BOUNDS FOR GRADIENT TRAJECTORIES AND GEODESIC DIAMETER OF REAL ALGEBRAIC SETS

D. D'ACUNTO AND K. KURDYKA

Abstract

Let $M \subset \mathbb{R}^n$ be a connected component of an algebraic set $\varphi^{-1}(0)$ where φ is a polynomial of degree d. Assume that M is contained in a ball of radius r. We prove that the geodesic diameter of M is bounded by $2r\nu(n)d(4d-5)^{n-2}$, where $\nu(n) = 2\Gamma(\frac{1}{2})\Gamma(\frac{n+1}{2})\Gamma(\frac{n}{2})^{-1}$. This estimate is based on the bound $r\nu(n)d(4d-5)^{n-2}$ for the length of the gradient trajectories of a linear projection restricted to M.

Introduction

Explicit bounds for the topology of real algebraic sets have been subject of an intensive study. For instance, if $X \subset \mathbb{R}^n$ is the zero set of a polynomial of degree d then the number of connected components of X, or more generally the sum of the Betti numbers of X, can be bounded by cd^n, where $c = c(n)$ is an explicit constant (see e.g. [**2**], [**3**], [**4**], [**6**], [**15**], [**16**]).

Bounds for the geometry of algebraic (or more generally semialgebraic) sets, in particular estimates for the geodesic diameter, appeared recently in quite various contexts. In theoretical robotics, "Robot Motion Planning" is the problem of finding a collision-free path for a robot (with n degrees of freedom) moving among obstacles. It is natural to consider that the obstacles and possible positions of the robot are semialgebraic sets (i.e. representable by a finite number of polynomial equations, inequations and set-theoretical operations). This leads (cf. e.g. [**16**]) to the general problem of constructing a path joining two given points x, y in a connected component A of a semialgebraic set in $\mathbb{R}^n$. Actually, this problem appears also in smooth analysis [**16**], quantitative transversality [**9**] or algorithms computing connected components of semialgebraic sets [**1**] and [**2**]. Of course the goal is to find a path as simple as possible, for instance depending on the context: the shortest one, described by few polynomials of low degree, with a simple parametrisation.

The usual manner such a path is constructed (see Canny [**5**], Donaldson [**9**], Bassu-Pollack-Roy [**1**], [**2**], Yomdin-Comte [**16**]) is the following: one first constructs so called "roadmap", i.e. a connected semialgebraic curve $C \subset A$ in such a way that it is easy to join each point of A with the curve C. Since C is arc-connected we can join x with y via C. Actually the constructed path is semialgebraic. The construction of C is in general quite involved; one applies induction on n which requires several projections. There are some algorithms to compute a roadmap, but

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they don't give a sharp estimate for the path's length in terms of n and degrees of the polynomials involved in a description of A.

In this paper we propose to join two points in a compact connected component of a smooth algebraic hypersurface by pieces of gradient trajectories of a linear projection restricted to the hypersurface. In Theorem 4.1 we give an explicit and quite sharp bound for length of such trajectories. This result seems to be new and of its own interest.

Theorem 2.1 provides an explicit estimate for the geodesic diameter of a smooth compact connected component of an algebraic hypersurface. Finally in the general case we approximate a connected component of an algebraic set by the union of some connected components of smooth algebraic hypersurfaces. To the best of our knowledge the estimate in Theorem 2.1 is the first explicit result in this direction. In the last section, we produce a sequence of examples which shows that the bound is asymptotically sharp.

Our method extends to exponential polynomials (cf Khovanskiĭ theory in [**12**]) and we give an explicit bound for the geodesic diameter depending only on the dimension, the degree of the polynomial and the number of involved exponential functions.

1. *Notation*

In this paper we respectively denote by $\langle\cdot,\cdot\rangle$, $|\cdot|$ and $d(\cdot,\cdot)$ the canonical Euclidean scalar product in $\mathbb{R}^n$, the associated norm and distance. We denote by $B^n(a,r) = \{x \in \mathbb{R}^n : |x-a| < r\}$ the open ball in $\mathbb{R}^n$ of centre a and radius r.

We recall that a subset $M \subset \mathbb{R}^n$ is *semialgebraic* if it is a finite Boolean combination of sets of the form $\{x \in \mathbb{R}^n : f(x) = 0\}$ and $\{x \in \mathbb{R}^n : g(x) > 0\}$, where f and g are polynomials on $\mathbb{R}^n$. By the Tarski-Seidenberg Theorem the projection of a semialgebraic set is again a semialgebraic set. We say that a function is semialgebraic if its graph is a semialgebraic set.

Let M be a compact connected semialgebraic set. We equip M with the geodesic distance d_M, this means that if x, y are two points in M then

$$d_M(x,y) = \inf\{\text{length of } \gamma_{x,y} : \gamma_{x,y} \subset M \text{ connected rectifiable path joining } x \text{ to } y\}.$$

We denote by $D(M) = \sup\{d_M(x,y) : x, y \in M\}$ the *geodesic diameter of* M.

In this paper, by *line* $L \subset \mathbb{R}^n$ we mean a linear subspace of $\mathbb{R}^n$ of dimension 1.

2. *Main result*

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree $d \geq 2$. Let M be a compact connected component of $\varphi^{-1}(0)$. Assume that $M \subset B^n(a,r)$. Under these hypotheses we state the main result of this paper:

THEOREM 2.1. *The geodesic diameter of M is bounded by*

$$rA(n,d) = r\nu(n)2d(4d-5)^{n-2},$$

where $\nu(n) = 2\Gamma(\frac{1}{2})\Gamma(\frac{n+1}{2})\Gamma(\frac{n}{2})^{-1}$ *and* Γ *is the Euler Gamma function.*

The proof of Theorem 2.1 for a smooth hypersurface M is based on our previous paper [**8**]. In the loc. cit. paper we have shown a similar result for the length of the trajectories of ∇f, where f is the restriction to an open ball $B^n(a,r)$ of a polynomial of degree d. In the present paper, for a smooth hypersurface M, we will define a vector field Y on M, tangent to M with a finite number of singularities. We then give a bound for the length of the trajectories of Y (i.e. the solutions of $x' = Y(x)$ with $x \in M$) which depends only on n, d and r. From this result we deduce the desired bound for $D(M)$ in the general case, that is when M is possibly a singular connected component of an algebraic subset of $\mathbb{R}^n$.

REMARK 2.2. Actually $rA(n,d)$ bounds the sum of geodesic diameters of all compact connected components $M_1, \ldots, M_s$ of $\varphi^{-1}(0)$ included in $B^n(a,r)$. That is

$$\sum_{i=1}^{s} D(M_i) \le rA(n,d). \tag{2.1}$$

3. *Gradient field of a Morse function*

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree d. Let M be a smooth compact connected component of $\varphi^{-1}(0)$ of dimension $n-1$ and let L be a line passing through the origin in $\mathbb{R}^n$. We denote by $\pi_L : \mathbb{R}^n \to L$ the orthogonal projection on L and by $f_L : M \to L$, the restriction of π_L to the manifold M. The Euclidean metric of $\mathbb{R}^n$ induces an analytic Riemannian metric on M and we denote by ∇_M the gradient with respect to this metric. Actually, for every $x \in M$, $\nabla_M f_L(x)$ is just the projection of $\nabla \pi_L(x)$ on the tangent space T_xM. The symbol ∇ stands for the gradient with respect to the Euclidean metric of $\mathbb{R}^n$. Note that $\nabla \pi_L$ is constant on $\mathbb{R}^n$ and can be chosen as a unit vector in the direction of L which we denote by v_L. For $x \in M$ we naturally obtain

$$\nabla_M f_L(x) = v_L - \langle v_L, \frac{\nabla\varphi(x)}{|\nabla\varphi(x)|}\rangle \frac{\nabla\varphi(x)}{|\nabla\varphi(x)|}. \tag{3.1}$$

The function $f_L : M \to L$ is semialgebraic. Hence the set of critical values of f_L is finite. We can easily check that $x \in M$ is a critical point of f_L if and only if $T_xM = L^{\perp}$.

For a suitable choice of $L \in \mathbb{P}^{n-1}$, the function f_L is a Morse function. Here $\mathbb{P}^{n-1}$ stands for the projective space of lines in $\mathbb{R}^n$. In other words we have the following

PROPOSITION 3.1. *There exists a semialgebraic set $\mathcal{E} \subset \mathbb{P}^{n-1}$ of codimension at least 1, such that for all $L \in \mathbb{P}^{n-1} \setminus \mathcal{E}$, the function f_L has only Morse singularities.*

Proof. The proposition is a consequence of the following result (cf. [**11**] pp 43-44): if M is a smooth submanifold of $\mathbb{R}^n$ and g is a C^∞ function on M, then for almost all $a \in \mathbb{R}^n$, the function $h_a = g + \langle a, \cdot \rangle$ is a Morse function on M. So this holds in particular for $g = 0$. Clearly, the set $\{a : h_a \text{ is not Morse}\}$ is semialgebraic in this case. $\square$

By Proposition 3.1, we can assume that π_L is the projection on the last coordinate (with respect to the canonical basis $(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})$ of $\mathbb{R}^n$), and thus $\nabla \pi_L = \frac{\partial}{\partial x_n}$. For simplicity we denote by π the projection π_L and by f the function f_L. The vector field $\nabla_M f$ is well defined on M and has only finitely many singularities. We will investigate some properties of the trajectories of $\nabla_M f$ in order to prepare the proof of Theorem 2.1.

DEFINITION 3.2. Let I be an interval in $\mathbb{R}$. We say that $x : I \to M$ is a *trajectory (or an integral curve)* of the vector field $\nabla_M f$ if $t \mapsto x(t)$ is a continuous, piecewise C^1, map satisfying (except at finitely many points) the differential equation

$$x'(t) = \frac{\nabla_M f(x(t))}{|\nabla_M f(x(t))|}. \tag{3.2}$$

In fact we consider trajectories of $\frac{\nabla_M f}{|\nabla_M f|}$ in the set of regular points of f but we allow trajectories to pass also by singular points (i.e. where $\nabla_M f = 0$). Recall that by Łojasiewicz's gradient inequality [**13**], [**14**], if $I =]a, b[$ is an open interval and $x(t)$ satisfies equation (3.2) for any $t \in I$, then the length of $x(t)$ is finite, consequently $x(t)$ has limits at a and b. Moreover, if $x(t)$ is maximal (it cannot be extended at its end-points by another piece of trajectory satisfying (3.2)), then the limit points are necessarily critical points of f. So, in Definition 3.2, we may assume that the interval I is closed and bounded, since M is compact. The exceptional values $t \in I$ are those for which $x(t)$ is a critical point of f. Clearly, there are finitely many of them since f has finitely many critical values and $t \mapsto f(x(t))$ is strictly increasing.

4. *Length of the trajectories of* $\nabla_M f$

Throughout this section φ is still a polynomial of degree d in n variables and $M \subset B^n(a, r)$ is a smooth compact connected component of the algebraic hypersurface $\varphi^{-1}(0)$. The goal of this section is to give a universal upper bound (depending only on d, n and r) for the length of the trajectories of $\nabla_M f$. Namely,

THEOREM 4.1. *The length of any trajectory of* $\nabla_M f$ *is bounded by*

$$r\nu(n)d(4d-5)^{n-2},$$

where $\nu(n) = 2\Gamma(\frac{1}{2})\Gamma(\frac{n+1}{2})\Gamma(\frac{n}{2})^{-1}$.

In order to prove Theorem 4.1 we adapt the method developed in our former paper (cf. [**8**]). The main idea is to replace the study of transcendental objects (trajectories of an ordinary differential equation) by the study of a one-dimensional semialgebraic object (and furthermore an algebraic one) contained in M. This leads to the following

Comparison principle

Let $\Gamma \subset M$ be a semialgebraic curve meeting each level set of f. Assume for every point $y \in \Gamma$ we have $|\nabla_M f(y)| \leq |\nabla_M f(x)|$ for all $x \in f^{-1}(f(y))$. Then

LEMMA 4.2 (Comparison principle, [**8**]). *For every couple of values $a < b$ taken by f, the length of any trajectory of $\nabla_M f$ lying in $f^{-1}(]a, b[)$ is bounded by the length of $\Gamma \cap f^{-1}(]a, b[)$.*

Proof. Since Γ is semialgebraic we can reduce the problem to the case where $\Gamma \cap f^{-1}(]a, b[)$ is smooth connected and transverse to the level sets of f. Our function

f has finitely many critical values, so we may assume as well that f has no critical point in $M \cap f^{-1}(]a, b[)$. We still denote by Γ the curve $\Gamma \cap f^{-1}(]a, b[)$.

Let $x(s)$ be the arc-length parametrisation of a trajectory X of $\nabla_M f$ and let $\gamma(u)$ be the arc-length parametrisation of the curve Γ. We fix orientations so that both functions $s \mapsto (f \circ x)(s)$ and $u \mapsto (f \circ \gamma)(u)$ are strictly increasing.

Let $\eta : X \to \Gamma$ be the map $\eta = (f|_\Gamma)^{-1} \circ (f|_X)$. We now compute η in our arc-length charts, that is we consider $h(s) = \gamma^{-1} \circ \eta \circ x(s)$. Clearly, to prove Lemma 4.2 it is enough to show that $h'(s) \geq 1$.

Taking the derivative with respect to s in the equality $(f \circ x)(s) = (f \circ \eta \circ x)(s)$ we obtain the following:

$$\langle \nabla f(x(s)), x'(s) \rangle = \langle \nabla f((\eta \circ x)(s)), (\eta \circ x)'(s) \rangle.$$

But $x'(s) = \dfrac{\nabla f(x(s))}{|\nabla f(x(s))|}$, hence $|\nabla f(x(s))| \leq |\nabla f((\eta \circ x)(s))| \cdot |(\eta \circ x)'(s)|$. Since $\eta(x(s)) \in \Gamma$, we have $|\nabla f(x(s))| \geq |\nabla f((\eta \circ x)(s))|$, thus $|(\eta \circ x)'(s)| \geq 1$. But γ is an arc-length parametrisation, so $h'(s) = (\gamma^{-1} \circ \eta \circ x)'(s) = |(\eta \circ x)'(s)| \geq 1$. $\square$

Length of the trajectories of $\nabla_M f$

We consider here $|\nabla_M f|^2$ as a function defined on some open neighbourhood of M in $\mathbb{R}^n$, which is always possible, by formula (3.1). Note that the curve Γ is contained in the algebraic set $\Theta_M(f)$ of critical points of the function $|\nabla_M f|^2$ on the level sets of f, which we write

$$\Theta_M(f) = \{x \in M : \mathrm{d}(|\nabla_M f|^2) \wedge \mathrm{d}\pi \wedge \mathrm{d}\varphi = 0\}. \tag{4.1}$$

Actually, we shall also consider the set $\Gamma(f)$ of points $x \in M$ such that $|\nabla_M f|$ restricted to $f^{-1}(f(x))$, has a local minimum at x. Clearly, $\Gamma(f)$ is semialgebraic and $\Gamma \subset \Gamma(f) \subset \Theta_M(f)$. Our goal is to show, that for a generic line L and a generic polynomial φ, the set $\Gamma(f)$ is of dimension 1. Moreover, we shall give an explicit description of $\Gamma(f)$.

Note that for all $a \in L$ such that a is not a critical value of f, the critical points of the function $|\nabla_M f|^2$ over $\pi^{-1}(a) \cap M$ are the points where the gradient of $|\nabla_M f|^2$ is a linear combination of $\nabla\pi$ and $\nabla\varphi$. This implies the above expression for $\Theta_M(f)$.

Observe that a point $x \in M$ belongs to $\Theta_M(f)$ if and only if there exists $\lambda \in \mathbb{R}$ such that $\nabla_M \left(|\nabla_M f|^2\right)(x) = \lambda \nabla_M f(x)$. Recall that f is the restriction to M of the projection π on the last coordinate, obviously $\nabla\pi = \frac{\partial}{\partial x_n}$. Computing the partial derivatives of $|\nabla_M f|^2$ we get

$$\nabla \left(|\nabla_M f|^2\right) = -2 \frac{\langle \frac{\partial}{\partial x_n}, \nabla\varphi \rangle}{|\nabla\varphi|^2} Hess(\varphi) \nabla_M f, \tag{4.2}$$

where $Hess(\varphi)$ denotes the Hessian matrix of φ.

In general, the set $\Theta_M(f)$ is a hypersurface in M, because of the factor $\langle \frac{\partial}{\partial x_n}, \nabla\varphi \rangle$. We explain below that this factor can actually be neglected. Let us fix $v \in \mathbb{S}^{n-1}$ and define $J_M(f) = \{x \in M : \langle v, \nabla\varphi(x) \rangle = 0\}$, where $f = f_L$ is the restriction to M of the orthogonal projection on the line L spanned by v. Observe that for any $x \in M$ we have

$$x \in J_M(f) \Leftrightarrow \nabla_M(f)(x) = v.$$

Clearly, for a generic line $L \in \mathbb{P}^{n-1}$, the set $J_M(f)$ is a hypersurface in M, that is $\dim J_M(f) = n-2$. In such a case, there are only finitely many values $y_1, \ldots, y_k \in \mathbb{R}$

such that $\dim(f^{-1}(y_i)\cap J_M(f)) = n-2$. Now, let $y \in f(M)\setminus(\{y_1,\dots,y_k\}\cup K(f))$, where $K(f)$ stands for the set of critical values of f, then $f^{-1}(y)$ is a smooth manifold of dimension $n-2$. Let C_y be a connected component of $f^{-1}(y)$, then $\nabla_M f \not\equiv v$ on C_y. Hence, all local minima of the function $|\nabla_M f|$ on C_y are not in $J_M(f)$. Let ω be the dual 1-form of the polynomial vector field $|\nabla\varphi|^2 Hess(\varphi)\nabla_M f$. Recall that $\nabla\varphi \neq 0$ in a neighbourhood of M. Finally, put

$$\theta_M(f) = \{x \in M : \omega \wedge \mathrm{d}\pi \wedge \mathrm{d}\varphi = 0\}. \tag{4.3}$$

Remark 4.3. Note that the coefficients of ω are polynomials of degree bounded by $4d-5$. Moreover, $\theta_M(f)\setminus J_M(f) = \Theta_M(f)\setminus J_M(f)$. Hence, we have

$$(\Gamma \setminus f^{-1}(Y)) \subset (\Gamma(f)\setminus f^{-1}(Y)) \subset \theta_M(f),$$

where $Y = \{y_1,\dots,y_k\}\cup K(f)$ is a finite set. In particular $\Gamma \subset \theta_M(f)$ except possibly finitely many points.

We now introduce some notations: by $T_xM = (\nabla\varphi(x))^{\perp}$ we denote the tangent space to M at x, then by $i(x) : T_xM \to \mathbb{R}^n$ the inclusion map and by $p(x) : \mathbb{R}^n \to T_xM$ the orthogonal projection. Finally we put $\widetilde{Hess}(\varphi)(x) = p(x)\circ Hess(\varphi)(x)\circ i(x)$. Also recall that

$$\nabla_M\left(|\nabla_M f|^2\right) = p(x)\left(\nabla\left(|\nabla_M f|^2\right)(x)\right).$$

So applying the orthogonal projection $p(x)$ to the both sides of equality (4.2) we obtain

Lemma 4.4. *A point x belongs to $\theta_M(f)$ if and only if $\nabla_M f(x)$ is an eigenvector of $\widetilde{Hess}(\varphi)(x)$.*

The set $\theta_M(f)$ is crucial in the estimate on the length of the trajectories of $\nabla_M f$. When this set is a curve, we can easily estimate its length, and we have length $\theta_M(f) \geq$ length Γ . The following proposition shows that $\theta_M(f)$ is in general of dimension 1.

Proposition 4.5. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree $d \geq 2$. Let U be an open bounded subset of $\mathbb{R}^n$. Assume that $|\nabla\varphi| \geq c > 0$ in U and that $M = U\cap\varphi^{-1}(0)$ is compact (hence M is the smooth compact manifold, the union of some connected components of $\varphi^{-1}(0)$). Then, for any $\delta > 0$, there exists $\eta : \mathbb{R}^n \to \mathbb{R}$ a polynomial of degree 2 and $\mathcal{L}$ an open dense semialgebraic subset of the projective space $\mathbb{P}^{n-1}$ such that:*

(i) *the norm (max. of absolute values of the coefficients) of η is bounded by δ;*

(ii) *if a line L belongs to $\mathcal{L}$, then for $\tilde\varphi = \varphi + \eta$ and $\tilde M = U \cap \tilde\varphi^{-1}(0)$ we have*

$$\dim \theta_{\tilde M}(f_L) = 1,$$

where f_L is the orthogonal projection on L.

Let us introduce an incidence variety which is crucial for the proof. We put

$$\Sigma = \{(S,H,V) \in \mathcal{S}_n \times \mathbb{G}_{n-1,n} \times \mathbb{R}^n : \exists\lambda \in \mathbb{R}, p_H(S\cdot p_H(V)) = \lambda p_H(V)\},$$

where $\mathcal{S}_n$ is the space of $n \times n$ symmetric matrices, $\mathbb{G}_{n-1,n}$ the Grassmannian of vector hyperplanes of $\mathbb{R}^n$ and p_H the orthogonal projection on H. Proposition 4.5 is a consequence of the following lemma.

LEMMA 4.6. *The set Σ is algebraic of codimension $n-2$.*

Proof. Fix a hyperplane $H = \mathbb{R}^{n-1} \times \{0\}$ and define

$$\Sigma_H = \{(S, V) \in \mathcal{S}_n \times \mathbb{R}^n : \exists \lambda \in \mathbb{R},\ p_H(S \cdot\ p_H(V)) = \lambda p_H(V)\}.$$

Then Σ_H is of codimension $n-2$ in $\mathcal{S}_n \times \mathbb{R}^n$. Indeed, it is known (by [**8**] for instance) that the set

$$\Sigma'_{n-1} = \{(S', V') \in \mathcal{S}_{n-1} \times \mathbb{R}^{n-1} : \exists \lambda \in \mathbb{R},\ S' \cdot\ V' = \lambda V'\}$$

is algebraic of codimension $n-2$ in $\mathcal{S}_{n-1} \times \mathbb{R}^{n-1}$. Here $\mathcal{S}_{n-1}$ stands for the space of $(n-1) \times (n-1)$ symmetric matrices. It is easily seen that Σ_H is the inverse image of Σ'_{n-1} by the projection π which associates to each matrix S the matrix S' of the first $n-1$ rows and columns, and which associates to a vector V its orthogonal projection V' on $H = \mathbb{R}^{n-1} \times \{0\}$. Hence the codimensions of Σ_H and Σ'_{n-1} are equal.

On the other hand we have the canonical projection

$$P_2 : \mathcal{S}_n \times \mathbb{G}_{n-1,n} \times \mathbb{R}^n \to \mathbb{G}_{n-1,n}.$$

Clearly, $\Sigma_H \times \{H\} = P_2^{-1}(H) \cap \Sigma$. Above we have proved that $P_2^{-1}(H) \cap \Sigma$ is of codimension $n-2$ in the full fibre $P_2^{-1}(H)$. Hence (see for instance Theorem 3.18 in [**7**]), Σ is of codimension $n-2$. $\square$

Proof of Proposition 4.5. We will consider quadratic deformations of the polynomial φ. For $\alpha = (\alpha_i) \in \mathbb{R}^n$ and $\varepsilon = (\varepsilon_{jk}) \in \mathbb{R}^{\frac{n(n+1)}{2}}$ we put

$$\tilde{\varphi}_{\alpha,\varepsilon}(x) = \varphi(x) + \sum_{i=1}^{n} \alpha_i x_i + \sum_{1 \le j \le k \le n} \varepsilon_{jk} x_j x_k.$$

Let $Hess(\tilde{\varphi}_{\alpha,\varepsilon})$ stand for the Hessian matrix of $\tilde{\varphi}_{\alpha,\varepsilon}$. Since U is bounded and $|\nabla \varphi| \ge c$ in U, for some $c > 0$, so there exists $\delta > 0$ such that $\nabla \tilde{\varphi}_{\alpha,\varepsilon}(x)$ does not vanish in U, if $\|\alpha\| < \delta$ and $\|\varepsilon\| < \delta$. Hence the hyperplane $H_{\alpha,\varepsilon}(x)$, which is the orthogonal of $\nabla \tilde{\varphi}_{\alpha,\varepsilon}(x)$, is well defined. We now consider the map

$$\psi : (x, \alpha, \varepsilon, V) \mapsto (Hess(\tilde{\varphi}_{\alpha,\varepsilon})(x), H_{\alpha,\varepsilon}(x), V).$$

Note that ψ is a submersion on $U \times Q$, where $Q = \{(\alpha, \varepsilon, V) \in \mathbb{R}^n \times \mathbb{R}^{\frac{n(n+1)}{2}} \times \mathbb{R}^n;\ \|\alpha\| < \delta,\ \|\varepsilon\| < \delta\}$.

We deduce from the transversality Theorem with parameters (see e.g. [**11**]) that there exists an open dense semialgebraic set $\Omega \subset Q$ such that the map

$$\psi_{\alpha,\varepsilon,V} = \psi(\cdot, \alpha, \varepsilon, V) : U \to \mathbb{R}^n \times \mathbb{R}^{\frac{n(n+1)}{2}} \times \mathbb{R}^n$$

is transverse to Σ, for any $(\alpha, \varepsilon, V) \in \Omega$. Clearly, for such a choice of parameters, the set $\psi^{-1}_{\alpha,\varepsilon,V}(\Sigma)$ is algebraic (possibly empty) of dimension at most 2. So there is $t \in (-\delta, \delta)$ such that

$$\tilde{M} = M_{\alpha,\varepsilon,t} = U \cap \tilde{\varphi}^{-1}_{\alpha,\varepsilon}(t)$$

is a smooth compact manifold, moreover $\dim\left(\psi^{-1}_{\alpha,\varepsilon,V}(\Sigma)\cap M_{\alpha,\varepsilon,t}\right)\le 1$. Let us fix (α,ε) so that the set $L_{\alpha,\varepsilon}=\{V\in\mathbb{R}^n : V\neq 0,\ (\alpha,\varepsilon,V)\in\Omega\}$ is open and dense in $\mathbb{R}^n$. Now let $\mathcal{L}$ denote the set of lines spanned by all vectors $V\in L_{\alpha,\varepsilon}$. For any line $\mathbb{R}V=L\in\mathcal{L}$ it is easily seen that

$$\theta_{\tilde{M}}(f_L)=\psi^{-1}_{\alpha,\varepsilon,V}(\Sigma)\cap M_{\alpha,\varepsilon,t}.$$

So we have proved that $\dim\theta_{\tilde{M}}(f_L)\le 1$. But on each level set of f_L the function $|\nabla_M f_L|$ reaches its minimum at a point belonging to $\theta_{\tilde{M}}(f_L)$, therefore $\dim\theta_{\tilde{M}}(f_L)\ge 1$. Hence Proposition 4.5 follows. □

Recall that M is a compact connected component of $\varphi^{-1}(0)$ contained in a ball $B^n(a,r)$. The following proposition provides the bound announced in Theorem 4.1 for the length of the trajectories of $\nabla_M f$, in the case of a generic (in the sense of Proposition 4.5) polynomial φ.

PROPOSITION 4.7. *For a generic polynomial φ of degree $d\ge 2$, the length of the trajectories of $\nabla_M f$ is bounded by the length of $\theta_M(f)=\{x\in M : \omega\wedge d\pi\wedge d\varphi=0\}$. In particular this length is not greater than*

$$rA(n,d)=r\nu(n)d(4d-5)^{n-2}.$$

Recall that $\theta_M(f)$ contains the points in M at which $|\nabla_M f|$ is minimal on the level sets of f. So, Proposition 4.7 is a straightforward consequence of Lemma 4.2 and of an estimate for the length of $\theta_M(f)$ which we obtain below (cf. Lemma 4.8). In fact, we shall consider the set

$$\theta(f)=\{x\in\varphi^{-1}(0)\cap B^n(a,r) : \omega\wedge \mathrm{d}\pi\wedge \mathrm{d}\varphi=0\}.$$

Clearly $\theta_M(f)\subset\theta(f)$. We have

LEMMA 4.8. *For a generic polynomial φ the length of $\theta(f)$ is bounded by*

$$rA(n,d)=r\nu(n)d(4d-5)^{n-2}.$$

Proof. Let φ be a generic polynomial, by Proposition 4.5 we may assume that $\theta(f)$ is of dimension 1. From equations (4.2) and (4.3) we obtain that $\theta(f)$ is contained in the common zeros of $n-1$ polynomials as follows:
(i) the polynomial φ;
(ii) $n-2$ independent coefficients of the 3-form $\omega\wedge\mathrm{d}\pi\wedge\mathrm{d}\varphi$.
By Remark 4.3 the coefficients of the form $\omega\wedge\mathrm{d}\pi\wedge\mathrm{d}\varphi$ are polynomials of degree bounded by $4d-5$. Hence $\theta(f)$ is contained in an algebraic set described by one polynomial equation of degree d and $n-2$ polynomial equations of degree bounded by $4d-5$.

We will compute the length of $\theta(f)$ using the Cauchy-Crofton formula which we recall below. Let C be a compact rectifiable curve and let $\mathcal{H}$ denote the set of affine hyperplanes in $\mathbb{R}^n$. Then, for almost all hyperplanes $H\in\mathcal{H}$ the set $C\cap H$ is finite. Let $i(\mathrm{C},H)$ denote the cardinal of $C\cap H$. There exists a normalisation $\mathrm{d}\mu$ of the canonical measure $\mathrm{d}\tilde{\mu}$ on $\mathcal{H}$ such that the length of C can be expressed by the following formula (Cauchy-Crofton):

$$\text{Length } (C) = \int_{\mathcal{H}} i(C, H)\mathrm{d}\mu. \tag{4.4}$$

We are now in a position to complete the proof of Lemma 4.8. By the generalised version of Bezout's Theorem, the number of irreducible components of the intersection of $\theta(f)$ with a generic hyperplane H is bounded by the product of degrees of the polynomials defining $\theta(f)$. Note that for a generic hyperplane H and for a generic polynomial φ all the real points of $H \cap \theta(f)$ are non-degenerate. Hence we get $i(\theta(f), H) \leq d(4d-5)^{n-2}$ and

$$\text{Length } (\theta(f)) \leq rd(4d-5)^{n-2} \int_{\mathcal{H}_1} \mathrm{d}\mu,$$

where $\mathcal{H}_1$ is the set of affine hyperplanes that cut the unit ball (see [**8**] for more details). We denote by $\nu(n)$ the integral $\int_{\mathcal{H}_1} \mathrm{d}\mu$ and thus we obtain the desired bound for $\theta(f)$. The volume $\nu(n)$ can be explicitly computed (cf. [**16**]). Namely, $\nu(n) = 2\Gamma(\frac{1}{2})\Gamma(\frac{n+1}{2})\Gamma(\frac{n}{2})^{-1}$ where Γ is the Euler *Gamma* function. Hence Lemma 4.8 follows. $\square$

Proof of Theorem 4.1 in the general case. Let M be a compact connected component of an algebraic set $\varphi^{-1}(0)$ and assume that

$$|\nabla\varphi| \geq \delta > 0 \text{ in a neighbourhood of } M. \tag{4.5}$$

Let $f : M \to \mathbb{R}$ denote a linear projection restricted to M. Finally let us fix a trajectory $x(t)$ of $\nabla_M f$ and $\varepsilon > 0$. Recall that $K(f)$ – the set of critical values of f– is finite. By Łojasiewicz's Gradient Inequality (see also [**8**], Theorem 7.10) there exists a neighbourhood V of $K(f)$ such that the length of all parts of $x(t)$ lying in $f^{-1}(V)$ is not greater than $\frac{\varepsilon}{2}$. Note that there exists an $\eta > 0$ such that

$$|\nabla_M f| \geq \eta \text{ in } M \setminus f^{-1}(V). \tag{4.6}$$

Let $\pi : T(M) \to M$ be a tubular neighbourhood of M, that is $T(M) \subset \mathbb{R}^n$ is an open neighbourhood of M and π is the orthogonal projection (retraction) on M. According to Proposition 4.5, by a small change $\tilde{L}$ of the line L and by a small perturbation of the coefficients of φ we obtain a polynomial $\tilde{\varphi}$ such that Proposition 4.7 applies to $\tilde{M} = T(M) \cap \tilde{\varphi}^{-1}(0)$ and the projection on $\tilde{L}$. We have a mapping $h : \tilde{M} \to M$ which is the restriction of π to $\tilde{M}$. If the perturbation $\tilde{\varphi}$ is very small then h is very close to the identity. More precisely, for any $\sigma > 0$ there exists a perturbation $\tilde{\varphi}$ such that the differential $d_x h : T_x\tilde{M} \to T_{h(x)}M$ satisfies

$$|d_x h(y) - y| \leq \sigma|y|,\ y \in T_x\tilde{M}. \tag{4.7}$$

It follows that a similar estimate holds for the differential of $g = h^{-1}$. That is

$$|d_y g(x) - x| \leq \sigma'|x|,\ x \in T_y M, \tag{4.8}$$

with $\sigma' = \dfrac{\sigma}{1-\sigma}$.

Thus the direct image $\xi = g_*(\nabla_M f)$ is a vector field on $\tilde{M}$ which is very close to the gradient field of $f = f_L$ on $\tilde{M}$, more precisely is very close on the set $g(M \setminus f^{-1}(V))$. The same will hold if we slightly perturb the line L. So we may assume that Proposition 4.7 applies to $\tilde{M}$ and the projection $f = f_L$ on the line L. Arguing as in the proof Theorem 7.10 of [**8**] we can prove that the length of the trajectories of ξ is bounded by $(1+\sigma')\times rA(n,d)$. Hence the length of $x(t)$ in $M\setminus f^{-1}(V)$ is bounded by $(1+\sigma')^2\times rA(n,d)$. We choose σ such that

$$((1+\sigma')^2-1)rA(n,d)\leq\frac{\varepsilon}{2}.$$

Hence Theorem 4.1 follows. □

5. *Proof of the main result when M is a smooth hypersurface*

We will use the trajectories of $\nabla_M f$ to bound the geodesic diameter of M. The first approach could be the following:
If φ has degree d then the function f defined above has at most $(d-1)^{n-1}$ critical points. This implies that we can join two critical points of f by a trajectory of $\nabla_M f$, and thus the geodesic diameter of M is bounded by $2r\nu(n)(d-1)^{n-1}d(4d-5)^{n-2}$.

Actually this bound can be improved. We explain it in details throughout the rest of this section. The key point to make the bound sharper is the following idea which is similar to the proof of Theorem 9.3 of [**8**]. Let us be more explicit.

Let $M^{(t)}=\{x\in M:f(x)>t\}$ for $t\in f(M)=[a,b]$. We will prove Theorem 2.1 for each connected component of $M^{(t)}$ for all t. We can choose the line L so that f has exactly one global maximum and one global minimum. We can also assume that all the critical points $c_i,\ i=1,\dots,k$ of f lie in distinct levels of f. Let $M^{(t)}=\bigcup\limits_{i\in I_t}M_i^{(t)}$, where $M_i^{(t)}$ are the connected components of $M^{(t)}$. Let $\theta_M(f)$ be the set defined in section 4. The following observation is important for the proof of Theorem 2.1.

Lemma 5.1. *For any $t\in[a,b]$, any connected component of $f^{-1}(t)$ contains a point of $\theta_M(f)$.*

Indeed, $|\nabla_M f|^2$ has a minimum in each connected component of $f^{-1}(t)$, and $\theta_M(f)\cap f^{-1}(t)$ contains all local minima of $|\nabla_M f|^2$ on $f^{-1}(t)$. Note that the length of any trajectory of $\nabla_M f$ in $M_i^{(t)}$ is bounded by the length of $\theta_M(f)\cap M_i^{(t)}$. We can now state the key result of this section.

Lemma 5.2. *For all $t\in[a,b]$ and for all $i\in I_t$, any two points in $M_i^{(t)}$ can be joined by a curve of length bounded by* $2\text{Length}\,(\theta_M(f)\cap M_i^{(t)})$.

Proof. We proceed by induction on the number of critical points of f that lie in $M^{(t)}$. Let

$$a=y_1=f(c_1)<y_2=f(c_2)<\ldots<y_k=f(c_k)=b$$

be the critical values of f. Assume that $t\in]y_{k-1},b[$, and let $x_0,x_1\in M^{(t)}$. The set $M^{(t)}$ is connected and the trajectories of $\nabla_M f$ starting at x_0 and x_1 have the same limit point $c_k=\omega(x_1)=\omega(x_2)$ and Lemma 5.2 holds in this situation.

Let $c_j<c_k$ be a critical point of f and let $t\in]y_{j-1},y_j[$. Let us also fix a connected component $M_i^{(t)}$. If $c_j\notin M_i^{(t)}$ then any two points $x_0,x_1\in M_i^{(t)}$ can be joined by pieces of gradient trajectories. More precisely, if $f(x_0)\leq y_j$ and $f(x_1)\leq y_j$, the integral curves γ_{x_0} and γ_{x_1} cross the level set $f^{-1}(y_j)$ and their intersections with some level $s>y_j$ belong to the same connected component of $M^{(s)}$. Using the induction hypothesis and Lemma 5.1, the total length of the trajectories joining x_0 to x_1 does not exceed 2Length $(\theta_M(f) \cap M_i^{(t)})$.

Assume that $c_j \in M_i^{(t)}$. If c_j is a **local maximum**, then we are in the first case described above.

Assume c_j is a **local minimum**. First we examine the situation around c_j. Let $\varepsilon > 0$ be small enough so that y_j is the only critical value of f in the interval $[t, y_j + \varepsilon]$. Let D_{c_j} be the connected component of $f^{-1}(]t, y_j + \varepsilon])$ containing c_j. We take ε small enough so that any trajectory of $-\nabla_M f$, with initial condition in D_{c_j}, has its limit point at c_j. If we want to join, in $M_i^{(t)}$, two points that belong to D_{c_j}, we can do it as in the case of the local maximum. That is we follow the trajectories of $-\nabla_M f$ which meet at c_j. Clearly, the total length of these curves does not exceed 2Length $(\theta_M(f) \cap D_{c_j})$.

Suppose now that we want to join a point $x_0 \in D_{c_j}$ with a point $x_1 \in M_i^{(t)} \setminus D_{c_j}$ such that $f(x_1) < y_j$. Following the trajectories of $\nabla_M f$ starting at points x_0, x_1 we arrive at the points $x_0', x_1' \in M_i^{(y_j+\varepsilon)}$. Then the total length of these curves does not exceed 2Length $(\theta_M(f) \cap f^{-1}([t, y_j + \varepsilon]))$. Next we apply the induction to join x_0' with x_1' in $M_i^{(y_j+\varepsilon)}$.

The case when both points x_0, x_1 belong to $M_i^{(t)} \setminus D_{c_j}$ is similar.

The last case, when c_j is a **saddle point**, needs more care. If a trajectory ends at some critical point, we extend it by a trajectory starting at this point. Fix $s \in]y_j, y_{j+1}[$ and take any two points $x_0, x_1 \in M_i^{(t)}$ and assume that both trajectories γ_{x_0} and γ_{x_1} reach the level $f^{-1}(s)$ at points x_0' and x_1' in two different connected components of $M^{(y_j)}$. We can issue from the critical point c_j two trajectories η_0 and η_1 reaching the level $f^{-1}(s)$ at two points x_0'' and x_1''. We mark them in the way that x_0' and x_0'', respectively x_1' and x_1'', are in the same connected component of $M^{(y_j)}$. Now, by induction, we can join x_0' with x_0'', respectively x_1' with x_1''. Gluing together we obtain a curve (piecewise trajectory of $\nabla_M f$ and $-\nabla_M f$) joining x_0 to x_1 in $M_i^{(t)}$. Lemma 5.1 and the induction hypothesis imply the desired estimate for the length of the curve. This completes the proof of Lemma 5.2. $\square$

By connectedness of M, Lemma 5.2 implies that any two points in M can be joined by an arc of length bounded by $2r\nu(n)d(4d-5)^{n-2}$. This completes the proof of Theorem 2.1 when M is a smooth hypersurface.

To justify the bound (2.1) in Remark 2.2 for the sum of the geodesic diameters of all compact connected components of $\varphi^{-1}(0 \cap B^n(a,r))$ it is enough to observe that in Lemma 5.2 we only use the set $\theta_M(f) = \{x \in M : \omega \wedge \mathrm{d}\pi \wedge \mathrm{d}\varphi = 0\}$ which is a subset of $\theta(f) = \{x \in \varphi^{-1}(0) \cap B^n(a,r) : \omega \wedge \mathrm{d}\pi \wedge \mathrm{d}\varphi = 0\}$. But the estimate for the length of $\theta(f)$ is actually $r\nu(n)d(4d-5)^{n-2}$, by Lemma 4.8. On the other hand, if M and M' are two distinct compact connected components of $\varphi^{-1}(0)$, then the corresponding sets $\theta_M(f)$, $\theta_{M'}(f)$ are disjoint.

6. *Proof of the main result when M is not a smooth hypersurface*

In this section $M \subset B^n(a,r)$ is again a compact connected component of an algebraic set $\varphi^{-1}(0)$ but possibly singular. It means φ may have critical points on

M. We shall prove that the bound $2r\nu(n)d(4d-5)^{n-2}$ still holds for the geodesic diameter of M. The main idea is to represent M as a limit, for the Hausdorff distance, of a sequence of unions of some components of smooth hypersurfaces in $\mathbb{R}^n$ of the same degree as $\deg\varphi$. This kind of perturbation of a singular set, called "thickening", was discovered by H. Hironaka and proved (for instance) in [**6**]. We present below the statement, adapted to our context. We begin with some elementary properties of geodesic diameters and Hausdorff limits.

We thank Michel Coste who suggested some improvements in the singular case.

LEMMA 6.1. *Let M be a connected subset of $\mathbb{R}^n$. Assume that $M = X_1 \cup \cdots \cup X_s$, where $X_1, \dots, X_s$ are closed, connected subsets of M, then*

$$D(M) \leq \sum_{i=1}^{s} D(X_i).$$

LEMMA 6.2. *Suppose that a compact set $X \subset \mathbb{R}^n$ is the Hausdorff limit of a sequence of smooth compact connected submanifolds $X^k \subset \mathbb{R}^n$, $k \in \mathbb{N}$. Moreover, assume there exists $L > 0$ such that $D(X^k) \leq L$ for every $k \in \mathbb{N}$. Then*

$$D(X) \leq \varliminf_{k\to\infty} D(X^k).$$

Proof. Let $x, y \in X$, then there are two sequences of points $x_k \in X^k$ and $y_k \in X^k$ such that $x_k \to x$ and $y_k \to y$. Let $\gamma_k : [0, \beta_k] \to X^k$ be a smooth curve $\gamma_k(0) = x_k$, $\gamma_k(\beta_k) = y_k$ of length not greater than $D(X^k) \leq L$. By rescaling the arc-length parametrisation we may assume that $\beta_k = L$ and $|\gamma_k'| \leq 1$, for all $k \in \mathbb{N}$. So $\gamma_k : [0, L] \to \mathbb{R}^n$ is a bounded sequence of 1-Lipschitz arcs, hence by Ascoli's Theorem it has a convergent subsequence. Thus, we may assume that γ_k converges uniformly to an arc $\gamma : [0, L] \to \mathbb{R}^n$. Hence $\gamma([0, L]) \subset X$ and $\gamma(0) = x$, $\gamma(L) = y$. Moreover, γ is a 1-Lipschitz arc, so its length is not greater than L. To conclude we may suppose that $L = \varepsilon + \varliminf_{k\to\infty} D(X^k)$, where $\varepsilon > 0$ is arbitrary small. Hence Lemma 6.2 follows. □

We now state a simplified version of Hironaka's Thickening Lemma (cf. [**6**]).

THEOREM 6.3. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree d. Then there exist a semialgebraic family $\{\varphi_t\}_{t\in[0,1)}$ of polynomials of degree d such that:*
(i) *$\varphi_0 = \varphi$;*
(ii) *for any $t \in]0, 1[$ the set $\varphi_t^{-1}(0)$ is a nonsingular hypersurface;*
(iii) *the closure of the family $\varphi_t^{-1}(0)$, $t \in]0, 1[$ over $t = 0$ is equal to $\varphi^{-1}(0)$.*

Semialgebraic family means here that the coefficients of φ_t are continuous semialgebraic functions on $[0, 1[$ and actually we may assume that they are analytic on $]0, 1[$. Point (iii) of the statement means the following: let $\Sigma = \{(x, t) \in \mathbb{R}^n \times]0, 1[: \varphi_t(x) = 0\}$, then

$$\overline{\Sigma} \cap (\mathbb{R}^n \times \{0\}) = \varphi^{-1}(0) \times \{0\},$$

where the closure is taken in the strong topology in $\mathbb{R}^n \times \mathbb{R}$.

Assume now M is a compact connected component of $\varphi^{-1}(0)$ and U is a neighbourhood of M such that $U \cap \varphi^{-1}(0) = M$. Property (iii) in Theorem 6.3 says that

M is the Hausdorff limit of the family $U \cap \varphi_t^{-1}(0)$ as $t > 0$ tends to 0. Observe that, for $t > 0$ small enough, $U \cap \varphi_t^{-1}(0)$ is the union of compact connected submanifolds $X_1^t, \dots, X_s^t$. Moreover, since the family $\varphi_t^{-1}(0)$, $t \in]0,1[$ may be chosen topologically trivial (cf. [**4**]), the number of components is constant. We may arrange these components in such a way that, for each $i = 1, \dots, s$, the family X_i^t, $t \in]0,1[$ is topologically trivial. In particular, the Hausdorff limit, as $t \to 0$, of the family X_i^t, $t \in]0,1[$ exists and we denote it by X_i. Clearly each X_i is compact and connected, moreover $M = X_1 \cup \dots \cup X_s$. Assume now that $U \subset B(a,r)$. So, by Theorem 2.1 in the smooth case (more precisely by Remark 2.2), we know that $\sum_{i=1}^s D(X_i^t) \le rA(n,d)$. Hence Lemmas 6.1 and 6.2 yield estimate $D(M) \le rA(n,d)$ in the case where M is singular. Thus Theorem 2.1 follows. We leave to the reader a proof of the estimate (2.1) in Remark 2.2 in the singular case.

REMARK 6.4. In [**8**] we gave an explicit bound for the length of the trajectories of the gradient of an exponential polynomial. In both works ([**8**] and the present one) we needed to count the number of zeros of a system of equations. Let φ be an exponential polynomial of degree d containing k exponentials. By Khovanskiĭ's theory [**12**], the geodesic diameter of a compact connected component $M \subset B(a,r)$ of $\varphi^{-1}(0)$ is bounded by

$$rB(n,d,k) = 2r\nu(n)3^{n-2}d^{n-1}\left((3n-5)d+1\right)2^{\frac{k(k-1)}{2}}.$$

7. *Asymptotic sharpness of the bound via some examples*

Let us fix integers d and n. We will denote by $D(d,n)$ the supremum of geodesic diameters of connected components – included in a ball of radius 1 in $\mathbb{R}^n$– of sets $\varphi^{-1}(0)$ where φ is a polynomial of degree d. We will now show that

THEOREM 7.1.
(1) For any $d \ge 1$ we have $D(2d,2) = D(2d+1,2) = \pi d$;
(2) for any $n, d \in \mathbb{N}$ such that $n \ge 3$ and $d \ge 1$

$$D(2d,n) \ge 2d^{n-1}.$$

Proof of Theorem 7.1 statement (1). Let φ be a polynomial in 2 variables of even degree $2d \ge 2$ and let $\mathcal{C}$ be a compact connected component of $\varphi^{-1}(0)$ contained in the unit ball of $\mathbb{R}^n$. This means $\mathcal{C}$ is a closed plane curve.

From Cauchy-Crofton Formula (4.4) we obtain $D(\mathcal{C}) = \dfrac{1}{2}\mathrm{Length}\,\mathcal{C} \le \pi d$ and thus $D(2d,2) \le \pi d$. In order to prove the converse inequality we first assume

$$\varphi(x,y) = \prod_{i=1}^{d} \varphi_i(x,y)$$

where $\varphi_i(x,y) = (x-x_i)^2 + (y-y_i)^2 - (1-e)^2$, for $i = 1\dots,d$ and $0 < e \ll 1$. The centres of the circles $\mathcal{C}_i = \varphi_i^{-1}(0)$ are chosen in such a way that $\mathcal{C}_i \subset \mathbb{B}^n$ and the only singularities of $\varphi^{-1}(0)$ are ordinary double points. Note that $\sum_{i=1}^d D(\mathcal{C}_i) = (1-e)\pi d$. By Brusotti's Theorem (cf. [**3**]) for all sufficiently small $\varepsilon > 0$ there exists a deformation φ_ε, with $\deg \varphi_\varepsilon = d$, of φ such $\mathcal{C}_\varepsilon = \varphi_\varepsilon^{-1}(0)$ is a smooth connected curve contained in $\mathbb{B}^n$ verifying

$$|D(\mathcal{C}_\varepsilon) - (1-e)\pi d| \leq \varepsilon.$$

Thus, for all sufficiently small $e > 0$, we have $D(2d, 2) \geq (1-e)\pi d$. Hence statement (1) of Theorem 7.1 is proved when $\deg \varphi$ is even.

To complete the proof when $\deg \varphi$ is odd, it suffices to notice that in such a case the projective closure of $\varphi^{-1}(0)$ always has a component called pseudo-line. Recall that its complement in $\mathbb{P}^2$ has only one connected component. The pseudo-line contains only non compact components of $\varphi^{-1}(0)$ and is intersected by every line. Thus $D(2d+1, 2) \leq \pi d$ while the same arguments as above can be reproduced to obtain $D(2d+1, 2) \geq \pi d$. □

Proof of Theorem 7.1 statement (2). We are going to construct a sequence of examples which will confirm the estimate from below. First we recall "sinusoidal like" properties of Chebyshev's polynomials. Recall that the d-th Chebyshev polynomial (of the first kind) $T_d(x)$ is determined by

$$T_d(\cos\theta) = \cos(d\theta). \tag{7.1}$$

In particular it has the following properties:

Lemma 7.2.
(i) $|T_d(x)| \leq 1$, *for* $x \in [-1, 1]$*;*
(ii) T_d *has* $d+1$ *extrema on* $[-1, 1]$*, and the values at each extremum is* ± 1.
Hence the length of the graph of T_d *restricted to* $[-1, 1]$*, is greater than* $2d$.

Now let us return to the construction in $\mathbb{R}^n$. Let us denote $x = (x_1, \ldots, x_n)$. Let $0 < \varepsilon < 1$ and, for $i = 1, \ldots, n-2$, let $f_i^\varepsilon(x) = x_{i+1} - \varepsilon T_d(x_i/\varepsilon)$. Let us also define $f_{n-1}^\varepsilon(x) = x_n - T_d(x_{n-1}/\varepsilon)$. Then we put

$$p_{d,n}^\varepsilon(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n-1} (f_i^\varepsilon(x))^2.$$

Note that $p_{d,n}^\varepsilon$ is a polynomial of degree $2d$ in n variables. The zero set of $p_{d,n}^\varepsilon$ is a smooth curve which is the intersection of the hypersurfaces $\{x \in \mathbb{R}^n : f_i^\varepsilon(x) = 0\}$, for $i = 1, \ldots, n-1$. By Lemma 7.2, it is not difficult to see that the length of the curve $(p_{d,n}^\varepsilon)^{-1}(0) \cap [-1, 1]^n$ is at least $2d^{n-1}$. We now choose $\alpha > 0$ small enough and we put

$$\tilde{f}^\varepsilon(x) = \alpha(\varepsilon^2 - x_1^2) - p_{d,n}^\varepsilon(x).$$

Note that the set $\{\tilde{f}^\varepsilon > 0\}$ is a thin neighbourhood of the curve

$$C_\varepsilon = (p_{d,n}^\varepsilon)^{-1}(0) \cap [-\varepsilon, \varepsilon]^{n-1} \times [-1, 1]$$

so its geodesic diameter (as well as its boundary's) is almost equal to the length of $(p_{d,n}^\varepsilon)^{-1}(0) \cap [-\varepsilon, \varepsilon]^{n-1} \times [-1, 1]$ which is at least $2d^{n-1}$, by Lemma 7.2.

Finally, to obtain the desired algebraic set in the unit ball, we take the set $\{x \in \mathbb{R}^n : f^\varepsilon(x) = 0\}$, where $f^\varepsilon(x) = \tilde{f}^\varepsilon((1 + (n-1)\varepsilon^2)^{-1/2}x)$, so Theorem 7.1 holds when taking $\varepsilon \to 0$. □

References

1. S. Basu, R. Pollack and M-F. Roy, *Computing roadmaps of semi-algebraic sets on a variety*, J. Am. Math. Soc. 13, No.1, (2000), 55-82.

2. S. Basu, R. Pollack and M-F. Roy, *Algorithms in real algebraic geometry*, Algorithms and Computation in Mathematics, 10. Springer-Verlag, Berlin, 2003

3. R. Benedetti and J.-J. Risler, *Real algebraic and semialgebraic sets*, Hermann (1990).

4. J. Bochnak, M. Coste and M.F. Roy, *Real algebraic geometry*, E.M.G vol. 36 (1998) Springer.

5. J. Canny, *Constructing roadmaps of semi-algebraic sets. I: Completeness*, Artif. Intell. 37, No.1-3 (1988), 203-222 .

6. M. Coste, *Epaissisement d'une hypersurface algébrique réele*, Proc. Japan Acad. Ser. A Math. Sci. 68 (1992), no. 7, 175–180.

7. M. Coste, *An Introduction to O-minimal Geometry*, Dip. Mat. Univ. Pisa, Dottorato di Ricerca in Matematica, Istituti Editoriali e Poligrafici Internazionali, Pisa (2000).

8. D. D'Acunto and K. Kurdyka, *Bounds for gradient trajectories of polynomial and definable functions with application* , Preprint de l'Université de Savoie, France (2003).

9. S. K. Donaldson, *Symplectic submanifold and almost-complex geometry*, J. Diff. Geometry 44 (1996), 666-705.

10. H. Federer, *Geometric measure theory*, Grundlehren Math. Wiss, vol. 153 Springer-Verlag (1969).

11. V. Guillemin and A. Pollack, *Differential topology*, Prentice-Hall (1974).

12. A.G. Khovanskiĭ, *Fewnomials*, Translations of Mathematical Mongraphs 88, AMS Providence, R.I. (1991).

13. S. Łojasiewicz, *Une propriété topologique des sous-ensembles analytiques réels*, Equ. Derivées partielles, Paris 1962, Colloques internat. Centre nat. Rech. sci. 117 (1963), 87-89.

14. S. Łojasiewicz, *Sur les trajectoires du gradient d'une fonction analytique*, Semin. Geom. Univ. Studi Bologna 1982/1983 (1984), 115-117.

15. J. Milnor, *On the Betti numbers of real varieties*, Proc. Amer. Math. Soc. 15 (1964), 275-280.

16. Y. Yomdin and G. Comte, *Tame geometry with applications in smooth analysis*, Lecture Notes in Mathematics, 1834. Springer-Verlag, Berlin, 2004.

D. D'Acunto
Dipartimento di Matematica
via Buonarroti, 2
56127 Pisa
ITALY

didier.dacunto@univ-savoie.fr

K. Kurdyka
Laboratoire de Mathématiques
UMR 5127 CNRS
Université de Savoie, Campus scientifique
F-73376 Le Bourget du Lac cedex
FRANCE

kurdyka@univ-savoie.fr