Constructing Fibred Double Categories
Towards New Sheaf Models of Programming Languages

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Playgrounds

Game semantics: semantics = interaction.

- Pseudo double category:
  - with extra structure
  - and properties.

- Strategies:
  - sheaf semantics $\approx$ presheaf semantics + innocence
  - or concurrent game strategies.
Building playgrounds from signatures

Until now:
- automatic: playground $\leftrightarrow$ notion of strategy,
- playground = pseudo double category + properties,
- problem: prove properties by hand for each language.

This work:
- signatures $\rightarrow$ playgrounds $\rightarrow$ strategies,
- prove properties abstractly,
- $\rightsquigarrow$ new playground for HO games.
Overview

1. Multi-party HO
2. A playground for HO
3. Fibredness
Arena = finite forest of “moves”.

- Polarity (O or P) of a move = parity of its depth.
- Roots have polarity O.
- Notation:
  - A an arena,
  - $m$ a root of $A$,
  - $A \cdot m$ is the forest below $m$.

Play = $P$-visible, alternated, justified sequence of even length.
View = non-empty play such that $\lceil s \rceil = s$. 
A framework for HO

First: construct a multi-party framework for HO games.
Then: organise it into a pseudo double category.

Overview:

- Positions
- Generator moves
- Moves
- Plays
Positions

Position: (kind of) graph:

\[
\begin{align*}
A_1 & \quad \vdash \quad B \\
A_n & \quad \vdash \quad C
\end{align*}
\]
Positions

Position: (kind of) graph:

Vertices are players.
Positions

Position: (kind of) graph:

- Vertices are players
- Edges are "gameboards"

Vertices are labeled with arenas:
- \( A_1 \)
- \( \ldots \)
- \( A_n \)

Edges connect vertices and are labeled with sequents:
- \( A_1, \ldots, A_n \vdash B \)
Positions

Position: (kind of) graph:

- multiple incoming edges
- vertices are players
- edges are “gameboards”

Vertices are players, edges are "gameboards". Each player has a sequent $A_1, \ldots, A_n \vdash B$ or $A_1, \ldots, A_n \vdash \text{of arenas.}$
Positions

Position: (kind of) graph:

- Multiple incoming edges
- At most one outgoing edge
- Vertices are players
- Edges are “gameboards”

Vertices represent players, with multiple incoming edges and at most one outgoing edge. Edges are labeled with arenas and may be dangling. Each player has a sequent of arenas.
Positions

Position: (kind of) graph:

- Multiple incoming edges
- At most one outgoing edge
- Vertices are players
- Edges are “gameboards”
- Labelled with arenas

Vertices are players, and edges are "gameboards". The diagram shows multiple incoming edges and at most one outgoing edge, with vertices labelled with arenas.
Positions

Position: (kind of) graph:

- Multiple incoming edges
- At most one outgoing edge
- Vertices are players
- Edges are “gameboards”
- May be dangling
- Labelled with arenas
Position: (kind of) graph:

- Vertices are players.
- Edges are “gameboards.”
- Multiple incoming edges.
- At most one outgoing edge.
- Edges may be dangling.
- At most one source.
- Labelled with arenas.
Positions

Position: (kind of) graph:

- Multiple incoming edges
- At most one outgoing edge
- Vertices are players
- Edges are "gameboards"
- Multiple targets
- Labelled with arenas
- May be dangling
- At most one source

\[ A_1, \ldots, A_n \vdash B \text{ or } A_1, \ldots, A_n \vdash \text{of arenas.} \]
Positions

Position: (kind of) graph:

Vertices are players. Edges are “gameboards”.

- Multiple incoming edges
- At most one outgoing edge
- Multiple targets
- Labelled with arenas
- May be dangling
- At most one source

⇝ each player has a sequent $A_1, \ldots, A_n \vdash B$ or $A_1, \ldots, A_n \vdash$ of arenas.
Moves (1)

Typical move:

**final position:**

**initial position:**
Moves (1)

Typical move:

![Diagram of moves]

Final position:

Initial position:

Analogy with sequent calculus:

\[
\frac{A, B \cdot q}{A \vdash B} \\
\frac{B, C \vdash B \cdot q}{B, C \vdash}
\]
Moves (2)

More generally, embed into any context.
Example:

final position:

initial position:
Plays

- Sequences of moves?
- Too naive.
- Need expressive morphisms between plays (e.g., Ong-Tsukada).

Solution: retain only causal dependencies between moves.
Problem: how to represent this formally?
Representing positions

Consider the following category $\mathcal{C}_1$:
- objects = sequents and arenas,
- morphisms = occurrences of arenas in sequents.

\[
A_1, \ldots, A_n \vdash A
\]

Any position yields a presheaf on $\mathcal{C}_1$!
Representing positions: examples

\[ A_1, \ldots, A_n \vdash A \]

\[ A_1, \ldots, A_n \vdash \ldots \]

- A dangling edge labelled with \( A \): the representable presheaf \( y_A \).
- A player on any sequent: the representable presheaf on that sequent.
- The position \( A \rightarrow B \leftarrow C \):

\[ y_B \xrightarrow{s_1} y_B, C \vdash \]

\[ t \]

\[ y_A \vdash B \xrightarrow{.} \]
Diagram associated to a position

The presheaf representing

\[ A_1 \rightarrow A_n \rightarrow \cdots \rightarrow A \rightarrow B \rightarrow C \]

is a colimit of

\[ A_1 \rightarrow A_n \rightarrow \cdots \rightarrow (A_1, \ldots, A_n \vdash A) \rightarrow A \rightarrow (A, B \vdash C) \rightarrow C. \]
Representing moves

- How to represent plays?
- How to represent moves?
- Moves are more than just relations.
- Add objects to $\mathbb{C}_1$ to represent moves!
- Glue them together to form plays.

Which objects?
Adding objects to $\mathbb{C}_1$: fill the shell!

New object $\mu$ for move:

final position:

initial position:

Add green and red arrows making the diagram commute:
The augmented base category

Add an object for each instance

\[
\Gamma, B \cdot q \vdash \\
\Gamma \vdash B
\]

\[
\Delta_1, B, \Delta_2 \vdash B \cdot q \\
\Delta_1, B, \Delta_2 \vdash
\]

with associated morphisms.

Definition

Let $\mathbb{C}$ denote the augmented category.
Initial and final positions

Representable presheaf $y_{\mu}$: shapeless “blob”.

![Diagram](image-url)
Initial and final positions

Representable presheaf $y_\mu$: shapeless “blob”.

Organise it as a cospan $Y \to y_\mu \leftarrow X!$
Summary

For each intended move:

- new object $\mu$ in $\mathbb{C}$,
- cospan $Y \rightarrow y_{\mu} \leftarrow X$, the generator move,
- $X$ initial position,
- $Y$ final position.

Next step: want moves to occur inside larger context:

- glue the generator move
- along some stable part
- to some position (the context).
Stable part of a generator move

Any position \( I \) equipped with morphisms making

\[
\begin{array}{ccc}
I & \longrightarrow & y_{\mu} \\
\uparrow & & \uparrow \\
Y & \longrightarrow & X
\end{array}
\]

commute.
Move

Gluing of a generator move to some context $Z$:

\[
\begin{array}{ccc}
Y & \rightarrow & Y' \\
\uparrow & & \uparrow \\
\downarrow & & \downarrow \\
Y_{\mu} & \rightarrow & Y' \\
\downarrow & & \downarrow \\
I & \rightarrow & Z \\
\downarrow & & \downarrow \\
X & \rightarrow & Z \\
\end{array}
\]

(\text{levelwise pushout}).
Example move

final position:

initial position:
Play

Definition

Play = finite composite of moves in $\text{Cosp}(\hat{C})$. 
A (pseudo) double category

- Objects:
- Horizontal morphisms:
- Vertical morphisms:
- Cells:

\[
\begin{array}{c}
Y \xrightarrow{k} Y' \\
\downarrow \downarrow \\
U \xrightarrow{\alpha} U' \\
\downarrow \downarrow \\
X \xrightarrow{h} X'
\end{array}
\]
A (pseudo) double category

- Objects: positions.
- Horizontal morphisms: morphisms between positions.
- Vertical morphisms: plays.
- Cells: morphisms \( l \) making the diagram commute.
Composition

- Horizontal compositions: straightforward.
- Vertical composition: by pushout.
- Vertical composition of cells: by universal property of pushout.

Interchange law:

\[
\begin{array}{cccc}
Z & \rightarrow & Z' & \rightarrow & Z'' \\
\downarrow & \alpha & \downarrow & \alpha' & \downarrow \\
Y & \rightarrow & Y' & \rightarrow & Y'' \\
\downarrow & \beta & \downarrow & \beta' & \downarrow \\
X & \rightarrow & X' & \rightarrow & X''
\end{array}
\]
Comparison with Ong-Tsukada

<table>
<thead>
<tr>
<th>Ong-Tsukada</th>
<th>playyards</th>
</tr>
</thead>
<tbody>
<tr>
<td>views</td>
<td>=</td>
</tr>
<tr>
<td>plays</td>
<td>↔</td>
</tr>
<tr>
<td>no players, no positions</td>
<td>↔</td>
</tr>
<tr>
<td>time, sequential</td>
<td>↔</td>
</tr>
<tr>
<td>$P$-view</td>
<td>↔</td>
</tr>
<tr>
<td>$P$-visibility</td>
<td>↔</td>
</tr>
<tr>
<td>composition, hiding</td>
<td>↔</td>
</tr>
</tbody>
</table>
Views

- For Ong-Tsukada: $\mathcal{V}_{A,B} = \text{category of views on } A \rightarrow B$ (ordered by prefix).
- In our sense: views are some (ad-hoc) non-empty plays on $A \vdash B$, $\mathcal{E}^\mathcal{V}_{A \vdash B} = \text{category of views on } A \vdash B$, where morphisms are “temporal inclusion”.

Lemma

$\mathcal{V}_{A,B}$ and $\mathcal{E}^\mathcal{V}_{A \vdash B}$ are equivalent.
Plays

Wilder notion of play in our setting.
However:

Conjecture

Innocent strategies in our sense are equivalent to innocent strategies in the sense of Ong-Tsukada.

Idea: prove that

\[
\begin{align*}
\forall_{A,B} & \rightarrow \mathbb{P}_{A,B} \\
\mathbb{E}_{A \vdash B} & \rightarrow \mathbb{E}_{A \vdash B}
\end{align*}
\]

is an exact square.

Enough to prove: \( \mathbb{P}_{A,B} \rightarrow \mathbb{E}_{A \vdash B} \) fully faithful.
Behaviours

Important construction of playgrounds: $\mathbb{E}$.
Innocent strategies $\approx$ sheaves over $\mathbb{E}$.

- objects: $\textcolor{red}{U} : Y \rightarrow X$,
- morphisms $(U : Y \rightarrow X) \rightarrow (U' : Y' \rightarrow X')$: 

\[
\begin{array}{c}
Z \xrightarrow{l} Y' \\
W \downarrow \quad \Downarrow \alpha \\
Y \xrightarrow{\alpha} U' \\
U \downarrow \quad \Downarrow h \\
X \rightarrow X'.
\end{array}
\]
Composition in $\mathcal{E}$

By pasting

\[
\begin{array}{c}
\tilde{Z} \xrightarrow{\tilde{l}} Z' \xrightarrow{l''} Y'' \\
\downarrow \quad \quad \downarrow \quad \quad \quad \downarrow \\
\tilde{W} \quad \quad \quad W' \quad \quad \quad U''
\end{array}
\]

\[
\begin{array}{c}
\downarrow \quad \quad \downarrow \quad \quad \quad \downarrow \\
\tilde{Z} \quad \quad \quad Z' \quad \quad \quad Y'
\end{array}
\]

\[
\begin{array}{c}
\downarrow \quad \quad \quad \downarrow \\
W \quad \quad \quad Y' \quad \quad \quad U''
\end{array}
\]

\[
\begin{array}{c}
\downarrow \quad \quad \quad \downarrow \\
W \quad \quad \quad Y \quad \quad \quad X''
\end{array}
\]

where $\gamma$ is a cartesian lifting of $W'$ along $l$. 
Crucial property: fibredness

**Fibredness**

A pseudo double category $\mathbb{D}$ is fibred if its “vertical codomain” functor $\text{cod}$ is a fibration.
Crucial property: fibredness

Fibredness

A pseudo double category $\mathbb{D}$ is fibred if its “vertical codomain” functor $\text{cod}$ is a fibration.
Factorisation systems

\((\mathcal{L}, \mathcal{R})\) two classes of morphisms of \(C\) such that:

- every morphism of \(C\) can be decomposed as \(r \circ l\), for some \(l \in \mathcal{L}\) and \(r \in \mathcal{R}\),
- \(\mathcal{L} \perp \mathcal{R}\), i.e., for all \(l \in \mathcal{L}\), \(r \in \mathcal{R}\):

\[
\begin{array}{c}
\mathcal{L} \perp \mathcal{R} \\
\exists! \end{array}
\]

Properties:

- \(\mathcal{L}\) and \(\mathcal{R}\) contain all isos, are stable under composition,
- \(\mathcal{L}\) is stable under pushouts,
Cofibrant generation

\[ \mathcal{M} \perp = \text{class of morphisms } g \text{ such that for all } f \text{ of } \mathcal{M}, f \perp g. \]
\[ \perp \mathcal{M} = \text{idem.} \]

**Small object argument (Bousfield)**

For any set \( J \) of morphisms of \( \mathcal{C} \), \( (J \perp, \perp(J \perp)) \) is a factorisation system.

Cofibrantly generated factorisation system on \( \text{Cosp}(\hat{\mathcal{C}}) \): \( J \) set of \( X \to y_\mu \text{ for } Y \to y_\mu \leftarrow X \) generator move.
Construction of the candidate lifting

\[
\begin{array}{ccc}
Y & \xrightarrow{U} & X \\
| & | & | \\
\downarrow & \downarrow & \downarrow \\
Y' & \xrightarrow{U'} & X' \\
\end{array}
\quad
\begin{array}{ccc}
Y & \xrightarrow{U} & X \\
| & | & | \\
\downarrow & \downarrow & \downarrow \\
Y' & \xrightarrow{U'} & X' \\
\end{array}
\quad
\begin{array}{ccc}
Y & \xrightarrow{U} & X \\
| & | & | \\
\downarrow & \downarrow & \downarrow \\
Y' & \xrightarrow{U'} & X' \\
\end{array}
\]
Proof of fibration

- $U'' \rightarrow U'$ by the lifting property of factorisation systems,
- $Y'' \rightarrow Y'$ by universal property of pullback.
Conclusion

Done:
- general construction of playgrounds,
- better understanding of the link between playgrounds and classical game semantics.

Work to do:
- prove full faithfulness of $\mathbb{P}_{A,B} \to \mathbb{E}_{A\vdash_B}$,
- influence of space,
- influence of time,
- get the “right” views back,
- make hiding work.