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Uncountable ZF-Ordinals

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Let T be a theory such as ZF, KP, KP_n ($= \Sigma_n$-admissibility). Say that α is a *T-ordinal* if L_α is a model of T. For a subset x of some cardinal κ, let $\alpha_T(x)$ be the least ordinal $\alpha > \kappa$ such that $L_\alpha(x)$ is a model of T.

Assume $V = L$. In [**3**, **4**] the second author gave a characterization of the ordinals $\alpha_{\mathrm{KP}_n}(x)$ ($n \geq 1$, $x \subset \kappa$) for every cardinal κ. This is a generalization of a theorem of Sacks which says that every countable KP-ordinal is an $\alpha_{\mathrm{KP}}(x)$ for some $x \subset \omega$.

In [**2**] the first author showed that every countable ZF-ordinal is an $\alpha_{\mathrm{ZF}}(x)$ for some $x \subset \omega$. This result has been proved independently by A. Beller (see [**1**]).

In this paper we give a characterization of the ordinals $\alpha_{\mathrm{ZF}}(x)$ ($x \subset \kappa$) for every cardinal κ.

We use both the techniques of [**3**, **4** and **2**]. The situation for ZF is very different from that of KP. For the latter the ordinals have cofinality equal to the cofinality of κ whereas in the present case they have cofinality ω.

Let us mention that to prove this characterization much of the work of R. Jensen on the fine structure of L is used: the usual tools for fine structure but also the coding theorem and even the covering theorem, although we are working inside L.

THEOREM. *Assume $V = L$. Let α be a* ZF*-ordinal of cardinality κ, $\alpha > \kappa$. Then α is an $\alpha_{\mathrm{ZF}}(x)$ for some $x \subset \kappa$ if and only if one of the following holds*:

(1) *$L_\alpha \models \kappa$ is singular and α is a successor* ZF *ordinal and $L_\alpha \models$ the* sup *of the ZF-ordinals has cardinality κ.*

(2) *κ is regular and there is a $\beta < \alpha$ and a sequence $(X_n | n < \omega)$ such that*

(i) $\forall \gamma < \kappa \forall f \colon \gamma \to \beta$ (*f bounded* $\to f \in L_\alpha$);

(ii) *$X_n \in L_\alpha$ and $L_\alpha -$* card$(X_n) < \beta$ *for $n \in \omega$, $L_\alpha = \bigcup_n X_n$;*

(iii) *β is a regular cardinal in L_α.*

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(3) *κ is singular but $L_\alpha \models \kappa$ is regular, and there is a $\beta < \alpha$ and a sequence $(X_n | n < \omega)$ such that* (ii) *and* (iii) *of* (2) *hold and*

(i) $\forall \lambda < \beta \exists f: L_\lambda \to \kappa$, *$f$ one-one and tame, i.e.,*

$$\forall \gamma < \kappa f^{-1}[\gamma] \in L_\alpha.$$

(*Note*. (3(i)) can be replaced by (3(i′)): $\exists f: L_\alpha \to \kappa$, f one-one and tame, when cof $\kappa = \omega$. (2(i)) $\leftrightarrow$ (3(i)) when κ is regular.)

We shall deal with the three cases separately.

I. The following lemma will be often used:

LEMMA I.1. *Let κ be a cardinal $\geqslant \omega_2$, $x \subset \kappa$ such that $x \in L$ and $L_\alpha(x) \models$ ZF. Let $f \in L_\alpha(x)$ $f: \gamma < \kappa \to \alpha$.*

Then $f \in L_\alpha$.

PROOF. Let $A = \{\langle i, f(i)\rangle | i < \gamma\}$ where $\langle \cdot, \cdot \rangle$ is the Gödel pairing function. By Jensen's covering theorem there is a B such that: $B \in L_\alpha$, $B \supset A$ and $L_\alpha(x) \models \overline{\overline{B}} = \text{Max}(\omega_1, \overline{\overline{A}}) < \kappa$. Since $x \in L$, $L_\alpha \models \mu = \overline{\overline{B}} < \kappa$. Let $g \in L_\alpha$ $g: \mu \to B$ bijective, and $c = g^{-1}[A]$; then c is a subset of μ and so $c \in L_\alpha$. It follows that A and $f \in L_\alpha$. $\square$

LEMMA I.2 *Let κ be a cardinal, $x \subset \kappa$, $x \in L$ such that $L_\alpha(x) \models$ ZF + κ singular. Then $x \in L_\alpha$.*

PROOF. By Jensen's covering theorem, $L_\alpha \models \kappa$ is singular and (since $x \in L$) $L_\alpha - \text{cof}(\kappa) = L_\alpha(x) - \text{cof}(\kappa)$. Let $(\kappa_i | i < \lambda) \in L_\alpha$ be a normal sequence converging to κ where $\lambda = L_\alpha - \text{cof}(\kappa)$. Define $f: \lambda \to L_\alpha$ by $f(i) =$ the L-code for $x \cap \kappa_i$. By Lemma I.1, $f \in L_\alpha$ and so $x \in L_\alpha$. $\square$

Case (1) of the Theorem is now clear: If $L_\alpha \models \kappa$ singular by Lemma I.2, $x \in L_\alpha$. Let $\beta < \alpha$ be least such that $x \in L_\beta$. Then clearly α is the least ZF-ordinal greater than β and (since $x \subset \kappa$) $L_\alpha \models \beta < \kappa^+$.

The opposite is trivial: it is enough to take for x a code for an ordinal greater than the ZF-ordinals below α.

II.

LEMMA II.1. *Let κ be a cardinal and α be $\alpha_{\text{ZF}}(x)$ for some $x \subset \kappa$. Then there is a $\beta < \alpha$ and a sequence $(X_n | n < \omega)$ such that* (2(ii)) *and* (2(iii)) *of the Theorem hold.*

PROOF. Let β be such that $L_\alpha(x) \models \beta = \kappa^+$ and set $y_n = \{t \in L_\alpha(x) | t$ is Σ_n-definable in $L_\alpha(x)$ with parameters from $\kappa \cup \{x\}\}$. Then clearly $y_n \in L_\alpha(x)$; $y_n \in y_{n+1}$ and $L_\alpha(x) - \text{card}(y_n) = \kappa$.

Set $y = \bigcup_n y_n$. Clearly $y \prec L_\alpha(x)$.

Set $\pi: y \to^{\cong} L_\gamma(x)$. Then $\gamma = \alpha$ since $L_\alpha(x) \models$ ZF and $L_\alpha(x) \models \forall \delta\, L_\delta(x) \not\models$ ZF. So $y = L_\alpha(x)$ since every element of y is y-definable from $\kappa \cup \{x\}$.

Now let $x_n \in L_\alpha$ be such that $x_n \supset y_n \cap L_\alpha$ and $L_\alpha(x) \models \text{card}(x_n) = \kappa$. Then clearly $L_\alpha = \bigcup x_n$ and $L_\alpha - \text{card}(x_n) < \beta$. $\square$

To prove that (2(i)) is true in the case κ is regular, we shall first assume $\kappa \geqslant \omega_2$. This proof does not work for $\kappa = \omega_1$ since it uses Lemma I.1 which is not true for ω_1 by a theorem of Bukovsky. The proof we shall give for ω_1 works for every regular cardinal κ, but since it is a bit more complicated, it seems useful to give first the simplest one.

LEMMA II.2. *Let κ be a regular cardinal and α be $\alpha_{ZF}(x)$ for some $x \subset \kappa$. Then* (2) *of the Theorem holds.*

PROOF. It remains to show (2(i)); let $L_\alpha(x) \vDash \beta = \kappa^+$.

(*) Assume first $\kappa \geqslant \omega_2$: Let $f\colon \gamma < \kappa \to \mu < \beta$ and let $g \in L_\alpha(x)$ $g\colon \mu \to \kappa$ bijective, and $h = g \circ f\colon \gamma \to \kappa$. Since κ is regular, h is bounded and so $h \in L_k$ and $f \in L_\alpha(x)$. Now using Lemma I.1, $f \in L_\alpha$. Note that we have used here not only the fact that $L_\alpha(x) \vDash \kappa$ is regular, but also the fact that κ is regular.

(**) Assume now $\kappa = \omega_1$: the proof uses the second author's notion of critical projecta defined in [3]. We prove exactly as in [3, Lemmas 9–11] that the ρ_i, ρ_i' have cofinality ω_1, where the ρ_i, ρ_i' are the critical projecta of β and then (this is Theorem 13 in [3]) that (2(i)) holds. □

We now have to prove the converse part. So assume from now on that (2) of the Theorem holds. We have to find $x \subset \kappa$ such that $\alpha = \alpha_{ZF}(x)$. We shall build x by a 3-step forcing iteration over L_α. The main problems are to show that we can find *in L* the generics we need.

We first find an $x_0 \subset \beta$ such that

$$L_\alpha(x_0) \vDash \mathrm{ZF} + \beta = \kappa^+.$$

Since β is regular in L_α it is either a successor cardinal or an inaccessible one.

(*) Assume first $L_\alpha \vDash \beta = \theta^+$ for some $\theta < \beta$ let $\mathbf{P}$ be the usual poset to collapse θ on κ. By (2(i)), $\mathbf{P}$ is $< \kappa$-closed (that means: if $(p_i | \, i < \gamma < \kappa)$ is a decreasing sequence of conditions *in or out of* L_α, then there is a $p \in \mathbf{P}$ such that $p \leqslant p_i \forall i < \gamma$). Since $\overline{\overline{L_\alpha}} = \kappa$ it is easy to find in L a $\mathbf{P}$ generic over L_α and from that an x_0 such that $L_\alpha(x_0) \vDash \mathrm{ZF} + x_0 \subset \kappa + \beta = \kappa^+$.

(**) Assume next $L_\alpha \vDash \beta$ is inaccessible. Let $\mathbf{P}$ be the usual poset to collapse all the cardinals between κ and β: more precisely let $I = L_\alpha$-card $\cap \,]\kappa, \beta[$ and

$$\mathbf{P} = \Big\{ p = (p_j)_{j \in J} | J \subset I \overline{\overline{J}} < \kappa, \mathrm{dom}(p_j) \subset \kappa, \mathrm{card}(\mathrm{dom}\, p_j) < \kappa \\ p_j\colon \mathrm{dom}\, p_j \to j \Big\}.$$

Note that we do not ask $J \in L_\alpha$. Also note that (by (2(i))) $\forall j \in I p_j \in L_\alpha$.

Set $\tilde{\mathbf{P}} = \mathbf{P} \cap L_\alpha$. $\tilde{\mathbf{P}}$ is, in L_α, the usual poset to make $\beta = \kappa^+$.

LEMMA II.3. *Let $D \subset \tilde{\mathbf{P}}$, $D \in L_\alpha$ be dense in $\tilde{\mathbf{P}}$. Then D is predense in* $\mathbf{P}$.

PROOF. Let $A \subset D$ be a maximal antichain in $\tilde{\mathbf{P}}$, $A \in L_\alpha$. Since it is well known that $\tilde{\mathbf{P}}$ has, in L_α, the $< \beta$ chain condition there is a $\theta < \beta$ such that for $p \in A$ $J_p \subset \theta$.

Now let $q \in \mathbf{P}$ and define $\tilde{q}$ by $J_{\tilde{q}} = J_q \cap \theta$ and for $j \in J_{\tilde{q}}\, \tilde{q}_j = q_j$; then, by (2(i)), $\tilde{q} \in \tilde{\mathbf{P}}$. Since A is a maximal antichain $\tilde{q}$ is compatible with some $r \in A$ but since $J_r \subset \theta$, q also is compatible with r. □

Using this lemma it is not difficult to find a $\tilde{\mathbf{P}}$ generic over L_α. Let $(D_i | \, i < \kappa)$ be an enumeration of the open dense subsets of $\tilde{\mathbf{P}}$ in L_α. Define a decreasing sequence $(p_i | \, i < \kappa)$ of elements of $\mathbf{P}$ such that: $\forall i \; p_i \in D_i$ as follows: $p_0 = \varnothing$. Assume $(p_j | \; j < i < \kappa)$ has been defined. Set $p = \bigcup_{j<i} p_j$. Then $p \in \mathbf{P}$. By Lemma II.3 let p_i be the least q such that $q \leqslant p$ and $q \in D_i$.

Set $p_\kappa = \bigcup_{i<\kappa} p_i$. Then $G = \{ q \in \tilde{\mathbf{P}} | \; \forall j \in J_q \, q_j = (p_\kappa)_j \upharpoonright \operatorname{dom} q_j \}$. It is clear that G is $\tilde{\mathbf{P}}$ generic over L_α.

So we have proved

Lemma II.4. *There is a subset x_0 of β such that*

(1) $L_\alpha(x_0) \vDash \mathrm{ZF} + \beta = \kappa^+$;

(2) $L_\alpha(x_0) = \bigcup X_n$ *where* $X_n \in L_\alpha(x_0)$ *and* $L_\alpha(x_0) - \operatorname{card}(X_n) = \kappa$.

In the second step we use the results of [2] to find a subset x_1 of β to kill all the ZF-ordinals. Let $\mathbf{P} = \mathbf{P}_\kappa$ with the notations of [2]. $\mathbf{P}$ is a class in $L_\alpha(x_0)$. It is shown in [2] that in a $\mathbf{P}$ generic extension of $L_\alpha(x_0)$ all the ZF-ordinals are killed and that this extension satisfies $V = L_\alpha(x_1)$ for some $x_1 \subset \beta$. Moreover, $\mathbf{P}$ is κ-distributive in $L_\alpha(x_0)$.

For $n < \omega$ let $(\Delta_i^n | \, i < \kappa)$ be an enumeration of the open dense subclasses of $\mathbf{P}$ definable by a Σ_n-formula with parameters from X_n. By the distributivity of $\mathbf{P}$, $D_n = \bigcap_{i<\kappa} \Delta_i^n$ is an open dense subclass of $\mathbf{P}$.

Define a sequence $(p_n | n < \omega)$ of elements of $\mathbf{P}$ by $p_0 = \varnothing$, $p_{n+1} =$ some $p \leqslant p_n$ such that $p \in D_n$. Then clearly $\bigcup p_n$ is $\mathbf{P}$ generic over $L_\alpha(x_0)$.

It remains now to code x_1 by a subset of κ. So it is enough to show

Lemma II.5. *Let κ be a regular cardinal, α, β be ordinals of cardinality κ, and x a subset of β such that*

$$L_\alpha(x) \vDash \mathrm{ZF} + \beta = \kappa^+.$$

Then there is, in L, a subset y of κ such that

$$L_\alpha(y) \vDash \mathrm{ZF} + \beta = \kappa^+ + x = \{ \xi < \beta | \, S_\xi \cap y \text{ is bounded} \},$$

where $(S_\xi | \; \xi < \beta)$ is some nice sequence of almost disjoint subsets of κ; i.e., S_ξ is uniformly $L_\alpha(x \cap \xi)$-definable.

(*Note.* If β had (true) cofinality κ, there would be no problems since then the forcing that gives y would be $< \kappa$-closed. But here β has cofinality ω!)

Proof. We use Solovay's trick (see [1, p. 12]); the S_ξ are $S(b_\xi)$ where the b_ξ are mutually generic. Let $\mathbf{P}$ be the poset of conditions (not necessarily in $L_\alpha(x)$) to code x by a subset of κ. Let $\tilde{\mathbf{P}} = \mathbf{P} \cap L_\alpha(x)$. The lemma similar to Lemma II.3 with the new forcing is proved in [1, Lemma 1.3, p. 13]. From that it is easy to find the generic we need: Do as after Lemma II.3. □

The proof of the second case is now complete.

III. We assume now that κ is a singular cardinal but $L_\alpha \models \kappa$ is regular.

For the "only if" part of the Theorem we have to prove (3(i)). The proof of that is exactly as in [**4**, Theorem 9], using the critical projecta of β.

To prove (3(i′)) in the case $\operatorname{cof} \kappa = \omega$ we use the following, which is proved in [**4**] (see Theorem 3).

Claim. Assume $\operatorname{cof}(\alpha) = \omega$ and for all $y \in L_\alpha$ there is a $\lambda < \alpha$ and $(y_n | \, n < \omega)$ such that $y = \bigcup y_n$ and $\forall n < \omega \; y_n \in L_\lambda$ and $\operatorname{card}(y_n) < \kappa$. Then there is a tame injection from L_α into κ.

So it is enough to prove the hypothesis of that claim. By Lemma II.1 and by (3(i)), if $y \in L_\alpha$ we can write $y = \bigcup y_n$ with $y_n \in L_\alpha$ and $L_\alpha - \operatorname{card}(y_n) < \kappa$. Now since $y \subset L_\mu$ for some $\mu < \alpha$, $\forall n \; y_n \in L_{\mu^+}$.

To prove the converse part of the Theorem, it is enough to show that we can get a generic for the first and third steps of the iteration given in §II. (The second one is exactly as in §II.) This is done as follows: In each case we have to meet the open dense subsets of some poset $\mathbf{P}$ which is (inside L_α) $< \kappa$-closed (since in L_α, κ is regular).

It is enough to prove

LEMMA III.1. *Let* $\lambda = \operatorname{cof}(\kappa) < \kappa$ *and* $f\colon \gamma < \lambda \to \mu < \beta$. *Then* $f \in L_\alpha$ *(note that —in fact—in the "only if" part of the Theorem this is proved before proving* (3(*i*)), *but it turns out that it is a consequence of it).*

LEMMA III.2. *There is a sequence* $(D_i | \, i < \lambda)$ *of open dense subsets of* $\mathbf{P}$ *such that every subset of* $\mathbf{P}$ *that meets all the* D_i *is* $\mathbf{P}$ *generic over* L_α.

From these lemmas we can find—by the same techniques as in §II—the generics we need.

PROOF OF LEMMA III.1. Let $f\colon \gamma < \lambda \to \mu < \beta$. By (3(i)), there is a $g\colon \mu \to \kappa$ one-one and tame. Let $h = g \circ f\colon \gamma \to \kappa$. Then h is bounded in κ and $h \in L_\alpha$.

But then $f = (g^{-1} \restriction \rho) \circ h$ for some $\rho < \kappa$ and so $f \in L_\alpha$ since g is tame. □

Note that we have used here that $g^{-1} \restriction \rho \in L_\alpha$ for $\rho < \kappa$ and not only $g^{-1}[\rho] \in L_\alpha$. This comes from the fact that $g^{-1} \restriction \rho = g_0 \circ g'$ where $g'\colon \rho \to \rho' \in L_\kappa$, $\rho' = \operatorname{ordertype}(g^{-1}[\rho]) < \kappa$ and $g_0\colon \rho' \to g^{-1}[\rho]$ lists $g^{-1}[\rho]$ in increasing order.

PROOF OF LEMMA III.2. By (2(ii)) there is a sequence $(\Delta_n | \, n < \omega)$ such that $\Delta_n \in L_\alpha$, $L_\alpha - \operatorname{card}(\Delta_n) < \beta$, and $\bigcup_n \Delta_n$ is the set of the open dense subsets of P. Now by (3(i)) there is an enumeration $(\Delta^n_\xi | \, \xi < \kappa)$ of Δ_n for each n such that $(\Delta^n_\xi | \, \xi < \nu) \in L_\alpha$ for each $n < \omega$ and $\nu < \kappa$.

Let $(\kappa_i | \, i < \lambda)$ be a normal sequence converging to κ. Set $D_{n,i} = \bigcap_{\xi < \kappa_i} \Delta^n_\xi$. Then $D_{n,i} \in L_\alpha$ for $n < \omega$ and $i < \lambda$ and since $\mathbf{P}$ is, in L_α, $< \kappa$-closed: $D_{n,i}$ is open dense. It is then enough to rearrange the $\mathrm{D}_{n,i}$'s into a λ-sequence. □

This achieves the proof of the Theorem.

IV. Some final comments.

(1) The Theorem can be easily generalized to sequences of ZF ordinals: following [**2**] we can give *sufficient* conditions for a sequence of length $< \kappa^+$ of ZF

ordinals of cardinality κ to be an initial segment of the α, such that $L_\alpha(x)$ is a model of ZF, for some $x \subset \kappa$. As in [2] the essential fact is to assume $\operatorname{Sup} Q \cap \alpha < \alpha$ for $\alpha \in Q$, where Q is the given sequence.

(2) It would be interesting to find some classes A (or for which classes?) for which there is a subset x of ω such that A is exactly the class of the α such that $L_\alpha(x)$ is a model of ZF. This is done in [5] for KP instead of ZF.

(3) Finally note that in the Theorem (2(ii)) cannot be replaced by a simple condition on the cofinality of α and β; for example $\operatorname{cof}(\alpha) = \operatorname{cof}(\beta) = \omega$. To see that, assume that there is a $\bar{\beta}$ such that

$$\omega_1 < \bar{\beta} < \omega_2 \quad \text{and} \quad L_{\bar{\beta}+3} \models \bar{\beta} \text{ is inaccessible.}$$

We shall find α of cofinality ω for which there is no sequence $(X_n | \, n < \omega)$ satisfying (2(ii)): we first find a γ such that

(*) ($L_{\gamma+2} \models \gamma$ is inaccessable) and ($\operatorname{cof} \gamma = \omega_1$) and (for $\delta < \gamma$ if $L_\gamma \models \delta$ regular then $\operatorname{cof}(\delta) = \omega_1$). (Define $x_0 =$ the Skolem Hull of ω_1 in $L_{\bar{\beta}+2}$;

$$x_{i+1} = \operatorname{SH}\big(x_i \cup \{x_i\}, L_{\bar{\beta}+2}\big); \qquad x_i = \bigcup_{j<i} x_j \quad \text{for limit } i.$$

Let $\pi\colon x_{\omega_1} \to^{\cong} L_{\gamma+2}$. It is easy to see that γ has the desired properties.)

Now define the sequence $(\alpha_n)_{n<\omega}$ as follows: $\alpha_0 = \omega_1$; $\alpha_{2n+1} = \alpha_{2n}^+$ in the sense of L_γ; $\alpha_{2n+2} =$ the least $\alpha > \alpha_{2n+1}$ such that $L_\alpha \prec L_\gamma$ (such an α exists since $L_{\gamma+2} \models \gamma$ is inaccessible). Set $\alpha = \bigcup \alpha_n$. Let $L_\alpha \prec L_\gamma$ so α is a ZF ordinal and $\operatorname{cof}(\alpha) = \omega$. Assume there is a $\beta < \alpha$ and a sequence $(x_n | \, n < \omega)$ such that (2(ii)) holds. Choose $\mu = \alpha_{2n+1} > \beta$; then, by (*) $\operatorname{cof}(\mu) = \omega_1$ but $\mu = \bigcup_n (x_n \cap \mu)$, and since $L_\alpha - \operatorname{card}(x_n \cap \mu) < \mu$, $\operatorname{cof}(\mu) = \omega$, a contradiction.

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