Annals of Mathematical Logic 23 (1982) 121–125
North-Holland Publishing Company

Δ_3^1 REALS

René DAVID

Départment de Mathématique, Université Toulouse-le-Mirail, 109 bis, rue Vauquelin, 31058 Toulouse Cedex, France

Received 8 September 1982

In this paper I use the method developed in [2] to give an answer to a question of L. Harrington (see [3]).

Theorem. *The following theories are equiconsistent*:

(1) ZF+ *there is an inaccessible cardinal.*

(2) ZFC+GCH+$\forall\, x, y \subset \omega$ $(y \in L(x) \rightarrow y$ *is* $\Delta_3^1(x))$.

One part of this result is trivial since in a model of (2) the cardinal χ_1 is easily seen to be inaccessible in L.

Also note that (2) is a consequence of ZFC+GCH+$\forall\, x \subset \omega\, x^{\#}$ exists; the theorem shows that it is a *strict* consequence of it.

To prove the other part of the theorem I start with a model M_0 of ZF+ $V=L$. Let κ be the least inaccessible cardinal in M_0. I shall construct a generic extension N of M_0 in which κ will be χ_1 and that satisfies (2).

The idea is the following: I shall add a sequence $(a_\alpha)_{\alpha<\kappa}$ of subsets of ω such that for all $\alpha < \kappa$:

(1) $\chi_1^{L(a_\alpha)} = \chi_{\alpha+2}^L$ and there is a $\Delta_1^1(a_\alpha)$ code for $\chi_{\alpha+1}^L$;

(2) a_α is—in $L((a_\beta) \mid \beta < \kappa)$—$\Delta_3^1(x)$ for every code x of α.

(To ensure (2) I use the method developed in [2] to make a_α 'very absolutely' $\Delta_3^1(x)$, i.e. $\Delta_3^1(x)$ even when $\chi_1^{L(a_\alpha)}$ is collapsed.) Then in $N = L((a_\beta) \mid \beta < \kappa)$ if $x, y \subset \omega$ and $\chi_1^{L(x)} = \chi_{\alpha+1}^L$ and $y \in L(x)$, then y is Δ_3^1 in any code for α and so in x.

Now the proof is: Following [2] I can easily build a sequence $(T(n, \alpha) \mid n \in \omega, \alpha < \kappa)$ of Suslin trees of height κ^{++} such that if s is a finite subset of ω and $\alpha \in \kappa$ and

$$T^*(s, \alpha) = \prod_{\substack{n \in \omega \\ \beta \neq \alpha}} T(n, \beta) \times \prod_{n \in s} T(n, \alpha),$$

$T^*(s, \alpha)$ being ordered componentwise, then if N is a $T^*(s, \alpha)$ generic extension of M_0, for $n \in s T(n, \alpha)$ remains Suslin in N.

Let $P(n, \alpha)$ be the forcing to add a branch in $T(n, \alpha)$ and then 'nicely' code it—using Jensen's coding machinery—by a subset $A(n, \alpha)$ of $[\chi_{\alpha+1}, \chi_{\alpha+2}[$ in such a way that the following is true:

$$\forall \beta, \gamma, \delta \, [\textit{if } \chi_{\alpha+1} < \beta < \gamma < \delta \text{ and } L_\delta(A(n, \alpha) \cap \beta)$$
$$\models \mathrm{ZF}^- + \beta = \chi^L_{\alpha+2} + \gamma \text{ is the least inaccessible cardinal}$$
$$+ \text{every set has cardinality at most } (\gamma^+)^L,$$
$$\textit{then} \text{ when } A(n, \alpha) \text{ is decoded}$$
$$\text{—using Jensen's machinery}$$
$$\text{—a cofinal branch is found in the}$$
$$\text{tree } T(n, \alpha) \text{ calculated in } L_\delta].$$

Note that, in this formula, β and γ need not to be the true $\chi_{\alpha+2}$ and κ. In particular this is true even for $\delta <$ the true $\chi_{\alpha+2}$. (Here is one of the essential tricks in [2].)

Let P be the product of the $P(n, \alpha)$: i.e. $p \in P$ iff p is a function with domain $\omega \times \kappa$ and $\forall (n, \alpha) \in \omega \times \kappa \; p(n, \alpha) \in P(n, \alpha)$.

Lemma 1. *Forcing with P preserves cardinals and cofinalities.*

Proof. It follows from the standard methods in [4] and [2], using chain condition and distributivity. The fact that we have here a product causes no trouble: this is clear for cardinals $> \kappa$. Let me look at it for cardinals $< \kappa$. Let $\beta < \kappa$; each $P(n, \alpha)$ is the iteration of its 'high' part $P_{\chi_{\beta+2}}(n, \alpha)$ and of its 'low' part $P^{\chi_{\beta+1}}(n, \alpha)$.

Since the high part is $\chi_{\beta+1}$-distributive P is (isomorphic) to $C_1 * C_2$ (i.e. forcing first with C_1 and then with C_2) where:

$$C_1 = \prod_{n,\alpha} P_{\chi_{\beta+1}}(n, \alpha),$$

which is proved to be $\chi_{\beta+1}$-distributive as in [2], and

$$C_2 = \prod_{n,\alpha} P^{\chi_{\beta+1}}(n, \alpha) = \prod_{\substack{n<\omega \\ \alpha \leq \beta}} P^{\chi_{\beta+1}}(n, \alpha)$$

(the last fact comes from that for $\alpha > \beta$ $P^{\chi_{\beta+1}}(n, \alpha)$ is the trivial poset; remind that $P(n, \alpha)$ codes to a subset of $[\chi_{\alpha+1}, \chi_{\alpha+2}[$. Now for every n, α, β there is a function $G: P^{\chi_{\beta+1}}(n, \alpha) \to \chi_{\beta+1}$ such that: $G(p) = G(q) \to p$ and q are compatible and so $\prod_{n \in \omega, \alpha \leq \beta} P^{\chi_{\beta+1}}(n, \alpha)$ satisfies the $\leq \chi_{\beta+1}$ chain condition.

The lemma follows then easily. □

Let M_1 be a P generic extension of M_0. Note that M_1 has the same subsets of ω as M_0 and that it satisfies:

$$V = L(A(n, \alpha) \mid n < \omega, \alpha < \kappa) + \kappa \text{ is inaccessible.}$$

Now work in M_1.

For $\alpha<\kappa$ let Q_α be the notion of forcing that:

(1) First collapses $\chi_{\alpha+1}$ on ω in the usual way; let g_α be a subset of ω of order type $\chi_{\alpha+1}$.

(2) Next adds a subset a_α of ω such that:

(i) a_α codes—using almost disjoint subsets of ω—g_α in such a way that for some function F which is arithmetic and uniform in α $g_\alpha=F(a_\alpha)$;

(ii) if $\{\xi<\chi^L_{\alpha+2} \mid S(n,\xi)\cap a_\alpha \text{ is finite}\}=D^\alpha_n$, then $D^\alpha_n=A(n,\alpha)$ if $n=2i$ (resp. $2i+1$) and $a_\alpha(\mathrm{i})=1$ (resp. 0) and $D^\alpha_n=\emptyset$ if $n=2i$ (resp. $2i+1$) and $a_\alpha(i)=0$ (resp. 1) where $(S(n,\xi)\mid n<\omega,\ \xi<\chi^L_{\alpha+2})$ is—in $L(g_\alpha)$—a family of almost disjoint subsets of ω.

(This coding is the second trick in [2].)

Let $Q=\prod_{\alpha<\kappa}Q_\alpha$ where $q\in Q$ iff q is a function whose domain is a finite subset of κ and for $\alpha\in\mathrm{dom}(q)$: $q(\alpha)\in Q_\alpha$.

Let M_2 be a Q generic extension of M_1. It follows from usual method that the cardinals not less than κ are preserved and that κ is χ_1 in M_2 (this comes from the fact that $\mathrm{card}(Q_\alpha)<\kappa$ for every $\alpha<\kappa$).

Let N be $L(a_\alpha \mid \alpha<\kappa)$. N is the model that satisfies (2) of the theorem. Since N satisfies: $\exists\, A\subset\chi_1\ V=L(A)$ it satisfies ZFC+GCH so it remains to show that: $\forall\, x, y\subset\omega\ (y\in L(x)\to y \text{ is } \Delta^1_3(x))$, is true in N; first note that:

Lemma 2. *In N*:

$$\forall\, x\subset\omega\, \chi_1^{L(x)}<\chi_1.$$

The following is the key of the proof.

Lemma 3. *The following holds in N: For $n<\omega$, $\alpha<\kappa$, $T(n,\alpha)$ is Suslin iff ($n=2i$ and $a_\alpha(i)=0$) or ($n=2i+1$ and $a_\alpha(i)=1$).*

Proof. The 'only if' part is trivial since, for example, if $n=2i$ and $a_\alpha(i)=1$ a_α codes $A(n,\alpha)$ and so in $L(a_\alpha)$ there is a branch in $T(n,\alpha)$.

The 'if' part follows—as in [2]—from the next two lemmas.

Lemma 4. *Let $n<\omega$, $\alpha<\kappa$ and s be a finite subset of ω such that $n\in s$; there are posets C_i ($i=1,2,3$) such that:*

(1) $P*Q\simeq(C_1*C_2)\times C_3$ *where*

$$C_1=\prod_{\substack{m\in\omega\\ \beta\neq\alpha}}P(m,\beta)\times\prod_{m\notin s}P(m,\alpha),$$

$$C_2=\prod_{\beta\neq\alpha}Q_\beta\times Q^s_\alpha$$

where Q^ρ_α is the subset of Q_α such that if a_α is any generic over Q^s_α, then $a_\alpha\supset s$. (For

For $\alpha<\kappa$ let Q_α be the notion of forcing that:

(1) First collapses $\chi_{\alpha+1}$ on ω in the usual way; let g_α be a subset of ω of order type $\chi_{\alpha+1}$.

(2) Next adds a subset a_α of ω such that:

(i) a_α codes—using almost disjoint subsets of ω—g_α in such a way that for some function F which is arithmetic and uniform in α $g_\alpha=F(a_\alpha)$;

(ii) if $\{\xi<\chi^L_{\alpha+2} \mid S(n,\xi)\cap a_\alpha \text{ is finite}\}=D^\alpha_n$, then $D^\alpha_n=A(n,\alpha)$ if $n=2i$ (resp. $2i+1$) and $a_\alpha(i)=1$ (resp. 0) and $D^\alpha_n=\emptyset$ if $n=2i$ (resp. $2i+1$) and $a_\alpha(i)=0$ (resp. 1) where $(S(n,\xi)\mid n<\omega,\ \xi<\chi^L_{\alpha+2})$ is—in $L(g_\alpha)$—a family of almost disjoint subsets of ω.

(This coding is the second trick in [2].)

Let $Q=\prod_{\alpha<\kappa} Q_\alpha$ where $q\in Q$ iff q is a function whose domain is a finite subset of κ and for $\alpha\in\operatorname{dom}(q)$: $q(\alpha)\in Q_\alpha$.

Let M_2 be a Q generic extension of M_1. It follows from usual method that the cardinals not less than κ are preserved and that κ is χ_1 in M_2 (this comes from the fact that $\operatorname{card}(Q_\alpha)<\kappa$ for every $\alpha<\kappa$).

Let N be $L(a_\alpha\mid\alpha<\kappa)$. N is the model that satisfies (2) of the theorem. Since N satisfies: $\exists\, A\subset\chi_1\ V=L(A)$ it satisfies ZFC+GCH so it remains to show that: $\forall\, x,y\subset\omega\ (y\in L(x)\to y \text{ is } \Delta^1_3(x))$, is true in N; first note that:

Lemma 2. *In N:*

$$\forall\, x\subset\omega\ \chi_1^{L(x)}<\chi_1.$$

The following is the key of the proof.

Lemma 3. *The following holds in N: For $n<\omega$, $\alpha<\kappa$, $T(n,\alpha)$ is Suslin iff ($n=2i$ and $a_\alpha(i)=0$) or ($n=2i+1$ and $a_\alpha(i)=1$).*

Proof. The 'only if' part is trivial since, for example, if $n=2i$ and $a_\alpha(i)=1$ a_α codes $A(n,\alpha)$ and so in $L(a_\alpha)$ there is a branch in $T(n,\alpha)$.

The 'if' part follows—as in [2]—from the next two lemmas.

Lemma 4. *Let $n<\omega$, $\alpha<\kappa$ and s be a finite subset of ω such that $n\in s$; there are posets C_i $(i=1,2,3)$ such that:*

(1) $P*Q\simeq(C_1*C_2)\times C_3$ where

$$C_1=\prod_{\substack{m\in\omega\\ \beta\neq\alpha}} P(m,\beta)\times\prod_{m\notin s} P(m,\alpha),$$

$$C_2=\prod_{\beta\neq\alpha} Q_\beta\times Q^s_\alpha$$

where Q^s_α is the subset of Q_α such that if a_α is any generic over Q^s_α, then $a_\alpha\supset s$. (For

Now let x, $y \subset \omega$ be such that $y \in L(x)$. Let α be such that $\chi_1^{L(x)} = \chi_{\alpha+1}^L$ (remind that κ is the least inaccessible in L and so the regular cardinals below κ are—in L—the successor ones); let β be such that $L(x) \models \phi$ (y, β, x) that is, y is the βth real in $L(x)$; β is clearly $\Delta_1(g_\alpha)$ and so Δ_1 (a_α); so using Lemma 7 it is enough to show:

Lemma 9. *Let* $x \in N$ *be such that*: $\chi_1^{L(x)} = \chi_{\alpha+1}^L$; *then the predicates:* '$y \in \mathrm{WO}$ *and* $|y| = \alpha$' *and* '$y \in \mathrm{WO}$ *and* $|y| = \chi_{\alpha+1}^L$' *are* $\Sigma_3^1(x)$.

Proof. (1) $|y| = \alpha \leftrightarrow \exists\, t, z$ $(t, z \in \mathrm{WO}$ and $|t| = |y|$ and $t \in L(x)$ and $|z| > \chi_1^{L(x)}$ and $L_{|z|}(x) \models \chi_1 = \chi_{|t|+1}^L$ and this is $\Sigma_3^1(x)$ since $L_{|z|}(x) \models \psi(t)$ is $\Sigma_2^1(x, z, t)$ and $|z| = \chi_1^{L(x)}$ (and then $|z| > \chi_1^{L(x)}$ too) is $\Sigma_3^1(x, z)$.

(2) is as (1), replacing $\chi_1 = \chi_{|t|+1}$ by $\chi_1 = |t|$. □

This achieves the proof of the theorem.

References

[1] A. Beller, R.B. Jensen and P. Welch, Coding the Universe (Cambridge University Press, London, 1981).

[2] R. David, A very absolute Π_2^1 real singleton, Ann. Math. Logic 23 (1982) 101–120. [This issue.]

[3] H. Friedman, 102 problems in Mathematical Logic, J. Symbolic Logic 40 (2) (1975).

[4] R.B. Jensen, Coding the universe by a real, Notes (1975).

[5] R.B. Jensen and R. Solovay, Some applications of almost disjoint sets, in: Y. Bar Hillel, ed., Mathematical Logic and Foundation of Set Theory (North-Holland, Amsterdam, 1970).