A direct proof of the confluence of combinatory strong reduction

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Abstract

I give a proof of the confluence of combinatory strong reduction that does not use the one of λ-calculus. I also give simple and direct proofs of a standardization theorem for this reduction and the strong normalization of simply typed terms.

1 Introduction

Combinatory Logic (see [2], [3]) is a first order language that simulates the λ-calculus without using bounded variables. But, at present, the known proofs of confluence are all based on the confluence of the λ-calculus which has to be proved before and thus Combinatory Logic is not a self-contained theory. The question of getting a direct proof of this confluence was raised long ago in [2] and appears in the TLCA list of open problems. I give here such a proof.

The paper is organized as follows. Section 2 gives the main definitions of Combinatory Logic, states the theorem and the idea of the proof. Section 3 gives the proof of the confluence of an auxiliary system. Section 4 gives the equivalence of the two systems and deduce the confluence of the original one. Section 5 gives a standardization theorem and section 6 gives a direct proof of strong normalization for simply typed terms. Finally, I conclude in section 7 with some remarks.

2 The idea of the proof of confluence

2.1 Combinatory Logic

Definition 1 *The set C of combinators is defined by the following grammar (where x denotes a variable)*

$$C := x \mid K \mid S \mid I \mid (C\ C)$$

In the literature, the objects determined by this grammar are usually called CL-terms and the word combinator is given for *closed* CL-terms. However since, in section 3, the word term will be used for something slightly different, I prefer to keep the word combinator here.

Definition 2 *For $u \in C$, the term $[x]u$ is defined, by induction on u, by the following rules*

1. $[x]u = Ku$ *if* $x \notin u$
2. $[x]x = I$
3. $[x](u\ x) = u$ *if* $x \notin u$
4. $[x](u\ v) = (S\ [x]u\ [x]v)$ *if none of the previous rules apply.*

Definition 3 *The reduction on combinators is the closure by contexts of the following rules.*

1. $(K\ u\ v) \succ u \qquad (S\ u\ v\ w) \succ (u\ w\ (v\ w)) \qquad (I\ u) \succ u$
2. $[x]u \succ [x]v \qquad$ *if* $u \succ v$

I recall here usual notions about reductions.

Definition 4 *Let* $\rightarrow$ *be a notion of reduction.*

- *As usual,* $\rightarrow^*$ *denotes the reflexive and transitive closure of* $\rightarrow$*.*
- *The reduction* $\rightarrow$ *is locally confluent if, for any term* u*, the following holds. If* $u \rightarrow u_1$ *and* $u \rightarrow u_2$*, then* $u_1 \rightarrow^* u_3$ *and* $u_2 \rightarrow^* u_3$ *for some* u_3*.*
- *The reduction* $\rightarrow$ *commutes with the reduction* $\rightarrow_1$ *if, for any term* u*, the following holds. If* $u \rightarrow^* u_1$ *and* $u \rightarrow_1^* u_2$ *then* $u_1 \rightarrow_1^* u_3$*,* $u_2 \rightarrow^* u_3$ *for some* u_3
- *The reduction* $\rightarrow$ *is confluent if it commutes with itself.*
- *A term* u *is strongly normalizing (denoted as* $u \in SN$*) if there is no infinite reduction of* u*.*

The main result of this paper is the following theorem.

Theorem 5 *The reduction* $\succ$ *on combinators is confluent.*

Note that without rule (2) of definition 3 (this reduction is then called weak reduction), the confluence would be trivially proved by the method of parallel reductions. This rule is added to have the equivalence of combinatory logic (denote as LC) and λ-calculus (denoted as Λ) in the following sense. Let H be the translation between Λ and LC defined by $H(x) = x$, $H((u_1\ u_2)) = (H(u_1)\ Hu(_2))$ and $H(\lambda x.u) = [x]H(u)$. Without this rule, the compatibility property between Λ and LC (i.e. if t reduces to t', then $H(t)$ reduces to $H(t')$) would not be true. This is because, the reduction in LC will not allow a reduction below a λ. For example, let $t = \lambda x.(\lambda y.x\ x)$. Then $H(t) = [x](K\ x\ x) = (S\ K\ I)$ is normal whereas t is not.

Also note that the confluence would not be true if clause (3) of definition 2 (which corresponds, intuitively, to the η-equality of the λ-calculus) in the definition of $[x]u$ had been omitted. Here is an example. Let $t = [x](K\ y\ x)$ and $u = (S\ t\ (K\ z))$. If clause (3) is omitted, then t reduces in one step to $[x]y = (K\ y)$ whereas, with clause (3), $t = (K\ y)$). Thus $u \rightarrow u_1 = (S\ (K\ y)\ (K\ z))$. On the other hand $u = [x](K\ y\ x\ z) \rightarrow [x](y\ z) = (K\ (y\ z)) = u_2$ and u_1, u_2 are not reducible to a common term.

2.2 The idea of the proof

I want to prove the confluence by using the same method as in [1] i.e. by proving first a theorem on finiteness of developments. Then, by this theorem, Newman's Lemma and the local confluence of the developments we get the confluence of developments. Then it remains to show that the reduction itself is the transitive closure of the developments.

But the given system is quite hard to study because it is difficult to mark the redexes and thus to give a precise definition for a theorem on finiteness of developments. This is also because the form of a term does not determine easily its redexes. The main technical reason is the following. We should think that any reduct of $[x]u$ would have the form $[x]u'$ for some reduct u' of u. But this property, which is trivial in the λ-calculus, is not true here. Here is an example. Let u, v be combinators, x be a variable that occurs both in u and v and let $t = [x](u\ v) = (S\ [x]u\ [x]v)$. Then, it is easy to check that $t = ([y][x](u\ (y\ x))\ [x]v)$. Now if $u = (S\ u_1\ u_2)$ then t reduces to $t' = ([y][x](u_1\ (y\ x)\ (u_2\ (y\ x)))\ [x]v)$ and it is easy to check that t' cannot be written as $[x]w$ for some reduct w of $(u\ v)$. Note that, in the λ-calculus, the corresponding equality i.e. $\lambda x.(u\ v) = (\lambda yx.(u\ (y\ x))\ \lambda x.v)$ needs β-reductions and not only η-reductions whereas in Combinatory Logic it only comes from the η-rule.

Thus I will first prove the confluence of an auxiliary system. This system will be shown to be equivalent to the other one in the sense that the symmetric and transitive closure of both systems are the same. Then I will deduce the confluence of the first system from the one of the second.

The auxiliary system treats separately the reductions that, intuitively, corresponds in the λ-calculus to β and η. To prove the confluence of this system, I prove the confluence of β. This is done, as mentioned above, by proving a theorem on finiteness of developments. Note that the fact that the reduction is the transitive closure of developments (which is trivial in the λ-calculus) is not so easy here. I deduce the confluence of the whole system (intuitively β and η) by another commutation lemma.

Lemma 6 (Newman's Lemma) *Let $\rightarrow$ be a notion of reduction that is locally confluent and strongly normalizing. Then $\rightarrow$ is confluent.*

3 An auxiliary system

To define this new system, I first remove the η-equality in the definition of the abstraction.

Definition 7

1. $\lambda x.u = (K\ u)$ *if* $x \notin u$
2. $\lambda x.x = I$
3. $\lambda x.(u\ v) = (S\ \lambda x.u\ \lambda x.v)$ *if none of the previous rules apply.*

and I add new reduction rules. Rule (2) is necessary to have confluence. Rule (3) corresponds to the η-reduction and is necessary to have the equivalence with the other system.

Definition 8

1. $(K\ u\ v) \rightarrow u \qquad (S\ u\ v\ w) \rightarrow (u\ w\ (v\ w)) \qquad (I\ u) \rightarrow u$
2. $(S\ (K\ u)\ (K\ v)) \rightarrow (K\ (u\ v))$
3. $(S\ (K\ u)\ I) \rightarrow u$

4. $\lambda x.u \to \lambda x.v$ *if* $u \to v$

Theorem 9 *The reduction* $\to$ *on combinators is confluent.*

As mentioned before, to prove this theorem I first prove the confluence of the system where the η-reduction (i.e. rule (3) of definition 8) has been removed. The theorem on finiteness of developments of this system can be formalized as theorem 29 below. I need some new definitions.

3.1 Some definitions

Definition 10 *Let* V *be an infinite set of variables.*

- *Let* $A = V \cup \{S_i \ / \ i = 0,1,2,3\} \cup \{K_i \ / \ i = 0,1\} \cup \{I_i \ / \ i = 0,1\}$. *The elements of* A *will be called atoms.*
- *The set of terms is defined by the following grammar*

$$T := A \mid \quad (T\ T)$$

The meaning of the indices on S, K, I is the following. First, I want to mark the redexes that are allowed to be reduced. I do this by simply indexing the letters S, K, I. The index 0 means that the symbol is not marked (i.e. we are not allowed to reduce the corresponding redex), the index 1 means that the redex is allowed.

I also want to indicate whether or not a combinator S, K, I is the first symbol of a term of the form $\lambda x.u$ for which I want to reduce in u. Actually, for K, I there is nothing to do because a variable has no redex and, since $\lambda x.u = (K\ u)$ when x does not occur in u, the redexes in u are, in fact, already visible at the top level. But for S this will be useful and I need thus 4 indices.

- S_0 is an S that is neither marked nor introduced by a λ,
- S_1 is an S that is marked but not introduced by a λ,
- S_2 is an S that is not marked but introduced by a λ
- S_3 is an S that is marked and introduced by a λ.

Definition 11 *Let* u *be a term and* x *be a variable. I define, for* $i = 0, 1$ *the set of terms (denoted as* $\lambda_i x.t$*) by the following rules.*

1. *if* $t = x$, $\lambda_i x.t = \{I_i\}$
2. *if* $t \neq x$ *is an atom,* $\lambda_i x.t = \{(K_i\ t)\}$
3. *if* $t = (u\ v)$ *and* $x \notin t$, $\lambda_i x.t = \{(K_i\ t)\} \cup \{(S_{i+2}\ u'\ v') \mid u' \in \lambda_i x.u, v' \in \lambda_i x.v\}$
4. *if* $t = (u\ v)$ *and* $x \in t$, $\lambda_i x.t = \{(S_{i+2}\ u'\ v') \mid u' \in \lambda_i x.u, v' \in \lambda_i x.v\}$.

The reason of this unusual definition and, in particular, the fact that $\lambda_i x.t$ represents a set of terms instead of a single term, is the following. It will be useful to ensure that the set of terms of the form $\lambda x.u$ is closed by reduction. But this is not true if the abstraction is defined by the rules of definition 7.

Here is an example. Assume x occurs in u but not in v and u reduces to u' for some u' that does not contain x. Then $t = \lambda x.(u\ v)$ reduces to $\lambda x.(u'\ v) = (K\ (u'\ v))$. But $t = (S\ \lambda x.u\ (K\ v))$ reduces to $(S\ \lambda x.u'\ (K\ v)) = (S\ (K\ u')\ (K\ v))$ which is not of the form $\lambda x.w$. Allowing, in such a case, both $(K\ (u'\ v))$ and $(S\ (K\ u')\ (K\ v))$ to be in $\lambda x.(u'\ v)$ will repair this problem.

The given definition is then an indexed version of this idea. The index 1 (resp. 0) will mean that the S, K, I introduced by the definition are marked (resp. are not marked) and thus allow a redex to be reduced. Note that the $i+2$ indexing S means (depending whether $i = 1$ or $i = 0$) that S comes from a λ and is (or is not) marked.

Definition 12 *The reduction on terms is the closure by contexts of the following rules*

1. *(a) For* $i = 1, 3$ $(S_i\ u\ v\ w) \triangleright (u\ w\ (v\ w))$
   (b) $(K_1\ u\ v) \triangleright u$ *and* $(I_1\ u) \triangleright u$
   (c) For $i = 0, 1$ $(S_{i+2}\ (K_i\ u)\ (K_i\ v)) \triangleright (K_i\ (u\ v))$
   (d) For $i = 0, 1$, *if* $u \triangleright v$, $t \in \lambda_i x.u$ *and* $t' \in \lambda_i x.v$, *then* $t \triangleright t'$
2. *The level of a reduction (denoted as* $lvl(t \triangleright t')$*) is defined as follows.*
   - *If the reduction uses the rules (a,b,c), the level is 0.*
   - *If the reduction uses the rule (d), the level is* $lvl(u \triangleright v) + 1$.

These rules are the indexed version of the rules (1, 2, 4) of definition 8. Note that, for example, if x does not occur in $(u\ v)$ and $u \triangleright u'$, then $(K\ (u\ v)) \triangleright (S\ (K\ u')\ (K\ v))$. Also note that the level is a property of the whole reduction, not just its final conclusion. For example, if $u \triangleright u'$, the reduction $(S_3\ (K_1\ u)\ (K_1\ v)) \triangleright (K_1\ (u'\ v))$ can be deduced in 2 ways, giving level 1 and level 0.

3.2 Fair terms

Notation 13
- *Let* E *be a set of terms and* $\overrightarrow{u}$ *be a sequence of terms (resp.* f *be function into terms). I will write* $\overrightarrow{u} \in E$ *(resp.* $f \in E$*) to express the fact that each term of the sequence* $\overrightarrow{u}$ *(resp. in the image of* f*) is in* E.
- *Let* $\overrightarrow{u}$ *be a finite (possibly empty) sequence of terms and* v *be a term. I denote by* $(v\ \overrightarrow{u})$ *the term* $(v\ u_1\ ...\ u_n)$ *where* $\overrightarrow{u} = u_1, ..., u_n$.

Definition 14
- *An address is a finite list of elements of the set* $\{l, r\}$.
- *The empty list will be denoted by* ε *and* $[a :: l]$ *(resp.* $[l :: a]$*) will denote the list obtained from* a *by adding* l *at the end (resp. at the beginning) of* a *and similarly for* r.
- *If* a, a' *are addresses, I will denote by* $a < a'$ *the fact that* a *is an initial segment of* a'.
- *The* u *be a term. I will denote by* u_a *the sub-term of* u *at the address* a. *More precisely,* u_a *is defined by the following rules:* $u_\varepsilon = u$, $(u\ v)_{[l::a]} = u_a$ *and* $(u\ v)_{[r::a]} = v_a$.

Definition 15
- *Let* u *be a term and* f *be a function from a set* E *of addresses in* u *into terms. I say that* f *is adequate for* u *(I will also say* (u, f) *is adequate) if there are no addresses* a, a' *in* E *such that* $a < a'$.
- *Let* (u, f) *be adequate and* x *be a variable. Then* $\phi_x(u, f)$ *is a term obtained by replacing in* u, *for each* $a \in dom(f)$, *the term at address* a *by* $(w_a\ f(a))$ *for some* $w_a \in \lambda_1 x.u_a$.

- *Let u be a term, $x_1, ..., x_n$ (resp. $f_1, ..., f_n$) be a sequence (possibly empty) of variables (resp. of functions). The term $\phi_{x_1}(\phi_{x_2}(...(\phi_{x_n}(u, f_n), f_{n-1})...)f_1)$ will be denoted by $\phi(u, \overrightarrow{x}, \overrightarrow{f})$ or simply $\phi(u)$ if we do not need to mention explicitly $\overrightarrow{x}$, $\overrightarrow{f}$ or if they are clear from the context.*

In case f is constant and thus, for some v, we have $f(a) = v$ for each $a \in dom(f)$, a term of the form $\phi_x(u, f)$ is the term obtained from $(w\ v)$ for some $w \in \lambda_1 x.u$ by reducing the head redex but where the substitution $[x := v]$ has been only partly propagated in u.

Definition 16 *The set F of fair terms is defined by the following grammar.*

1. *x, S_0, K_0, I_0 are fair*
2. *If u, v are fair then so is $(u\ v)$.*
3. *If u is fair and $t \in \lambda_0 x.u$ then so is t.*
4. *If v_1, v_2, v_3 are fair, then so are $(S_1\ v_1\ v_2\ v_3)$, $(K_1\ v_1\ v_2)$ and $(I_1\ v_1)$*
5. *If x is a variable, $u, f \in F$ and (u, f) is adequate, then $\phi_x(u, f)$ is fair.*

Fair terms are thus combinators where we have marked the redexes that are allowed to be reduced. The terms of the form $\phi_x(u, f)$ are introduced for the following reason. If $t = (w\ v)$ for some $w \in \lambda_1 x.u$, I may want to reduce both a redex in u and t as a redex. Thus the set of fair terms must be closed by the following rule: (6) If u, v are fair then so is $t = (w\ v)$ for $w \in \lambda_1 x.u$. But, if I had defined fair terms by rules 1, 2, 3, 4 and 6, then F will not be closed by reduction because, if $w \in \lambda_1 x.u$, the reduct of $t = (w\ v)$ will not necessarily be fair. The reason is the following. Let $u = (u_1\ u_2)$ be such that u is fair but u_1 is not (for example $u_1 = (K_1\ y), u_2 = y$). Then $v = (\lambda_1 x.u\ z)$ is fair. But $v \triangleright v' = (\lambda_1 x.u_1\ z)\ (\lambda_1 x.u_2\ z)$ and v' may not be fair since u_1 is not.

Definition 17 *Let u be fair. I denote by $nb(u)$ the number of rules that have been used to prove that u is fair.*

3.3 Some properties of fair terms

Lemma 18 *The set of fair terms is closed by substitutions.*

Proof Note that, since x is a bound variable in t for $t \in \lambda_i x.u$ or $t = \phi_x(u, f)$, I assume, when computing $\sigma(t)$, that x does occur in the image of σ. The proof is by an immediate induction on $nb(u)$. Use the fact that, if $t \in \lambda_i x.u$, then $\sigma(t) \in \lambda_i x.\sigma(u)$. □

Lemma 19 *Let $t = (\alpha\ \overrightarrow{u}) \in F$ where α is an atom.*

1. *If α is S_2, then $lg(\overrightarrow{u}) \geq 2$. If α is S_1 or S_3, then $lg(\overrightarrow{u}) \geq 3$.*
2. *If α is K_1, then $lg(\overrightarrow{u}) \geq 2$. If α is I_1, then $lg(\overrightarrow{u}) \geq 1$.*

Proof By induction on $nb(t)$. I only look at the cases with S. The other ones are similar.

- If the last rule that has been used to prove $t \in F$ is (2) of definition 16, the result follows immediately from the *IH*. If it is rule (4) the result is trivial.
- If it is rule (3). If $\alpha = S_2$, the result is also trivial. The other cases are impossible.

- If it is rule (5) and $(\alpha\ \overrightarrow{u}) = \phi_y(v, f)$. Let a be the leftmost address in $dom(f)$. For $\alpha = S_1$ (resp. $\alpha = S_2$) we may not have $a = [l, l, ..., l]$ since this will imply that t begins with S_3. Thus $v = (S_1\ \overrightarrow{w})$ (resp. $v = (S_2\ \overrightarrow{w})$) and the result follows from the *IH*. For $\alpha = S_3$, if the leftmost address is not of the form $[l, l, ..., l]$ the result is as before. Otherwise, this implies that $t = (w_a\ f(a)\ \overrightarrow{s})$ for some $w_a \in \lambda_1 y.v_a$ and some $\overrightarrow{s}$ and the result is trivial.

□

Lemma 20 *Let u, u' be terms, $t \in \lambda_i y.u$ and $t' \in \lambda_j x.u'$. Assume t is a sub-term of t'. Then, either t is a sub-term of u' or $i = j$, $x = y$ and u is a sub-term of u'.*

Proof By induction on u'. □

Lemma 21

- *Let $t = (\alpha\ \overrightarrow{u}) \in F$ where $\alpha \in V \cup \{S_i, K_i, I_i\ /i = 0, 1\}$. Then, $\overrightarrow{u} \in F$.*
- *If $t = (S_2\ \overrightarrow{u}) \in F$, then $t = \phi((r\ \overrightarrow{w}))$ for some $r \in \lambda_0 y.v$ and some $v, \overrightarrow{w} \in F$.*

Proof By induction on $nb(t)$, essentially as in lemma 19. □

3.4 Some properties of reduction

Lemma 22 *Let u_1, u_2 be fair and assume $t = (u_1\ u_2) \triangleright t'$. Then $t' = (u'_1\ u_2)$ or $t' = (u_1\ u'_2)$ where $u_i \triangleright u'_i$.*

Proof It is enough to show that there is no possible interactions between u_1 and u_2. Such an interaction could occur in the following cases.
- $lvl(t \triangleright t') = 0$. This is forbidden by lemma 19.
- $lvl(t \triangleright t') > 0$ and, for example, $t \in \lambda_0 x.v$ and $t' \in \lambda_0 x.v'$ for some $v \triangleright v'$. This could occur if $u_1 = (S_2\ w_1)$ for some $w_1 \in \lambda_0 x.t_1$, $u_2 \in \lambda_0 x.t_2$ and $v = (t_1\ t_2)$. But this is again impossible by lemma 19. □

Lemma 23 *Let u_1, u_2, u_3 be terms.*

- *Assume $t = (I_1\ u_1) \triangleright t'$. Then either $t' = u_1$ or $t' = (I_1\ u'_1)$ for $u_1 \triangleright u'_1$.*
- *Assume $t = (K_1\ u_1\ u_2) \triangleright t'$. Then either $t' = u_1$ or $t' = (K_1\ u'_1\ u_2)$ or $t' = (K_1\ u_1\ u'_2)$ for $u_i \triangleright u'_i$.*
- *Assume $t = (S_1\ u_1\ u_2\ u_3) \triangleright t'$. Then either $t' = (u_1\ u_3\ (u_2\ u_3))$ or $t' = (S_1\ u'_1\ u'_2\ u'_3)$ where $u_i \triangleright u'_i$ for a unique i and $u'_j = u_j$ for $j \neq i$.*

Proof It is enough to show that the mentioned reductions are the only possibilities. I only look at the last case since the other ones are similar.

If $lvl(t \triangleright t') = 0$, the result is trivial. Otherwise, this means that there is a sub-term of $t \in \lambda_i x.v$ which reduces to a term in $\lambda_i x.v'$ for $v \triangleright v'$. But, this sub-term has to be a sub-term of some u_j because, otherwise we will have S_2 or S_3 instead of S_1, and the result follows immediately. □

Lemma 24 *Assume $t \in \lambda_0 x.u$ and $t \triangleright t'$. Then either $t' \in \lambda_0 x.u$ and $size(t') < size(t)$ or $t' \in \lambda_0 x.u'$ for some u' such that $u \triangleright u'$.*

Proof If $lvl(t \triangleright t') = 0$, the result is clear. Otherwise, this follows easily from Lemma 20. □

Lemma 25 *Assume $\phi(u, \overrightarrow{y}, \overrightarrow{f}) \in \lambda_0 x.v$. Then $u \in \lambda_0 x.w$ for some w such that $\phi(w, \overrightarrow{y}, \overrightarrow{f}) = v$.*

Proof By an immediate induction on the length of the sequence $\overrightarrow{y}$ it is enough to prove the result for $\phi_y(u, f)$. This is proved by induction on v. I only consider the case $v = (v_1\ v_2)$ and $\phi_y(u, f) = (S_2\ r_1\ r_2)$ where $r_j \in \lambda_0 x.v_j$ (the other cases are similar). The leftmost address in $dom(f)$ cannot be $[l, l, ..., l]$ because, otherwise, $\phi_y(u, f)$ will begin with S_3. Thus $u = (u_1\ u_2)$ and $\phi_y(u, f) = (S_2\ \phi_y(u_1, f_1)\ \phi_y(u_2, f_2))$. Thus $\phi_y(u_i, f_i) \in \lambda_0 x.v_i$ and we conclude by the *IH*. □

Lemma 26 *Let $u, f \in F$ be such that (u, f) is adequate. Then a redex in $t = \phi_x(u, f)$ is either in u or in some $f(a)$ or is $(w_a\ f(a))$ for some a and some $w_a \in \lambda_1 x.u_a$. Thus, if $t \triangleright t'$, one of the following cases holds.*

- *$t' = \phi_x(u', f')$ for some u', f' such that $u \triangleright u'$*
- *$t' = \phi_x(u, f')$ where $f \triangleright f'$*
- *t' is obtained from t by reducing the redex $(w_a\ f(a))$ for some $a \in dom(f)$ and some $w_a \in \lambda_1 x.u_a$. Then, $t' = \phi_x(u', f')$ and*
  - *If $u_a = x$, then u' is u where the occurrence of x at the address a has been replaced by $f(a)$ and $dom(f') = dom(f) - \{a\}$.*
  - *If $x \notin u_a$, then $u' = u$ and $dom(f') = dom(f) - \{a\}$.*
  - *If $u_a = (v_1\ v_2)$ then $dom(f') = dom(f) - \{a\} \cup \{[a :: l], [a :: r]\}$, $f'([a :: l]) = f'([a :: r]) = f(a)$ and, for $b \neq a$, $f'(b) = f(b)$.*

Proof By induction on $nb(u)$. The only thing to be shown is that the mentioned cases are the only possible ones. For $lvl(t \triangleright t') = 0$, this follows immediately from the fact that terms of the form $(w_a\ f(a))$ for some $w_a \in \lambda_1 x.u_a$ cannot introduce an interaction since they are redexes. For $lvl(t \triangleright t') > 0$, assume $r \in \lambda_i x.w$ is a subterm of $\phi_y(u, f)$ and the reduction takes places in w. Then, by lemma 25, either the reduction is actually in f or $w = \phi_y(v', f')$ for some adequate (v', f') and the result follows from the *IH*. □

Lemma 27

- *The set of fair terms is closed by reduction.*
- *Let u be fair and σ be a fair substitution. Assume $t = \sigma(u) \triangleright t'$, then either $t' = \sigma(u')$ for some $u \triangleright u'$ or $t' = \sigma'(u)$ for some $\sigma \triangleright \sigma'$.*

Proof By induction on $nb(u)$, using lemmas 22, 23, 24 and 26. □

3.5 Confluence of $\triangleright$ on fair terms

Lemma 28 *Let u be fair and σ be a fair substitution. If $u, \sigma \in SN$, then so is $\sigma(u)$.*

Proof This follows immediately from lemma 27. □

Theorem 29 *Any fair term t is in SN.*

Proof By induction on $nb(t)$.

- If $t = x, S_0, K_0, I_0$, the result is trivial.
- If $t = (t_1\ t_2)$, then, by the *IH*, $t_1, t_2 \in SN$ and, since $t = \sigma((x\ y))$ where $\sigma(x) = t_1$ and $\sigma(y) = t_2$, the result follows from lemma 27.
- If $t = (S_1\ t_1\ t_2\ t_3)$, $t = (K_1\ t_1\ t_2)$ or $t = (I_1\ t_1)$ the proof is similar, e.g. $(S_1\ t_1\ t_2\ t_3) = \sigma((S_1\ x_1\ x_2\ x_3)$ where $\sigma(x_i) = t_i$.

- If $t \in \lambda_0 x.v$, the result follows from lemma 24 and the *IH*.
- Finally, assume $t = \phi_x(u, f)$. Let t' be the term obtained from u by replacing, for each $a \in dom(f)$, u_a by $u_a[x := f(a)]$. It follows from lemma 28 that $t' \in SN$. But, by lemma 26, and infinite reduction of t would give an infinite reduction of t' since it is not possible to have infinitely many successive reductions of t of the form of the last case of lemma 26. Thus t is in SN.

□

Lemma 30 *Let u, v be terms. Then, for $w \in \lambda_1 x.u$, $(w\ v) \triangleright^* u[x := v]$.*

Proof By induction on u. □

Lemma 31 *The reduction $\triangleright$ is locally confluent on fair terms.*

Proof The only critical pairs are the following.

- $t = (w\ u_3)$, $w \in \lambda_1 x.(u_1\ u_2)$, $t \triangleright t_1 = (w_1\ u_3\ (w_2\ u_3))$ for $w_j \in \lambda_1 x.u_j$, and $t \triangleright t_2 = (w'\ u_3)$ for $w' \in \lambda_1 x.v$ and $(u_1\ u_2) \triangleright v$. Both t_1 and t_2 reduces to $v[x := u_3]$.
- $t = (S_{i+2}\ r_1\ r_2) \in \lambda_i x.(u_1\ u_2)$, $x \in u_1$, $x \notin u_2$ (for example), for some $u_1 \triangleright v_1$ such that $x \notin v_1$, $t \triangleright t_1 = (K_i\ (v_1\ u_2))$ and $t \triangleright t_2 = (S_{i+2}\ (K_i\ v_1)\ (K_i\ u_2)))$. But $t_2 \triangleright t_1$.

□

Theorem 32 *The reduction $\triangleright$ is confluent on fair terms.*

Proof By lemma 6 and 31. □

3.6 Proof of theorem 9

In this section I will still denote by $\triangleright$ the reduction on *combinators* given by rules (1, 2, 4) of definition 8.

Definition 33

- *Let u be a combinator. A labelling of u is a function that associates to each occurrence of S (resp. K, I) in u some S_i (resp. some K_i, I_i).*
- *If L is a labelling of u, I still denote by $L(u)$ the term obtained by replacing in u the symbols S (resp. K, I) by $L(S)$ (resp. $L(K)$, $L(I)$).*
- *Let u be a term. I denote by $\theta(u)$ the combinator defined by the following rules. $\theta(x) = x$, $\theta(S_i) = S$, $\theta(K_i) = K$, $\theta(I_i) = I$ and $\theta((u\ v) = (\theta(u)\ \theta(v))$*
- *Let u be a combinator and L, L' be labelling of u. I say that L' is an extension of L if the following holds.*

  1. *For each S in u,*
     - either $L(S) = L'(S)$
     - or $L(S) = S_0$ and $L'(S) = S_i$ for $i = 1, 2$ or 3
     - or $L(S) = S_2$ or $L(S) = S_1$ and $L'(S) = S_3$.
  2. *For each K in u, $L(K) = L'(K)$ or $L(K) = K_0$ and $L'(K) = K_1$.*
  3. *For each I in u, $L(I) = L'(I)$ or $L(I) = I_0$ and $L'(I) = I_1$.*

A labelling of u is thus a way of marking redexes in u. The function θ consists in un-marking terms to get combinators. Extending a labelling means allowing more redexes to be reduced.

Lemma 34 *Let u be a combinator and L be a labelling of u. If $L(u) \triangleright v$ then $u \triangleright \theta(v)$.*

Proof Immediate. □

Lemma 35 *Assume $t = L(\lambda x.r) \in F$ for some L, r. Then, there is an extension L' of L such that $L'(\lambda x.r) \in \lambda_0 x.v$ for some $v \in F$.*

Proof First note that, for combinators, $\lambda x.r$ represents a single term and thus having written $t = L(\lambda x.r)$ is not a typo !

L' is obtained by iterating the following algorithm.

- If x does not occur in r, choose $L' = L$. Since $t = (L(K)\ L(r))$, by lemma 19, $L(K)$ must be K_0 and thus, by lemma 21, $L(r) \in F$.

- If $r = x$, choose $L' = L$. The argument is similar.

- If $r = (r_1\ r_2)$. Then $\lambda x.r = (S\ \lambda x.r_1\ \lambda x.r_2)$. By lemma 19, $L(S)$ must be either S_0 or S_2.

If $L(S) = S_2$, by lemma 21, $t \in \phi(\lambda_0 x.v)$ for some $v \in F$. Thus L satisfies the desired property since, by lemma 25, t must be in $\lambda_0 x.\phi(v)$.

If $L(S) = S_0$, then, by lemma 21, $L(\lambda x.r_i) \in F$. Choose $L'(S) = S_2$ and iterate the algorithm with $L(\lambda x.r_j)$ for $j = 1, 2$.

□

Lemma 36 *Let t be a combinator and L be a labelling of t such that $L(t)$ is fair. Assume the $t \triangleright t'$. Then, there is an extension L' of L such that $L'(t)$ is fair and $L'(t) \triangleright v$ for some v such that $\theta(v) = t'$.*

Proof By induction on $nb(L(t))$. Look at the last rule that has been used to show that $L(t)$ is fair.

Rule (3) : a redex in $w \in \lambda_0 x.u$ is either a redex in u (and the result follows immediately from the *IH*) or it is of the form $(S_2\ (K_0\ u_1)\ (K_0\ u_2)) \triangleright (K_0\ (u_1\ u_2))$ and thus already appear in $L(t)$.

Rule (5) : a redex in $\phi_x(u, f)$ is either a redex in u or in some $f(a)$ or a redex already in $L(t)$ and the result follows immediately from the *IH*.

Rule (2) : then $t = (t_1\ t_2)$ and $L(t_1), L(t_2)$ are fair. If the reduced redex is either in t_1 or t_2, the result follows immediately from the *IH*. Otherwise it has been created by the application of t_2 to t_1. I will only look at the cases where the reduced redex starts with some S. The case of K and I are similar and much simpler. For sake of simplicity I will define L' by only mentioning the labels that are changed. We distinguish the different possible redexes.

(a) $t_1 = (S\ u\ v)$ and $t' = (u\ t_2\ (v\ t_2))$.

- If $L(S) = S_0$ then, setting $L'(S) = S_1$ gives the desired properties since, by lemma 21, $L(u), L(v)$ are in F and thus $L'(t)$ also is in F.

- $L(S)$ may not be S_1 or S_3 since, by lemma 19, it would have at least 3 arguments.

- If $L(S) = S_2$ then, by lemma 21, $L(t) = \phi(w)$ for some $w \in \lambda_0 x.v$ and some $v \in F$. Then, choosing L' in such a way that $L'(t) = \phi(w_1)$ for $w_1 \in \lambda_1 x.v$ will give the desired properties .

(b) $t_1 = (S\ (K\ u)$, $t_2 = (K\ v)$ and $t' = (K\ (u\ v))$. Then $L(S)$ must be S_0 because otherwise, by lemma 19, S would have at least two arguments. Similarly, we must have $L(K) = K_0$. Then, by lemma 21, u, v are fair and thus setting $L'(S) = S_2$ and $L'(K) = K_0$ gives the desired properties.

(c) $t_1 = (S\ w_1)$ for $w_1 \in \lambda x.u_1$, $t_2 \in \lambda x.u_2$ and $t' \in \lambda x.v$ where v is a reduct of $(u_1\ u_2)$. Again by lemma 19, we must have $L(S) = S_0$. By lemma 21, $L(w_1) \in F$. By lemma 35, extend L so that $L'(u_i) \in F$. Then setting L'' in such a way that $L''(t) \in \lambda_0 x.(u_1\ u_2)$ gives the desired properties.

Rule (4) : then $t = (S\ u_1\ u_2\ u_3)$, $L(S) = S_1$ and the $L(u_i)$ are fair. If $t' = (u_1\ u_3\ (u_2\ u_3))$ or if the reduced redex is in some u_i the result is trivial. Otherwise this means that, for $i = 1, 2$ $u_i \in \lambda x.v_i$ and $t' = (w\ u_3)$ for some $w \in \lambda x.v$ such that v is a reduct of $(v_1\ v_2)$. Then, by lemma 35, extend L so that $L'(v_i) \in F$ and choose L'' in such a way that $L''(t) = (w'\ u_3)$ for $w' \in \lambda_1 x.(v_1\ v_2)$. □

Lemma 37 *Let t be a combinator. Assume that $t \triangleright v$ and $t \triangleright^* u$. Then, there is a labelling L of u and a term w such that $L(u)$ is fair, $L(u) \triangleright^* w$ and $v \triangleright^* \theta(w)$.*

Proof By induction on the length n of the reduction $t \triangleright^* u$.

- If $n = 1$, this follows immediately from lemmas 36 and 31.
- Otherwise, let $t \triangleright^* u_1 \triangleright u$. By the *IH*, let L_1 be a labelling of u_1 and w_1 be a term such that $L_1(u_1)$ is fair, $L_1(u_1) \triangleright^* w_1$ and $v \triangleright \theta(w_1)$. By lemma 36, let L be a labelling of u_1 that is an extension of L such that $L(u_1)$ is fair and $L(u_1) \triangleright r$ for r such that $\theta(r) = u$. By theorem 32, let w be such that $r \triangleright^* w$ and $w_1 \triangleright^* w$. Then L, w have the desired properties.

□

Proposition 38 *The reduction $\triangleright$ is confluent.*

Proof It is enough to show that, if $t \triangleright u$ and $t \triangleright^* v$ then $u \triangleright^* w$ and $v \triangleright^* w$ for some w. This follows immediately from lemma 37. □

Definition 39 *I denote by $\supset$ the reduction defined by the following rules.*

1. $(S\ (K\ u)\ I) \supset u \qquad (K\ u\ v) \supset u \qquad (I\ u) \supset u$
2. $\lambda x.u \supset \lambda x.v$ *if* $u \supset v$

Lemma 40 *The reduction $\supset$ is confluent and commutes with $\triangleright$.*

Proof The reduction $\supset$ is strongly normalizing since it decreases the size. Thus to prove the confluence, it is thus enough to show the local confluence and this is straightforward. Since $\supset$ is also non duplicating, to prove the commutation with $\triangleright$, it is enough to show the local commutation and this is again straightforward. Note that the reductions $(K\ u\ v) \supset u$, $(I\ u) \supset u$ that are already present in $\triangleright$ are used here to ensure the confluence of the only critical pair i.e. $(S\ (K\ u)\ I\ w) \supset (u\ w)$ and $(S\ (K\ u)\ I\ w) \triangleright (K\ u\ w\ (I\ w))$.

□

Theorem 9 *The reduction $\rightarrow$ is confluent.*

Proof Since $\rightarrow$ is the union of $\triangleright$ and $\supset$, the result follows immediately from proposition 38 and lemma 40. □

4 Proof of theorem 5

I denote by $\equiv$ the equivalence relation induced by $\succ$ i.e. $u \equiv v$ iff there is a sequence $u_0, ..., u_n$ of combinators such that $u_0 = u$, $u_n = v$ and, for each i, either $u_i \succ u_{i+1}$ or $u_{i+1} \succ u_i$. The equivalence induced by $\rightarrow$ will be denoted by $\approx$.

Lemma 41 *1. For each u, v, $(S\ (K\ u)\ I) \succ^* u$ and $(S\ (K\ u)\ (K\ v)) \succ^* (K\ (u\ v))$*

2. For each u, $\lambda x.u \rightarrow [x]u$ and $\lambda x.u \succ [x]u$.

Proof

1. Let x be a fresh variable. Then, $(S\ (K\ u)\ I) = [x](S\ (K\ u)\ I\ x) \succ [x](K\ u\ x\ (I\ x)) \succ^* [x](u\ x) = u$ and $(S\ (K\ u)\ (K\ v)) = [x](S\ (K\ u)\ (K\ v)\ x) \succ [x](K\ u\ x\ (K\ v\ x)) \succ^* [x](u\ v) = (K\ (u\ v)))$.

2. This follows immediately from the first point.

□

Theorem 42 *Let u, v be combinators. Then $u \equiv v$ iff $u \approx v$.*

Proof It is enough to show that if $u \succ v$ then $u \approx v$ and if $u \rightarrow v$ then $u \equiv v$. Each point is proved by induction on the level of the reduction. The result is trivial for the level 0. Assume then that the level is at least 1. For the first direction, I have to show that, if $u \succ v$ then $[x]u \approx [x]v$. By the *IH* we know that $u \approx v$ and it is thus enough to show that, if $u \rightarrow v$, then $[x]u \approx [x]v$. By the previous lemma, we have $\lambda x.u \rightarrow [x]u$ and, since $\lambda x.u \rightarrow \lambda x.v \rightarrow [x]v$, we are done. For the other direction, we have to prove that, if $u \succ v$, then $\lambda x.u \equiv \lambda x.v$. This is because $\lambda x.u \succ [x]u \succ [x]v$ and $\lambda x.v \succ [x]v$. □

Remark

I have shown that the two reduction rules give the same equations on the combinators but the reductions are not the same i.e. there are combinators such that $u \rightarrow^* v$ for some v but u does not reduce to v by $\succ$ and, similarly, there are combinators such that $u \succ^* v$ for some v but u does not reduce to v by $\rightarrow$. Here are examples.

Let $u = [y][x](S\ x\ x\ (y\ x))$. Then $u \succ [y][x](x\ (y\ x)\ (x\ (y\ x)))$ and it is easy to check that u is normal for $\rightarrow$.

Let $u_1 = \lambda x.(S\ y\ x\ x) \rightarrow \lambda x.(y\ x\ (x\ x)) = v$ and it is not too difficult to check that u does not reduce to v by $\succ$.

Definition 43 *I denote by $\vdash$ the reduction defined by the following rules.*

1. $(S\ (K\ u)\ I) \vdash u$ $\quad (K\ u\ v) \vdash u$ $\quad (I\ u) \vdash u$

2. $[x]u \vdash [x]v$ if $u \vdash v$

Lemma 44 *The reduction $\vdash$ is confluent and commutes with $\succ$.*

Proof As in lemma 40 □

Lemma 45 *If $u \rightarrow^* v$ then $u \succ^* w$, $v \vdash^* w$ for some w.*

Proof By induction on the length of the reduction $u \rightarrow^* v$. Assume $u \rightarrow u_1 \rightarrow^* v$. If the level of the reduction $u \rightarrow u_1$ is 0, the result follows immediately from the *IH* since then we also have $u \succ u_1$. Otherwise, the reduction looks like $u = C[\lambda x.t] \rightarrow u_1 = C[\lambda x.t_1] \rightarrow^* v$ where $t \rightarrow t_1$. By the *IH*, we have $t \succ^* w_1$, $t_1 \vdash^* w_1$ for some w_1 and thus $u \succ^* w_2$, $u_1 \vdash^* w_2$ where $w_2 = C[w_1]$. By the *IH* we also have $u_1 \succ^* w$, $v \vdash^* w$ for some w. By lemma 44, we have $w_2 \succ^* w_3$ and $w \vdash^* w_3$ for some w_3 which is the term we are looking for. □

Theorem 5 *The reduction $\succ$ is confluent.*

Proof Assume $t \succ^* t_1$ and $t \succ^* t_2$. Then $t_1 \equiv t_2$ and thus, by theorem 42, $t_1 \approx t_2$. Since $\rightarrow$ is confluent we thus have $t_1 \rightarrow^* t_3$, $t_2 \rightarrow^* t_3$ for some t_3. By lemma 45, let v_i be such that $t_i \succ^* v_i$ and $t_3 \vdash^* v_i$. Since $\vdash$ is confluent, let t_3 be such that $v_1 \vdash t_3$ and $v_2 \vdash t_3$. Since $\vdash$ is clearly a restriction of $\succ$, we have $t_i \succ^* t_3$ □

5 A standardization theorem

In this section I prove a standardization theorem for the system of section 3. I study this system instead of the one of section 2 because, as already mentioned in section 2.2, in the original system, what could be the leftmost redex is not clear at all.

Note that the following definition of a standard reduction does not need the definition of the residue of a redex. It is a definition by induction on $\langle lg(t \rightarrow t'), size(t)\rangle$ where $lg(t \rightarrow t')$ is the number of steps of the reduction. It uses the idea that is implicit in [1] and simply says that a standard reduction either reduces the head redex at the first step or is not allowed to reduce it.

Definition 46 *A reduction $t \rightarrow^* t'$ is standard ($t \rightarrow_{st} t'$ for short) if it satisfies the following properties.*

1. $t = (x\ \overrightarrow{u})$, $t' = (x\ \overrightarrow{u'})$ *and, for each i, $u_i \rightarrow_{st} u'_i$*
2. $t = (K\ u)$, $t' = (K\ u')$ *and $u \rightarrow_{st} u'$.*
3. $t = (S\ u)$, $t' = (S\ u')$ *and $u \rightarrow_{st} u'$.*
4. $t = (S\ u\ v)$ *and*
   - *either $t' = (S\ u'\ v')$ for $u \rightarrow_{st} u'$ and $v \rightarrow_{st} v'$*
   - *or the reduction is $t \rightarrow t_1 \ldots \rightarrow t_k \rightarrow_{st} t'$ for some $k \geq 0$ such that $t_i = (S\ u_i\ v_i)$, $u \rightarrow_{st} u_k$, $v \rightarrow_{st} v_k$ and*
     - either $t_k = [x]w$, $t' = [x]w'$, $w \rightarrow_{st} w'$ and, for each $i < k$, t_i cannot be written as $[x]r$ for some r
     - or $u_k = (K\ u'_k), v_k = (K\ v'_k)$, the reduction $t_k \rightarrow_{st} t'$ is $t_k \rightarrow (K\ (u'_k\ v'_k)) \rightarrow_{st} t'$ and, for each $i < k$, t_i cannot be written as $(S\ (K\ u'_i)\ (K\ v'_i))$
     - or $u_k = (K\ u'_k), v_k = I$, the reduction $t_k \rightarrow_{st} t'$ is $t_k \rightarrow u'_k \rightarrow_{st} t'$ and, for each $i < k$, t_i cannot be written as $(S\ (K\ u'_i)\ I)$
5. $t = (I\ u_1\ \ldots\ u_n)$ *for $n \geq 1$ and*
   - *either $t' = (I\ u'_1\ \ldots\ u'_n)$ for $u_i \rightarrow_{st} u'_i$*
   - *or the reduction is $t \rightarrow (u_1\ \ldots\ u_n) \rightarrow_{st} t'$*
6. $t = (K\ u_1\ \ldots\ u_n)$ *for $n \geq 2$ and*
   - *either $t' = (K\ u'_1\ \ldots\ u'_n)$ for $u_i \rightarrow_{st} u'_i$*
   - *or the reduction is $t \rightarrow (u_1\ u_3\ \ldots\ u_n) \rightarrow_{st} t'$*
7. $t = (S\ u_1\ \ldots\ u_n)$ *for $n \geq 3$ and*
   - *either $t' = (r\ u'_3\ \ldots\ u'_n)$ where $(S\ u_1\ u_2) \rightarrow_{st} r$ and $u_i \rightarrow_{st} u'_i$ for $i \geq 3$*
   - *or the reduction is $t \rightarrow (u_1\ u_3\ (u_2\ u_3)\ u_4\ \ldots\ u_n) \rightarrow_{st} t'$*

Lemma 47
- *Assume $u_i \rightarrow_{st} u'_i$ for each i. Then $(u_1\ \ldots\ u_n) \rightarrow_{st} (u'_1\ \ldots\ u'_n)$*
- *Assume $u \rightarrow_{st} [x]u'$. Then $(u\ v) \rightarrow_{st} u'[x := v]$*

Proof Easy. □

Theorem 48 *If $t \rightarrow^* t'$ then $t \rightarrow_{st} t'$.*

Proof By induction on $lg(t \rightarrow^* t')$. It is enough to show that if $t \rightarrow_{st} t' \rightarrow t''$ then $t \rightarrow_{st} t''$. This is done by induction on $\langle lg(t \rightarrow_{st} t'), size(t)\rangle$ and by case analysis. We look at the rule that has been used to show $t \rightarrow_{st} t'$ and then what is the reduced redex in $t' \rightarrow t''$. I just consider two cases. The first one is typical and easy. The second one is similar but a bit more complex.

- $t = (K\ u_1\ ...\ u_n)$ for $n \geq 2$.
  - If the reduction is $t \rightarrow (u_1\ u_3\ ...\ u_n) \rightarrow_{st} t'$ we apply the *IH* to $(u_1\ u_3\ ...\ u_n) \rightarrow_{st} t' \rightarrow t''$ to get $(u_1\ u_3\ ...\ u_n) \rightarrow_{st} t''$ and thus $t \rightarrow (u_1\ u_3\ ...\ u_n) \rightarrow_{st} t''$ is standard.
  - If the reduction is such that $t' = (K\ u'_1\ ...\ u'_n)$ for $u_i \rightarrow_{st} u'_i$ then
    - either $t'' = (K\ u'_1\ ...\ u''_i\ ...\ u'_n)$ for $u'_i \rightarrow u''_i$ and we apply the *IH* to $u_i \rightarrow_{st} u'_i \rightarrow u''_i$ to get the result
    - or $t'' = (u'_1\ u'_3\ ...\ u'_n)$ and then $t \rightarrow (u_1\ u_3\ ...\ u_n) \rightarrow^* (u'_1\ u'_3\ ...\ u'_n)$ is standard by lemma 47.
- $t = (S\ u_1\ ...\ u_n)$ for $n \geq 3$ and $t' = (r\ u'_3\ ...\ u'_n)$ where $(S\ u_1\ u_2) \rightarrow_{st} r$ and $u_i \rightarrow_{st} u'_i$ for $i \geq 3$. Assume also that $r = [x]a$, $x \notin r$ and $t'' = (a\ u'_4\ ...\ u'_n)$. This means that, for $i = 1, 2$, $u_i \rightarrow_{st} [x]v_i$ and that $(v_1\ v_2) \rightarrow_{st} a$. But then, by lemma 47, $(u_i\ u_3) \rightarrow_{st} v_i[x := u_3]$. Thus, the following reduction is standard. $t \rightarrow (u_1\ u_3\ (u_2\ u_3)\ u_4\ ...\ u_n) \rightarrow_{st} (v_1[x := u_3]\ v_2[x := u_3]\ u_4\ ...\ u_n) \rightarrow_{st} (a[x := u_3]\ u_4\ ...\ u_n) \rightarrow_{st} (a\ u'_4\ ...\ u'_n) = t''$.

□

6 Strong normalization of the typed calculus

In this section I prove the strong normalization of the auxiliary system of section 3. Note that the system of section 2 is not strongly normalizing even though this is for the following bad reason. Let $t = (S\ x\ x)$. Then $t = [y](S\ x\ x\ y) \succ [y](x\ y\ (x\ y)) = t$.

The types are the simple types i.e. constructed from basic types with the arrow. The typing rules are the usual ones i.e. I has type $A \rightarrow A$, K has type $A \rightarrow B \rightarrow C$, S has type $(A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow A \rightarrow C$ for every types A, B, C and, finally, if u has type $A \rightarrow B$ and v has type A then $(u\ v)$ has type B.

Definition 49
- *A combinator t is highly normalizing ($t \in HN$ for short) if it can be obtained by the following rules.*
  1. *$t = S$ or $t = K$ or $t = I$ or $t = (x\ t_1\ ...\ t_n)$ for $t_1,\ ...,\ t_n \in HN$.*
  2. *$t = (K\ t_1)$ or $t = (S\ t_1)$ for $t_1 \in HN$*
  3. *$t = (S\ t_1\ t_2)$ for $(t_1\ x\ (t_2\ x)) \in HN$ where x is a variable.*
  4. *$t = (I\ t_1\ ...\ t_n)$ for $n \geq 1$ and $(t_1\ t_2\ ...\ t_n) \in HN$*
  5. *$t = (K\ t_1\ ...\ t_n)$ for $n \geq 2$, $(t_1\ t_3\ ...\ t_n) \in HN$ and $t_2 \in HN$*
  6. *$t = (S\ t_1\ ...\ t_n)$ for $n \geq 3$ and $(t_1\ t_3\ (t_2\ t_3)\ t_4\ ...\ t_n) \in HN$*
- *If $t \in HN$ we denote by $\eta(t)$ the number of rules that have been used to show $t \in HN$.*

We have introduced this notion of normalization which is stronger than the usual one (see the next lemma) because the proof of lemma 51 below would not work if HN was replaced by SN.

Lemma 50 *If $t \in HN$ then t is strongly normalizing.*

Proof By induction on $\eta(t)$. The non trivial cases are when the last rule that has been applied to prove $t \in HN$ is (3) or (6).

- Assume first $t = (S\ t_1\ t_2)$. Then, by the *IH*, $t' = (t_1\ x\ (t_2\ x)) \in SN$ and thus $t_1, t_2 \in SN$. Thus an infinite reduction of t must look like $t \rightarrow^* t'' \rightarrow^* ...$ where for some v_i, $t_i \rightarrow^* \lambda x.v_i$ and

  - either the reduction of t'' is in $(v_1\ v_2)$. But $(v_1\ v_2) \in SN$ since $t' \in SN$ and $t' \rightarrow^* (\lambda x.v_1\ x\ (\lambda x.v_2\ x)) \rightarrow^* (v_1\ v_2)$. Contradiction.

  - or $\lambda x.v_i = (K\ v_i)$ and the reduction is $t'' = (S\ (K\ v_1)\ (K\ v_2) \rightarrow (K(v_1\ v_2)) \rightarrow^*$ ... This is impossible since $t' \in SN$ and $t' \rightarrow^* (v_1\ v_2)$.

  -or $\lambda x.v_1 = (K\ v_1)$, $\lambda x.v_2 = I$ and the reduction is $t'' = (S\ (K\ v_1)\ I \rightarrow v_1 \rightarrow^*$ ... This is impossible since $t' \in SN$ and $t' \rightarrow^* v_1$.

- $t = (S\ t_1\ ...\ t_n)$. Again, by the *IH*, $t' = (t_1\ t_3\ (t_2\ t_3)\ t_4\ ...\ t_n) \in SN$. Thus the t_i are in SN and also $(S\ t_1\ t_2) \in SN$. The first point is clear. For the second, we argue as follows. Reasoning as in the previous case, it is enough to show that $(t_1\ x\ (t_2\ x)) \in SN$. If it was not the case then $(t_1\ t_3\ (t_2\ t_3)$ would also not been in SN and this contradicts the fact that $t' \in SN$. Thus an infinite reduction of t looks like $t \rightarrow^* (r\ t'_3\ ...\ t'_n) \rightarrow t'' \rightarrow^* ...$ where r is a reduct of $(S\ t_1\ t_2)$ and t'' is obtained by an interaction between r and its arguments. But we have shown (in the proof of theorem 48) that then t' reduces to t" and this is a contradiction.

□

Lemma 51 *Let t be a combinator and σ be a substitution such that all the variables in the domain of σ have the same type. Assume $t \in HN$ and the image of σ is included in HN. Then $\sigma(t) \in HN$.*

Proof By induction on $\langle type(\sigma), \eta(t)\rangle$. Look at the last rule that has been used to prove $t \in HN$. The only non trivial case is when $t = (x\ t_1\ ...\ t_n)$ and $x \in dom(\sigma)$. By the *IH*, $u_i = \sigma(t_i) \in HN$. We now have to distinguish the different possible values for $\sigma(x)$. The most difficult case (the other ones are similar or trivial) is when $\sigma(x) = (S\ a_1\ a_2)$. We have to show that $t' = (a_1\ u_1\ (a_2\ u_1)\ u_2\ ...\ u_n) \in HN$. But $t' = \tau((z\ u_2\ ...\ u_n))$ where z is a fresh variable such that $\tau(z) = (a_1\ u_1\ (a_2\ u_1))$. But $type(z) < type(x)$ and, by the *IH*, it is thus enough to show that $t'' = (a_1\ u_1\ (a_2\ u_1)) \in HN$. But $t'' = \tau'((a_1\ z'\ (a_2\ z')))$ where z' is a fresh variable such that $\tau'(z') = u_1$. Since $type(z') < type(x)$ and $(a_1\ z'\ (a_2\ z')) \in HN$ (because $(S\ a_1\ a_2) \in HN$), the result follows from the *IH*. □

Corollary 52 *Every typed combinator t is in HN and thus in SN.*

Proof By induction on the size of t using $(u\ v) = (x\ v)[x := u]$ and lemma 51. □

7 Final remarks

Though intuitively quite simple, the given proof of confluence is technically rather involved and, in particular, it is more elaborate than the one using the confluence of the λ-calculus. Thus, one may wonder about the real use of such a proof even if this is the condition to have a self contained theory. I will argue for another reason.

Combinatory Logic somehow looks like a calculus with explicit substitutions. Though $([x]u\ v)$ is not exactly the *explicit* substitution $u[x := v]$, it has often to be understood in this way. In particular, the reduction $([x](u_1\ u_2)\ v) \rightarrow ([x]u_1\ v\ ([x]u_2\ v))$

looks like the propagation of the substitution into the two branches of the application. But proving confluence for such calculi is usually not trivial simply because the usual methods (parallel reductions or finite developments) need definitions that are not clear.

I thus hope that the given proof will help in finding simple proofs for calculi with explicit substitutions.

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