_{M_r}} \mathbb{C}_r$ for all rules r in $\mathcal{R}$.

▸ Remark. In more simple terms, this is the category $\mathbb{C}$ with all the objects r in $\mathcal{R}$ and corresponding morphisms.

▸ **Example 11.** In our running example, if we call the first rule u and the second one v, we have that $\mathbb{C}[u, v]$, the base category of the corresponding pseudo double category will have:

- as objects: the set $\{a, b, u, v\}$,
- as non-trivial morphisms: $t_{id_a} : a \to u$, $s_{id_a} : a \to u$, $t_{i_l} : a \to v$, $t_{i_r} : b \to v$, $s_{id_b} : b \to v$,

or, up to renaming, and in more graphical terms:

$$\begin{tikzcd}
u & v \\
a \arrow[u, bend left, "t_a^u"] \arrow[u, bend right, "s_a^u"'] \arrow[ur, "t_a^v"] & b \arrow[u, bend left, "t_b^v"] \arrow[u, bend right, "s_b^u"']
\end{tikzcd}$$

We now build a pseudo double category that will describe rewriting on the signature. More precisely, the objects of this pseudo double category will be positions, horizontal morphisms will describe "spatial inclusion", i.e., how positions are included in larger positions, and vertical morphisms will be "rewriting traces", i.e., how a position can reach another position after a finite number of rewriting steps.

First of all, we need to define how we model a rewriting step.

In order to do this, we first define "seeds", which are the basic components of a rewriting step.

▸ **Definition 12.** The seed that corresponds to the rule $r = Y_r \xleftarrow{w_r} I_r \xrightarrow{u_r} X_r$ is the commuting square shown in (1).

▸ **Definition 13.** A rewriting step is any cospan $Y \to M \leftarrow X$ obtained by taking a pushout of a seed along some monomorphism $I_r \to Z$:

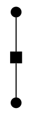
$$\begin{tikzcd}
& Y_r \arrow[rrr] \arrow[dr] & & & Y \arrow[dr] & \\
& & r \arrow[rrr] & & & M \\
I_r \arrow[uur] \arrow[urr] \arrow[rrr] \arrow[dr] & & & Z \arrow[uur] \arrow[urr] \arrow[dr] & & \\
& X_r \arrow[uur] \arrow[rrr] & & & X. \arrow[uur] &
\end{tikzcd}$$

We say that this rewriting step rewrites X into Y.

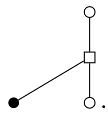
▸ **Example 14.** In our running example, there are two seeds, which correspond respectively to u and v. They are the pushouts

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & a \arrow[d, "\imath_2"] \\
a \arrow[r, "\imath_1"'] & u
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & b \arrow[d, "\imath_3"] \\
a + b \arrow[r, "\imath_1 + \imath_2"'] & v
\end{tikzcd}$$

We can represent them in a more graphical way. Recall the graphical notation for positions from Example 3. We can draw the seeds that correspond to u and v like so:



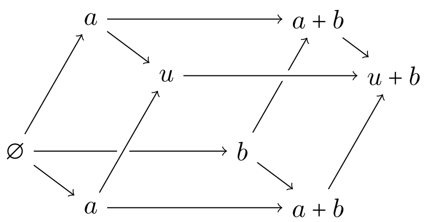
and



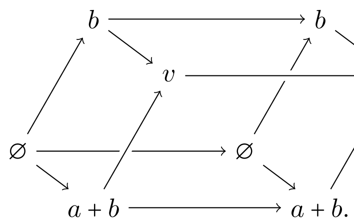
The initial position of each seed is drawn at the bottom, and its final position is drawn at the top, with another element in the middle that corresponds to the action of moving from the initial position to the final one itself.

▸ Remark. These drawings actually correspond to drawing the category of elements of the seeds that correspond to u and v, just like drawing a graph corresponds to drawing the category of elements of its corresponding presheaf. Here, we have drawn the categories of elements of the presheaves u and v, with the convention that $X_u = a$ and $X_v = a + b$ are drawn at the bottom, while $Y_u = a$ and $Y_v = b$ are drawn at the top. Therefore, in all our drawings, "time flows upwards".

▸ **Example 15.** Remember that our starting marking is $a + b$, so there are two possible rewriting steps on this marking:

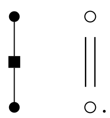


and



The first one corresponds to firing u on the marking $a + b$, which indeed gives back the marking $a + b$, while the second one corresponds to firing v on the same marking, which indeed gives the marking b.

We can draw these rewriting steps in a graphical way, very much like we did for seeds (and again, this corresponds to drawing the category of elements of the object we represent). Since the move that corresponds to v doesn't differ from the seed, it is represented exactly as in the example above; the move that corresponds to u is drawn like so:



The top and bottom positions indeed correspond to the initial and final positions of the rewriting step, and the equal sign between the two tokens in b express that they are actually the same token, but that this token is present both in the initial and final positions (in the category of elements, there is of course only one object corresponding to the token in b, but we draw it twice with an equal sign between them for increased readability).

▸ **Definition 16.** A rewriting trace is any cospan that is isomorphic to a finite composition of rewriting steps in the category of cospans on $\widehat{\mathbb{C}[\mathcal{R}]}$.

▸ **Example 17.** For example, on our running example, the different rewriting traces are (isomorphic to) either:

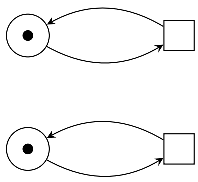
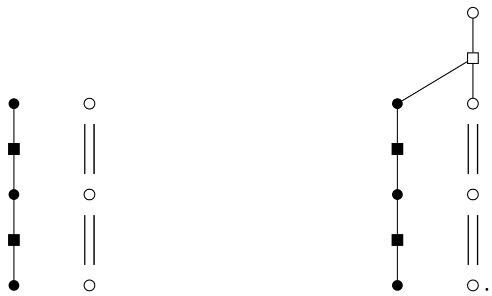


Figure 2 A less sequential Petri net.

- U_n for some n in $\mathbb{N}$, where U_n is the n-fold composition of the move that corresponds to u on the initial marking (note that they can indeed be composed because the final position is the same as the initial one),
- or V_n for some n in $\mathbb{N}$, where V_n is the composition of U_n with the move that corresponds to v.

They can be described in graphical terms, for example, here are the graphical descriptions of U_2 on the left and of V_2 on the right:



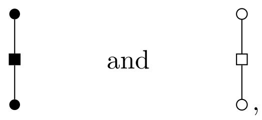
▸ **Example 18.** To see how this definition is not sequential, but retains only the minimal causality information between rewriting steps, we have to introduce another Petri net, because the one from our running example is entirely sequential.

Let us take as example the Petri net in figure 2, which is much more parallel: indeed, the two transitions do not interact anymore. If we still call a, b, u, and v the different places and transitions, we have that the possible moves from the initial marking are isomorphic to one of the following two moves:

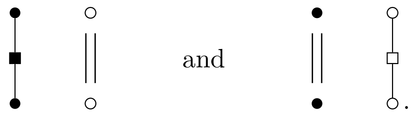
$$\begin{tikzcd}
 & a \arrow[rrr] \arrow[dr] & & & a+b \arrow[dr] & \\
 & & u \arrow[rrr] & & & u+b \\
\varnothing \arrow[uur] \arrow[rrr] \arrow[dr] & & & b \arrow[uur] \arrow[dr] & & \\
 & a \arrow[rrr] \arrow[uur] & & & a+b \arrow[uur] &
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
 & b \arrow[rrr] \arrow[dr] & & & a+b \arrow[dr] & \\
 & & v \arrow[rrr] & & & a+v \\
\varnothing \arrow[uur] \arrow[rrr] \arrow[dr] & & & a \arrow[uur] \arrow[dr] & & \\
 & b \arrow[rrr] \arrow[uur] & & & a+b. \arrow[uur] &
\end{tikzcd}$$

A possible execution in the Petri net consists in first firing u, then firing v. It is the composition of the two cospans $a+b \to u+b \leftarrow a+b$ and $a+b \to a+v \leftarrow a+b$, which gives $a+b \to u+v \leftarrow a+b$, which is the same as the rewriting trace obtained by first firing v, and then firing u, which corresponds to the fact that there is no causality relation between firing u and firing v.

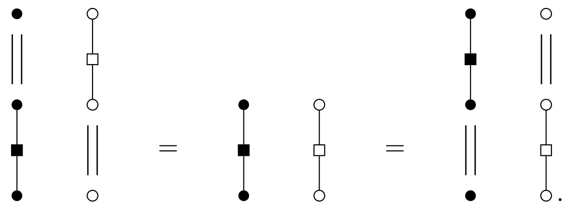
This can also be seen in graphical terms: if the seeds are drawn as



then the moves on the initial marking can be drawn as



Therefore, the execution that fires u then v and the one that fires v then u can be drawn as on the left and right respectively, which can also be drawn as in the middle (these drawings are again the graphical representation of the categories of elements associated to the different presheaves):



We can now define the pseudo double category $\mathbb{D}$ that will describe processes:

- its objects are the positions, i.e., the presheaves over $\mathbb{C}$ (more precisely, they're the presheaves over $\mathbb{C}[\mathcal{R}]$ that are of the form $\mathrm{Lan}_{i_U^{op}} X$ for some presheaf X over $\mathbb{C}$),
- its horizontal morphisms are monomorphisms of positions,
- its vertical morphisms are rewriting traces,
- its double cells are monomorphisms of traces, i.e., commuting diagrams of the form

$$\begin{tikzcd} Y \arrow[r, "l"] \arrow[d] & Y' \arrow[d] \\ U \arrow[r, "k"] & U' \\ X \arrow[r, "h"'] \arrow[u] & X' \arrow[u] \end{tikzcd},$$

where h, k, and l are mono.

▸ Remark. We can draw a parallel between rewriting and games (and indeed, the ideas used here to model rewriting stem from playgrounds, which are a general framework for game semantics):

- the objects our rewriting system works on (in the case of Petri nets, markings) are akin to positions,
- rewriting steps (in the case of Petri nets, the firing of a transition) to moves,
- and rewriting traces (in the case of Petri nets, executions) to plays.

▸ **Definition 19.** The category $\mathbb{E}(X)$ is the category of rewriting traces over X:

- its objects are the lower legs of rewriting traces: for each rewriting trace $Y \xrightarrow{s} U \xleftarrow{t} X$, $X \xrightarrow{t} U$ is an object of $\mathbb{E}(X)$,
- the morphisms from $X \xrightarrow{t} U$ to $X \xrightarrow{t'} U'$ are monomorphisms $U \xrightarrow{u} U'$ such that $ut = t'$.

▸ Remark. The morphisms in $\mathbb{E}(X)$ correspond to "prefix" inclusion of rewriting traces, i.e., there is a morphism $U \to U'$ in $\mathbb{E}(X)$ if the rewriting trace U can be followed by another rewriting trace such that the whole rewriting trace is isomorphic to U'.

▸ **Example 20.** On our running example, $\mathbb{E}(X)$ is composed of all the plays isomorphic to either U_n or V_n (as defined in Example 17), plus the morphisms that correspond to prefix inclusion, which are generated by:

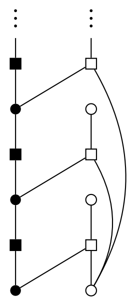
- for each n, the only morphism $U_n \to U_{n+1}$ that fixes X,
- for each n, the only morphism $U_n \to V_n$ that fixes X.

$\mathbb{E}(X)$ defines a diagram in $\widehat{\mathbb{C}[\mathcal{R}]}$ by mapping $X \xrightarrow{t} U$ to U (with the identity action on morphisms).

▸ **Definition 21.** The unfolding U_X of X is the colimit of the functor defined above.

▸ Remark. Since $X \xrightarrow{id_X} X$ is an object of $\mathbb{E}(X)$, the unfolding U_X of X comes with a map $X \to U_X$ obtained by universal property of U_X.

▸ **Example 22.** In our running example, if we draw the unfolding U_X of X, we obtain:


,

and the morphism $X \to U_X$ maps X to the bottom of this drawing, which is exactly what we expect.