

# BOUNDS FOR GRADIENT TRAJECTORIES OF POLYNOMIAL AND DEFINABLE FUNCTIONS WITH APPLICATIONS

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ABSTRACT. We show that gradient trajectories of a definable (in an o-minimal structure) family of functions are of uniformly bounded length. We show that the length of a trajectory of the gradient of a polynomial in  $n$  variables of degree  $d$  in a ball of radius  $r$  is bounded by  $rA(n, d)$ , where  $A(n, d) = \nu(n)((3d - 4)^{n-1} + 2(3d - 3)^{n-2})$  and  $\nu(n)$  is an explicit constant. We give explicit bounds for the length of gradient trajectories of quasipolynomials and trigonometric quasipolynomials. As an application we give a construction of curves (piece-wise gradient trajectory of a polynomial) joining two points in an open connected semialgebraic set. We give an explicit bound for its length. We also obtain an explicit and quite sharp bound in Yomdin's version of quantitative Morse-Sard Theorem.

## 1. INTRODUCTION

The trajectories of a gradient field, or more generally the flow of a gradient field, appear in various branches of mathematics and its applications. The case of the gradient of a polynomial function is of a particular interest. If  $f : U \rightarrow \mathbb{R}$  is a polynomial of degree  $d$  restricted to an open and bounded subset of  $\mathbb{R}^n$ , then the length of any trajectory of  $\nabla f$  is bounded by some constant  $A > 0$ . This follows from the famous Lojasiewicz's inequality [Lo1],[Lo2]: there exists  $\rho < 1$  and  $C > 0$  such that

$$(1.1) \quad |\nabla f(x)| \geq C|f(x) - c|^\rho,$$

for any  $x \in U$  such that  $f(x)$  is close to  $c$  which is a critical value of  $f$ . (In fact Lojasiewicz proved this for any analytic function in a neighborhood of  $\bar{U}$ .) However, for a given polynomial  $f$ , it does not seem possible to obtain by this method the bound  $A$  in the case where  $U$  is simple, for instance where  $U$  is a unit ball in  $\mathbb{R}^n$ . (The constant  $C$  is difficult to control). To our best knowledge no such estimate for  $A$  was known explicitly.

In this paper we prove that actually there is a constant  $A = A(n, d)$  depending only on the degree  $d$  of  $f$  and  $n$  - the number of variables of  $f$ , such that the length of any trajectory of  $\nabla f$  is bounded by  $A$ . More precisely, in Theorem 7.11, we prove that, if  $U$  is a unit ball in  $\mathbb{R}^n$ , then

$$(1.2) \quad A(n, d) = \nu(n)((3d - 4)^{n-1} + 2(3d - 3)^{n-2}),$$

where  $\nu(n)$  is an explicit constant depending only on the dimension  $n$  (see Remark 4.3).

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Our method is actually different from the argument of Łojasiewicz: we compare the length of any trajectory to the length of a semialgebraic curve  $\Gamma$  obtained by minimizing  $|\nabla f|$  on the fibres of  $f$ . First, by a transversality argument for quadratic forms, we prove in Proposition 7.2 that for a generic polynomial, the set  $\Gamma$  is actually a curve. Then we prove in Theorem 7.6 that, for a generic  $f$ , the bound (1.2) holds and finally that it holds in the general case. The estimate for the length of  $\Gamma$  is based on the Cauchy-Crofton formula which we recall in Chapter 4.

In fact our method applies in the more general situation of definable (in an o-minimal structure) families of  $C^2$  functions. O-minimal structures (see e.g., [Dr],[Co],[Ku]) introduced recently by model theorists, are natural generalisation of semialgebraic or subanalytic geometry. We recall some basic facts on o-minimal structures in Chapter 3.

The second-named author proved in [Ku] that for a  $C^2$  definable function  $f$  on a bounded open set  $U$ , there is a constant  $M > 0$  which bounds the length of the trajectories of the gradient vector field  $\nabla f$ . Let us now consider a definable family  $\mathcal{F} = \{f_p\}_{p \in \mathcal{P}}$  of functions defined on open subsets of  $\mathbb{R}^n$  contained in a compact  $K$ . One can naturally ask the following question: does there exist a constant  $M_{\mathcal{F}}$  such that for all  $p \in \mathcal{P}$ , the length of the trajectories of the gradient field  $\nabla f_p$  is bounded by  $M_{\mathcal{F}}$ ? We give the affirmative answer in Theorem 6.1.

Actually, in Theorem 6.1 we also consider the case where  $f_p$  cannot be extended to any neighborhood of  $\overline{U}$ , for instance  $f_p$  may be rational with poles on the boundary of  $U$ . So we have to consider not only critical values of  $f_p$ , but also generalized critical values which we recall in Chapter 5.

Some effective bounds for length of trajectories of gradient of quasipolynomials and trigonometric quasipolynomials are given in Chapter 8. They are based on Chapter 7 and Khovanskii's theory.

Some of these results were previously established in the Ph.D. thesis of the first-named author (see [D'A2]). We thank R. Moussu for pointing our attention to the paper [Ch], which was useful in the proof of Proposition 7.2.

The last two chapters (added in the second version of the paper) are due to the second named author. Chapter 10 is dedicated to the memory of Gérard Favre. In this chapter, we consider a problem of joining two points in a connected component  $A$  of a set  $\{f > 0\} \cap B(x_0, r)$ , where  $f$  is a polynomial of degree  $d$  and  $B(x_0, r)$  is a ball of radius  $r$  in  $\mathbb{R}^n$ . This is the simplest situation considered in robotics. We prove in Theorem 10.3 that any two points in  $A$  can be joined by a piecewise trajectory of  $\nabla g$ , where  $g(x) = (r^2 - |x - x_0|^2)f(x)$  is a polynomial of degree  $d + 2$ . Moreover we show that the length of the curve can be bounded by  $2r\nu(n)(3d + 2)^{n-1}$ . This bound seems to be quite optimal. In all classical papers on this problem the joining curve was semialgebraic and its construction was hard and quite involved, so the estimates for its length were rather coarse. The new idea here was to use trajectories of a polynomial hence transcendental curves. It seems also that for numerical applications our method is of interest since there are quite efficient algorithms to compute trajectories of gradient of a polynomial.

In the last chapter we consider a quantitative Morse-Sard Theorem. Let  $f : B(r) \rightarrow \mathbb{R}$  be a polynomial of degree  $d$  on a ball  $B(r) \subset \mathbb{R}^n$  of radius  $r$ . For any  $\varepsilon > 0$  we consider the set of nearly critical points  $\Sigma_\varepsilon = \{x \in B(r); |\nabla f(x)| < \varepsilon\}$ . Yomdin proved in [Yo2] that the set  $f(\Sigma_\varepsilon)$  can be covered by  $N(n, d)$  segments of length  $r\varepsilon$ . The number  $N(n, d)$  depends only on dimension  $n$  and degree  $d$ . In

many situations it is very important to have a good bound for  $N(n, d)$ . For instance Donaldson in his important paper [Do] needed the fact that for fixed  $n$  the function  $d \mapsto N(n, d)$  is bounded by a polynomial in  $d$ . We prove in Chapter 10 that  $N(n, d) \leq d(2d-1)^{n-1} + A(n, d)$  which is the first (to our knowledge) explicit bound for  $N(n, d)$  for any  $n$  and  $d$ . This bound is not far from optimal in the sense that always  $(d-1)^n \leq N(n, d)$ .

## 2. NOTATIONS

For simplicity we consider only gradient with respect to the standard Euclidean metric. If  $U$  is an open subset of  $\mathbb{R}^n$  and  $f : U \rightarrow \mathbb{R}$  is  $C^2$  function, then by *trajectory of  $\nabla f$* , we mean any piecewise smooth curve  $x : I \rightarrow U$ , where  $I$  is a finite union of open intervals  $\mathbb{R}$ , such that

$$x'(t) = \nabla f(x(t)) \text{ and } f \circ x : I \rightarrow \mathbb{R} \text{ is injective.}$$

Under these assumptions,  $X$  will denote the image of such a trajectory.

Throughout this paper  $\mathbb{B}^n$  stands for the open unit ball in  $\mathbb{R}^n$  and  $\mathbb{S}^{n-1}$  for the unit sphere in  $\mathbb{R}^n$ , both with respect to the Euclidean metric. For simplicity,  $|\cdot|$  denotes here the Euclidean norm corresponding to the standard inner product  $\langle \cdot, \cdot \rangle$ .

## 3. O-MINIMAL STRUCTURES

We recall below the definition of o-minimal structures expanding the real field. The rest of this Chapter is devoted to some examples and properties of o-minimal structures that will be useful for the proof of Theorem 6.1.

**Definition 3.1.** Let  $\mathcal{M} = \bigcup_{n \in \mathbb{N}} \mathcal{M}_n$ , where each  $\mathcal{M}_n$  is a family of subsets of  $\mathbb{R}^n$ . We say that the collection  $\mathcal{M}$  is an o-minimal structure expanding  $(\mathbb{R}, +, \cdot, <)$  if:

- (i) each  $\mathcal{M}_n$  is closed under finite unions, finite intersections and taking complement;
- (ii) if  $A \in \mathcal{M}_n$  and  $B \in \mathcal{M}_m$ , then  $A \times B \in \mathcal{M}_{n+m}$ ;
- (iii) let  $A \in \mathcal{M}_{n+m}$  and  $\pi_n : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$  be the projection on the first  $n$  coordinates, then  $\pi_n(A) \in \mathcal{M}_n$ ;
- (iv) let  $f, g_1, \dots, g_k \in \mathbb{R}[X_1, \dots, X_n]$ , then the set  $\{x \in \mathbb{R}^n : f(x) = 0, g_1(x) > 0, \dots, g_k(x) > 0\}$  belongs to  $\mathcal{M}_n$ ;
- (v)  $\mathcal{M}_1$  consists of all finite unions of open intervals and points.

For a fixed o-minimal structure  $\mathcal{M}$  expanding  $(\mathbb{R}, +, \cdot, <)$  we say that  $A$  is an  $\mathcal{M}$ -set if  $A \in \mathcal{M}_n$  for some  $n \in \mathbb{N}$ . We say that a function  $f : A \rightarrow \mathbb{R}^m$ , where  $A \subset \mathbb{R}^n$ , is an  $\mathcal{M}$ -function if its graph is an  $\mathcal{M}$ -set. Axiom (v) will be called the o-minimality of  $\mathcal{M}$ . We write for short *definable (in  $\mathcal{M}$ )* instead of  $\mathcal{M}$ -set.

**Example 3.2.** We give below a non exhaustive list of o-minimal structures on the real field  $(\mathbb{R}, +, \cdot, <)$  (see also [Dr-Mi] for detailed definitions and comparisons between the above examples) with examples of functions definable in those o-minimal structures:

- (i) Semialgebraic sets (by Tarski-Seidenberg);  $f(x, y) = \sqrt{x^4 + y^4}$ .
- (ii) Global subanalytic sets (by Gabrielov);  $f(x, y) = \frac{y}{\sin x}$ ,  $x \in (0, \pi)$ .
- (iii)  $\mathbb{R}_{\text{exp}}$ -definable sets (by Wilkie [Wi]);  $f(x, y) = x^2 \exp(-\frac{y^2}{x^4 + y^2}) \ln x$ .
- (iv)  $(\mathbb{R}_{\text{an}}^{\mathbb{R}})$ -definable sets (by Miller [Me]);  $f(x, y) = x^{\sqrt{2}} \exp(\frac{x}{y})$ ,  $0 < x < y < 1$ .

(v)  $\mathbb{R}_{an,exp}$ -definable sets (by van den Dries, Macintyre, Marker [D-M-M]);  $f(x, y) = x^{\sqrt{2}} \ln(\sin y)$ ,  $x > 0, y \in (0, \pi)$ .

**In the remaining of this paper  $\mathcal{M}$  will denote some fixed, but arbitrary, o-minimal structure expanding  $(\mathbb{R}, +, \cdot, <)$ .** We will give now several elementary properties of definable sets and definable functions.

*Remark 3.3.* Let  $E$  be a definable set in  $\mathbb{R}^{n+1}$ . Axioms (i) and (iii) imply that the sets

$$\{x \in \mathbb{R}^n : \exists x_{n+1} (x, x_{n+1}) \in E\} \text{ and } \{x \in \mathbb{R}^n : \forall x_{n+1} (x, x_{n+1}) \in E\}$$

are definable sets. Actually the first set is the image of  $E$  by projection, the second is the complement of the image of the complement of  $E$  by projection.

*Remark 3.4.* The sum, product, inverse, composition of definable functions is again a definable function. Also the image and inverse image of a definable set by a definable mapping are again definable sets. The proofs of these facts are quite standard applications of Remark 3.3 and axioms (i)-(iv) and actually the same as in the semialgebraic case (see e.g. [B-C-R]).

**Lemma 3.5.** *Let  $f : A \rightarrow \mathbb{R}$  be a definable function such that  $f(x) \geq 0$  for all  $x \in A$ . Let  $G : A \rightarrow \mathbb{R}^m$  be a definable mapping and define a function  $\varphi : G(A) \rightarrow \mathbb{R}$  by*

$$\varphi(y) = \inf_{x \in G^{-1}(y)} f(x).$$

*Then  $\varphi$  is definable.*

*Proof.* Write a formula for the graph of the function  $\varphi$  and apply Remark 3.3.

The two following properties of definable sets are essential for the proof of Theorem 6.1. The reader is here assumed to be familiar with the concept of o-minimal structure (for more details see for instance [Dr] or [Co]).

**Lemma 3.6** (Definable choice). *Let  $S \subseteq \mathbb{R}^{m+n}$  be a definable set and  $\pi_m : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^m$  be the natural projection on the first  $m$  coordinates. Then there exists a definable map  $f : \pi_m(S) \rightarrow \mathbb{R}^n$  such that the graph of  $f$  is contained in  $S$ .*

**Lemma 3.7** (uniform finiteness). *Let  $A \subset \mathbb{R}^{n+m}$  be a definable set and let  $A_y = \{x \in \mathbb{R}^n : (x, y) \in A\}$ . Assume that for all  $y \in \mathbb{R}^m$  the set  $A_y$  is finite, then there exists an integer  $N$  such that for all  $y \in \mathbb{R}^m$ ,  $\text{Card}A_y \leq N$ .*

#### 4. CAUCHY-CROFTON FORMULA

Let  $\Gamma$  be a compact definable curve and let  $\mathcal{H}$  denotes the set of affine hyperplanes in  $\mathbb{R}^n$ . Then, for almost every  $H \in \mathcal{H}$  (that is except maybe for a definable subset  $\mathcal{H}_1 \subset \mathcal{H}$  of codimension greater than or equal to 1), the set  $\Gamma \cap H$  is finite. Let  $i(\Gamma, H)$  denote the cardinal of  $\Gamma \cap H$ .

**Cauchy-Crofton formula 4.1** (see e.g. [Fe]). *There exists a normalization of the canonical measure  $d\mu$  on  $\mathcal{H}$  such that the length of  $\Gamma$  can be expressed with the following formula (Cauchy-Crofton):*

$$\text{Length}(\Gamma) = \int_{\mathcal{H}} i(\Gamma, H) d\mu.$$

Note that the Cauchy-Crofton formula is still valid if we just assume  $\Gamma$  to be a rectifiable curve. However, the definability hypothesis is crucial for the following Corollary:

**Corollary 4.2.** *Let  $K \subset \mathbb{R}^n$  be a compact set and let  $\mathcal{G} = \{\Gamma_p\}_{p \in \mathcal{P}}$  be a definable family of definable curves contained in  $K$ . Then, there exists a constant  $m_{\mathcal{G}} > 0$ , depending only on the family  $\mathcal{G}$ , such that for any  $p \in \mathcal{P}$ ,*

$$\text{Length}(\Gamma_p) \leq m_{\mathcal{G}}.$$

*Proof.* Note that  $\mathcal{C} = \{(p, H) \in \mathcal{P} \times \mathcal{H} : i(\Gamma_p, H) < \infty\}$  is a definable family. Moreover by the uniform finiteness Lemma there exists an integer  $i$  such that  $i(\Gamma_p, H) < i$  for any  $(p, H) \in \mathcal{C}$ . Now it is enough to apply the Cauchy-Crofton formula. q.e.d.

*Remark 4.3.* Note that in Corollary 4.2 the constant  $m$  is the product of some integer  $i$  by the normalized volume of the hyperplanes that intersect the compact set  $K$ . Let us denote by  $\nu(n)$  this volume when  $K = \overline{\mathbb{B}^n}$ . Then we have the following formula (see [Mo])

$$\nu(n) = \frac{n \text{Vol}_n(\mathbb{B}^n)}{\text{Vol}_{n-1}(\mathbb{B}^{n-1})}.$$

Note that  $\nu(n)$  can be computed in a different way using the Euler  $\Gamma$  function (see for instance [Fe]). We thus obtain the following alternative formula:

$$\nu(n) = 2\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n}{2}\right)^{-1}.$$

Note that in dimension 2, we obtain  $\nu(2) = \pi$  and, for any  $n \in \mathbb{N}$ ,  $\nu(n) \leq 2n$ . A simple computation shows that  $\nu(n) \sim \sqrt{2\pi n}$  for sufficiently large  $n$ .

## 5. GENERALIZED AND ASYMPTOTIC CRITICAL VALUES

**Definition 5.1.** Let  $f : U \rightarrow \mathbb{R}$  be a  $C^1$  function defined on an open bounded subset  $U$  of  $\mathbb{R}^n$ . We say that  $c \in \mathbb{R}$  is a **generalized critical value** of  $f$  if there exists a sequence  $x_\nu \in U$  such that  $f(x_\nu) \rightarrow c$  and  $\nabla f(x_\nu) \rightarrow 0$ . We denote by  $K(f)$  the set of all such values.

We say that  $c \in \mathbb{R}$  is an **asymptotic critical value** of  $f$  if there exists a sequence  $x_\nu \in U$  without any convergent subsequence in  $U$ , such that  $f(x_\nu) \rightarrow c$  and  $\nabla f(x_\nu) \rightarrow 0$ . We denote by  $K_a(f)$  the set of all such values.

Note that  $K(f) = K_a(f) \cup K_0(f)$ , where  $K_0(f)$  is the set of critical values of  $f$  in the usual sense. Moreover  $K(f)$  is always closed, contrarily to  $K_0(f)$  which may be non-closed if  $f$  is non-proper. Remark also that in the definition of asymptotic critical values we can take a sequence  $x_\nu \in U$  such that  $x_\nu \rightarrow b$ , where  $b$  is a point on the boundary of  $U$ . This is a consequence of compactness of  $\overline{U}$ .

In the o-minimal setting, if  $f$  satisfies the hypotheses of definition 5.1, we have the following result (cf. [Ku]):

**Proposition 5.2.** *Let  $f : U \rightarrow \mathbb{R}$  be a  $C^1$  function defined on an open bounded subset  $U$  of  $\mathbb{R}^n$ . Assume that  $f$  is definable in an o-minimal structure, then the set  $K(f)$  is finite.*

*Remark 5.3.* Let  $\mathcal{F} = \{f_p\}_{p \in \mathcal{P}}$  be a definable family such that each  $f_p$  is a  $C^1$  definable function. Let  $K(f_p)$  denote the set of generalized critical values of  $f_p$  and define  $K(\mathcal{F}) = \{(p, c) \in \mathcal{P} \times \mathbb{R} : c \in K(f_p)\}$ . By the uniform finiteness Lemma, the number of elements of  $K(f_p)$  is bounded independently of  $p \in \mathcal{P}$ . There exists a definable cell decomposition of the parameter space (i.e. into a finite number of definable cells), such that on each cell  $K(\mathcal{F})$  is a finite union of graphs of definable functions.

## 6. GRADIENT TRAJECTORIES OF A DEFINABLE FAMILY

Let  $F : U \rightarrow \mathbb{R}$  be a definable function defined on a definable set  $U \subset \mathbb{R}^n \times \mathbb{R}^k$ . Let us denote by  $\mathcal{P}$  the projection of  $U$  on the last coordinate. Let  $p \in \mathcal{P}$ , and define  $U_p = \{x \in \mathbb{R}^n : (x, p) \in U\}$ . For any  $p \in \mathcal{P}$  we denote by  $f_p$  the definable function  $F(\cdot, p)$  defined on the definable open set  $U_p$ . Throughout this section,  $\mathcal{F} = \{f_p\}_{p \in \mathcal{P}}$  denotes the definable family of such functions  $f_p$ . Under these hypotheses, we state the main result of this paper:

**Theorem 6.1.** *Let  $\mathcal{F}$  be a definable family of functions as above. Assume that for each  $p \in \mathcal{P}$ , the function  $f_p$  is of class  $C^2$  on  $U_p$ . Let  $K$  be a compact set in  $\mathbb{R}^n$  such that  $U_p \subset K$ . Then there exists a constant  $M_{\mathcal{F}} > 0$  such that, for all  $p \in \mathcal{P}$ , the length of any trajectory of  $\nabla f_p$  is bounded by  $M_{\mathcal{F}}$ .*

*Proof.* We build a definable set  $\Delta \subset U$  inside which, for each parameter  $p$ ,  $|\nabla f_p|$  is very small. For any  $p \in \mathcal{P}$ , let  $\varphi_p : \mathbb{R} \rightarrow \mathbb{R}_+ \cup \{\infty\}$  be the definable function given by:

$$\varphi_p(s) = \inf\{|\nabla f_p(x)| : x \in f_p^{-1}(s)\}.$$

By convention, we put  $\varphi_p(s) = +\infty$  if  $f_p^{-1}(s)$  is empty. Note that the function  $\varphi : \mathbb{R} \times \mathcal{P} \rightarrow \mathbb{R} \cup \{\infty\}$  defined by  $\varphi(s, p) = \varphi_p(s)$  is definable. Let  $\Delta_p$  and  $\Delta$  be the sets defined as follows:

$$\begin{aligned} \Delta_p &= \{x \in U_p \setminus f_p^{-1}[K(f_p)] : |\nabla f_p(x)| \leq 2\varphi_p[f_p(x)]\}, \\ \Delta &= \{(x, p) \in U : p \in \mathcal{P}, x \in \Delta_p\}. \end{aligned}$$

The sets  $\Delta$  and  $\Delta_p$  are definable. Moreover, it follows easily from the definition of  $K(f)$  that if  $s_0 \notin K(f_p)$ , then there exists a constant  $c(s_0) > 0$  such that  $\varphi_p \geq c(s_0) > 0$  in some neighborhood of  $s_0$ . Hence, if  $s_0 \in f_p(U_p) \setminus K(f_p)$ , then  $\Delta_p \cap f_p^{-1}(s_0) \neq \emptyset$ . Let  $\Phi$  be the definable map:

$$\Phi : \Delta \ni (x, p) \longmapsto (f_p(x), p) \in \mathbb{R} \times \mathcal{P}.$$

Then, according to the definable choice Lemma (cf Lemma 3.6), there exists a definable section  $\gamma$  of  $\Phi$ , that is a definable mapping  $\gamma : \Phi(\Delta) \rightarrow \Delta$  satisfying:  $\Phi \circ \gamma = \text{Id}_{\Phi(\Delta)}$ . Clearly we can write  $\gamma(t, p) = (\gamma_p(t), p)$ , where  $\gamma_p : f_p(U_p) \setminus K(f_p) \rightarrow \Delta_p \subset U_p$  is a definable mapping.

Let  $\Gamma_p$  denote the image of  $\gamma_p$ . For any  $p \in \mathcal{P}$ , by routine o-minimality arguments, the definable curve  $\Gamma_p$  is a finite union of points and connected  $C^1$  submanifolds. On each such component of  $\Gamma_p$ , the function  $f_p$  is  $C^1$  and injective, hence has only finitely many critical points (by o-minimality). This implies that  $\Gamma_p$  is transverse to the level sets of  $f_p$ , except maybe for finitely many levels.

**Lemma 6.2.** *Let  $X_p$  be a trajectory of  $\nabla f_p$ , then  $\text{Length}(X_p) \leq 2 \text{Length}(\Gamma_p)$ .*

*Proof.* Since  $\Gamma_p$  meets transversally all but a finite number of fibres of  $f_p$ , we will assume that the definable curve  $\Gamma_p$  is smooth, connected and transverse to every fibre of  $f_p$ . Moreover by deleting finitely many fibres  $f_p^{-1}(t)$ ,  $t \in K(f_p)$  we may assume that  $f_p$  has no critical values in  $U_p$ .

Let  $x(s)$  be the arc-length parametrization of the trajectory  $X_p$  and  $\theta(s')$  be the arc-length parametrization of the curve  $\Gamma_p$ . We fix orientations in the way that both functions  $s \mapsto (f_p \circ x)(s)$  and  $s' \mapsto (f_p \circ \theta)(s')$  are strictly increasing.

Let  $\eta : X_p \rightarrow \Gamma_p$  be the map defined as follows: for  $x \in X_p$ , the point  $\eta(x)$  is the unique intersection point of the fibre  $f_p^{-1}(f_p(x))$  with the definable curve  $\Gamma_p$ . In other words  $\eta = (f_p|_{\Gamma_p})^{-1} \circ (f_p|_{X_p})$ . We now compute  $\eta$  in our arc-length charts, that is we consider  $h(s) = \theta^{-1} \circ \eta \circ x(s)$ . Clearly, to prove Lemma 6.2 it is enough to show that  $h'(s) \geq \frac{1}{2}$ .

Taking derivative with respect to  $s$  in the equality  $(f_p \circ x)(s) = (f_p \circ \eta \circ x)(s)$  we obtain the following:

$$\langle \nabla f_p(x(s)), x'(s) \rangle = \langle \nabla f_p((\eta \circ x)(s)), (\eta \circ x)'(s) \rangle.$$

But  $x'(s) = \frac{\nabla f_p(x(s))}{|\nabla f_p(x(s))|}$ , hence

$$|\nabla f_p(x(s))| \leq |\nabla f_p((\eta \circ x)(s))| \cdot |(\eta \circ x)'(s)|.$$

Since  $\eta(x(s)) \in \Delta_p$ , we have  $2|\nabla f_p(x(s))| \geq |\nabla f_p((\eta \circ x)(s))|$ , thus  $|(\eta \circ x)'(s)| \geq \frac{1}{2}$ . But  $\theta$  is an arc-length parametrisation, so

$$h'(s) = (\theta^{-1} \circ \eta \circ x)'(s) = |(\eta \circ x)'(s)| \geq \frac{1}{2}$$

and Lemma 6.2 follows. q.e.d.

Note that in fact we have proved a more precise estimate for the length of any trajectory of  $\nabla f_p$ .

*Remark 6.3.* Let  $I$  be a finite union of open intervals in  $\mathbb{R}$  and let  $x : I \rightarrow U$  be a trajectory of  $\nabla f_p$ . Assume that  $x(I) \subset f_p^{-1}((t_1, t_2))$ . Then the length of  $x(I)$  is bounded by twice the length of  $\Gamma_p \cap f_p^{-1}((t_1, t_2))$ .

Now we can easily finish the proof of Theorem 6.1. Note that the definable family  $\{\Gamma_p\}_{p \in \mathcal{P}}$  satisfies the hypotheses of Corollary 4.2, so there exists a constant  $m_{\mathcal{F}} > 0$  such that for any  $p \in \mathcal{P}$ , the length of  $\Gamma_p$  is bounded by  $m_{\mathcal{F}}$ . Hence, by Lemma 6.2, the length of any trajectory of  $\nabla f_p$  is bounded by  $M_{\mathcal{F}} = 2m_{\mathcal{F}}$ . q.e.d.

In fact in Theorem 6.1 we can assume that all the definable sets  $U_p$  are bounded but not necessarily contained in a fixed compact set  $K$ . Let  $\mathcal{F}$  denote a definable family of functions verifying the same assumptions as in Theorem 6.1 except that the definable set  $U_p$  are bounded but no more contained in a fixed compact set  $K$ . We denote by  $d_p$  the diameter of  $U_p$ . Then we have the following Corollary:

**Corollary 6.4.** *There exists a constant  $M_{\mathcal{F}} > 0$  such that for every  $p \in \mathcal{P}$  the length of any trajectory of  $\nabla f_p$  is bounded by  $M_{\mathcal{F}} \cdot d_p$ .*

*Proof.* Let  $f_p \in \mathcal{F}$  and  $x_p \in \mathbb{R}^n$  such that  $U_p \subset B(x_p, d_p)$ . We may assume that the mappings  $p \mapsto x_p$  and  $p \mapsto d_p$  are definable. Let  $T_p : \mathbb{B}^n \ni X \mapsto x_p + d_p X \in B(x_p, d_p)$ . Define  $g_p = f_p \circ T_p$ . The bound is obtained by applying Theorem 6.1 to the definable family  $\mathcal{G} = \{g_p\}_{p \in \mathcal{P}}$ . q.e.d.

## 7. BOUNDS FOR GRADIENT TRAJECTORIES OF POLYNOMIALS

Throughout this Chapter  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  denotes a polynomial function of degree  $d$ . We will give an explicit upper bound for the length of a trajectory of  $\nabla f$  restricted to the unit ball  $\mathbb{B}^n$  (with respect to the Euclidean metric). First observe that the following Corollary follows easily from Theorem 6.1.

**Corollary 7.1.** *Let  $f$  be a polynomial in  $n$  variables of degree  $d$ . Then the length of any trajectory of  $\nabla f$  in a ball of radius  $r$  is bounded by  $rA(n, d)$ , where  $A(n, d)$  is a constant depending only on  $d$  and  $n$ .*

*Proof.* The family  $\mathcal{F}$  of polynomial functions in  $n$  variables and of degree less than or equal to  $d$  is semialgebraic and thus definable in any o-minimal structure. The space of parameters is the space of coefficients of these polynomials. Let  $\mathcal{F}_1$  be the semialgebraic family formed with the restriction to  $\mathbb{B}^n$  of all functions in  $\mathcal{F}$ . Theorem 6.1 applied to the family  $\mathcal{F}_1$  and  $K = \overline{\mathbb{B}^n}$  provides a bound  $A(n, d)$  on the length of the trajectories independent on the coefficients. Finally, by Corollary 6.4 we get the desired upper bound for the trajectories of  $\nabla f$  in any ball of radius  $r$ . q.e.d.

In order to compute  $A(n, d)$  explicitly, we shall construct a semi-algebraic curve  $\Gamma$  with the following property: if  $y \in \overline{\mathbb{B}^n}$ , then

$$(7.1) \quad |\nabla f(y)| \geq |\nabla f(x)|, \text{ for some } x \in \Gamma \cap f^{-1}(f(y)).$$

In other words we have to minimize  $|\nabla f|^2$  on the fibres of  $f$  restricted to  $\overline{\mathbb{B}^n}$ . More precisely, we shall prove that, for a generic polynomial  $f$  of degree  $d$ , the set

$$\Gamma_1 = \{x \in \mathbb{B}^n : |\nabla f|^2 \text{ has a loc. min. at } x \text{ on the regular fibre } f^{-1}(f(x)) \cap \mathbb{B}^n\}$$

is of dimension at most 1. We shall also prove that the set  $\Gamma_2 \subset \mathbb{S}^{n-1}$  defined by

$$\Gamma_2 = \{x \in \mathbb{S}^{n-1} : |\nabla f|^2 \text{ has a loc. min. at } x \text{ on the regular fibre } f^{-1}(f(x)) \cap \mathbb{S}^{n-1}\}$$

is of dimension 1. Then we take  $\Gamma = \Gamma_1 \cup \Gamma_2$  and for a generic polynomial we shall give explicit formulas for polynomials describing  $\Gamma_1$  and  $\Gamma_2$ .

The following Proposition shows that the points at which the level sets of the polynomials  $f$  and  $|\nabla f|^2$  are non transverse, generically define an algebraic curve. Let  $\mathbf{X} = (X_1, \dots, X_n)$  and denote by  $\mathbb{R}_d[\mathbf{X}]$  the space of polynomials in  $n$  variables of degree less than or equal to  $d$ .

**Proposition 7.2.** *Let  $n \geq 2$  and  $d \geq 2$ . Then, there exists a semialgebraic set  $E_d \subset \mathbb{R}_d[\mathbf{X}]$  of codimension greater than or equal to 1 such that, for any polynomial  $f \in \mathbb{R}_d[\mathbf{X}] \setminus E_d$ , the set*

$$\Theta_1(f) = \{x \in \mathbb{R}^n : d(|\nabla f|^2) \wedge df = 0\}$$

*is either empty or consists of a finite union of real algebraic curves and points.*

*Proof.* (inspired by [Ch]). Let us denote by  $S_n(\mathbb{R})$  the space of  $n \times n$  symmetric matrices with real coefficients. We define



$$\Sigma = \{(V, H) \in \mathbb{R}^n \times S_n(\mathbb{R}) : \exists \lambda \in \mathbb{R} : H \cdot V = \lambda V\}.$$

We need the following lemma:

**Lemma 7.3.** *The set  $\Sigma$  is algebraic of codimension  $n - 1$ .*

Note that  $(V, H) \in \Sigma$  if and only if  $V$  and  $H \cdot V$  are collinear. This means that all the  $2 \times 2$  minors of the  $2 \times n$  matrix formed by the coefficients of  $H \cdot V$  and  $V$  are equal to 0. It follows that  $\Sigma$  is algebraic. We now prove that  $\Sigma$  has codimension  $n - 1$ . Let  $\Lambda_k$  be the set of matrices  $H \in S_n(\mathbb{R})$  such that the maximum dimension of eigenspaces of  $H$  is equal to  $k$ , or equivalently that  $k$  is the maximum multiplicity of eigenvalues of  $H$ . Clearly the intersection of  $\Lambda_k$  with the subspace  $D_n$  of diagonal matrices is of codimension  $k - 1$  in  $D_n$ . Recall that  $O_n$ , the group of orthogonal  $n \times n$  matrices, is of dimension  $n(n - 1)/2$ . We can write

$$\Lambda_k = \bigcup_{A \in D_n \cap \Lambda_k} O_n A,$$

where  $O_n A$  is the orbit of the action of  $O_n$  by conjugation. Clearly  $\dim O_n A \leq \dim O_n$  for any matrix  $A \in S_n(\mathbb{R})$ . Hence  $\text{codim } \Lambda_k \geq k - 1$ . More precisely, if  $A \in \Lambda_k$  has one eigenvalue of multiplicity  $k$  (and the others of multiplicity 1), then the stabiliser of the action of  $O_n$  is of dimension  $k(k - 1)/2$ . Hence  $\dim O_n A = n(n - 1)/2 - k(k - 1)/2$ . It follows that

$$\text{codim } \Lambda_k = (k - 1) + k(k - 1)/2 = k(k + 1)/2 - 1.$$

Let  $\pi : \mathbb{R}^n \times S_n(\mathbb{R}) \rightarrow S_n(\mathbb{R})$  denotes the projection on the second coordinate. Observe that, if  $H \in \Lambda_k$ , then  $\pi^{-1}(H) \cap \Sigma$  is the union of eigenspaces of  $H$ , hence  $\pi^{-1}(\Lambda_k) \cap \Sigma$  is of dimension  $(\dim \Lambda_k + k)$ , thus of codimension  $k(k + 1)/2 - 1 + n - k$  in  $\mathbb{R}^n \times S_n(\mathbb{R})$ . But  $S_n(\mathbb{R}) = \bigcup \Lambda_k$ , so  $\Sigma$  is algebraic and  $\text{codim } \Sigma = \text{codim } (\pi^{-1}(\Lambda_1) \cap \Sigma) = n - 1$ . This ends the proof of Lemma 7.3.

Let  $Hf(x)$  be the Hessian matrix of  $f$  at the point  $x$ . Note that  $\nabla(|\nabla f|^2) = 2Hf \cdot \nabla f$ . Hence we have the following

**Lemma 7.4.** *The set  $\Theta_1(f) = \{x \in \mathbb{R}^n : d(|\nabla f|^2) \wedge df = 0\}$  is the set of points  $x \in \mathbb{R}^n$  such that  $\nabla f(x)$  is an eigenvector of  $Hf(x)$ .*

Let us fix a polynomial  $f$  of degree at most  $d$ . We are going to consider a deformation  $\tilde{f}$  of  $f$  by adding quadratic polynomials. For  $\alpha = (\alpha_i) \in \mathbb{R}^n$  and  $\varepsilon = (\varepsilon_{jk}) \in \mathbb{R}^{\frac{n(n+1)}{2}}$  we put

$$\tilde{f}_{\alpha, \varepsilon}(x) = f(x) + \sum_{i=1}^n \alpha_i x_i + \sum_{1 \leq j \leq k \leq n} \varepsilon_{jk} x_j x_k.$$

Let  $\Psi : \mathbb{R}^n \times \mathbb{R}^{\frac{n(n+1)}{2}} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times S_n(\mathbb{R})$  be the map defined as follows:

$$\Psi(\alpha, \varepsilon, x) = (\nabla \tilde{f}_{\alpha, \varepsilon}(x), H \tilde{f}_{\alpha, \varepsilon}(x)),$$

where  $H \tilde{f}_{\alpha, \varepsilon}$  stands for hessian matrix of  $\tilde{f}_{\alpha, \varepsilon}$ . Note that the differential of  $\Psi$  with respect to  $(\alpha, \varepsilon)$  is an isomorphism. Hence  $\Psi$  is a submersion. By the transversality Theorem with parameters (see e.g. [Gu-Po] or [Go-Ma] Chapter 1), we can deduce that there exists an open dense semialgebraic set  $\mathcal{E}_1 \subset \mathbb{R}^n \times \mathbb{R}^{\frac{n(n+1)}{2}}$  such that, for any  $(\alpha, \varepsilon) \in \mathcal{E}_1$ , the map

$$\Psi_{\alpha, \varepsilon} = \Psi(\alpha, \varepsilon, \cdot)$$

is transverse to  $\Sigma$ . More precisely we fix some stratification of  $\Sigma$  and we claim that  $\Psi_{\alpha,\varepsilon}$  is transverse to all strata of  $\Sigma$ . Recall that by Lemma 7.3 the strata of  $\Sigma$  are of codimension at least  $n-1$ . This means that for any  $(\alpha,\varepsilon) \in \mathcal{E}_1$  the set  $\Psi_{\alpha,\varepsilon}^{-1}(\Sigma)$  is an algebraic set (possibly empty) of dimension at most 1. But  $\Psi_{\alpha,\varepsilon}^{-1}(\Sigma) = \Theta_1(\tilde{f}_{\alpha,\varepsilon})$ , by Lemma 7.4. Hence Proposition 7.2 follows. q.e.d.

We now study  $\Gamma_2$ , the set of points of  $\mathbb{S}^{n-1}$  where  $|\nabla f|^2$  has a local minimum on the fibres of  $f$  restricted to the sphere. Let  $r(x) = |x|^2$  and define

$$\Theta_2(f) = \{x \in \mathbb{S}^{n-1} : d(|\nabla f|^2) \wedge df \wedge dr = 0\}.$$

Note that  $\Gamma_2 \subset \Theta_2(f)$  and  $\Theta_2(f)$  is the set of critical points of  $|\nabla f|^2$  on the fibres of  $f$  restricted to the sphere. For a generic polynomial  $f$  the set  $\Theta_2(f)$  is a curve, namely we have:

**Proposition 7.5.** *There exists a semialgebraic set  $F_d \subset \mathbb{R}_d[\mathbf{X}]$  of codimension greater than or equal to 1 such that, for any polynomial  $f \in \mathbb{R}_d[\mathbf{X}] \setminus F_d$ , the set  $\Theta_2(f)$  is nonempty and consists of a finite union of real algebraic curves and points.*

*Proof.* We proceed like before and define

$$\tilde{\Sigma} = \{(V, H, p) \in \mathbb{R}^n \times S_n(\mathbb{R}) \times \mathbb{S}^{n-1} : (H \cdot V) \wedge V \wedge p = 0, \}.$$

Let  $\pi : \mathbb{R}^n \times S_n(\mathbb{R}) \times \mathbb{S}^{n-1} \rightarrow \mathbb{R}^n \times S_n(\mathbb{R})$  denote the projection on the first two coordinates. Let  $H, V \in \mathbb{R}^n \times S_n(\mathbb{R})$  be such that  $(H \cdot V) \wedge V = 0$ . Then  $\pi^{-1}(H, V) = \{H\} \times \{V\} \times \mathbb{S}^{n-1}$  and by Lemma 7.3 the set

$$\tilde{\Sigma}_1 = \{(V, H, p) \in \mathbb{R}^n \times S_n(\mathbb{R}) \times \mathbb{S}^{n-1} : (H \cdot V) \wedge V = 0\}.$$

is of codimension  $n-1$  in  $\mathbb{R}^n \times S_n(\mathbb{R}) \times \mathbb{S}^{n-1}$ . Consider now the set  $\tilde{\Sigma}_2 \subset \tilde{\Sigma}$  defined by

$$\tilde{\Sigma}_2 = \{(V, H, p) \in \mathbb{R}^n \times S_n(\mathbb{R}) \times \mathbb{S}^{n-1} : (H \cdot V) \wedge V \wedge p = 0, (H \cdot V) \wedge V \neq 0\}.$$

Note that conditions  $(H \cdot V) \wedge V \wedge p = 0$  and  $(H \cdot V) \wedge V \neq 0$  just mean that  $p$  is contained in the plane spanned by the vectors  $H \cdot V$  and  $V$ . Hence,  $\pi^{-1}(H, V) \cap \tilde{\Sigma}_2$  has codimension  $n-2$  in  $\mathbb{R}^n \times S_n(\mathbb{R}) \times \mathbb{S}^{n-1}$ . Moreover, condition  $(H \cdot V) \wedge V \neq 0$  is generic in  $\mathbb{R}^n \times S_n(\mathbb{R})$ . Thus  $\tilde{\Sigma}_2$  has codimension  $n-2$  in  $\mathbb{R}^n \times S_n(\mathbb{R}) \times \mathbb{S}^{n-1}$ . Hence,  $\tilde{\Sigma} = \tilde{\Sigma}_1 \cup \tilde{\Sigma}_2$  is of codimension greater than or equal to  $n-2$ .

Once again we are done, using a quadratic perturbation of polynomials of fixed degree  $d$  and the transversality Theorem with parameters. More precisely, for  $\alpha = (\alpha_i) \in \mathbb{R}^n$  and  $\varepsilon = (\varepsilon_{jk}) \in \mathbb{R}^{\frac{n(n+1)}{2}}$  we put

$$\tilde{f}_{\alpha,\varepsilon}(x) = f(x) + \sum_{i=1}^n \alpha_i x_i + \sum_{1 \leq j \leq k \leq n} \varepsilon_{jk} x_j x_k.$$

Let  $\tilde{\Psi} : \mathbb{R}^n \times \mathbb{R}^{\frac{n(n+1)}{2}} \times \mathbb{S}^{n-1} \rightarrow \mathbb{R}^n \times S_n(\mathbb{R}) \times \mathbb{S}^{n-1}$  be the map defined as follows:

$$\tilde{\Psi}(\alpha, \varepsilon, p) = (\nabla \tilde{f}_{\alpha,\varepsilon}(p), H \tilde{f}_{\alpha,\varepsilon}(p), p),$$

where  $H \tilde{f}_{\alpha,\varepsilon}$  stands for hessian matrix of  $\tilde{f}_{\alpha,\varepsilon}$ . Note that the differential of  $\tilde{\Psi}$  is an isomorphism and thus  $\tilde{\Psi}$  is a submersion. This means that  $\tilde{\Psi}$  is transverse to  $\tilde{\Sigma}$ . By the transversality Theorem with parameters there exists an open dense semialgebraic set  $\mathcal{E}_2 \subset \mathbb{R}^n \times \mathbb{R}^{\frac{n(n+1)}{2}}$  such that for any  $(\alpha, \varepsilon) \in \mathcal{E}_2$ , the map  $\tilde{\Psi}_{\alpha,\varepsilon}$

is transverse to  $\tilde{\Sigma}$ . Thus,  $\text{codim}_{\mathbb{S}^{n-1}}(\tilde{\Psi}_{\alpha,\varepsilon})^{-1}(\tilde{\Sigma}) \geq n - 2$ . Hence  $\dim \theta_2(\tilde{f}_{\alpha,\varepsilon}) \leq 1$ . This completes the proof of Proposition 7.5. q.e.d.

At this point we are able to bound the length of any trajectory of the gradient field for a generic polynomial. According to Propositions 7.2 and 7.5 we can give an upper bound by estimating the lengths of  $\Theta_1(f)$  and  $\Theta_2(f)$  in the case where these sets are of dimension at most equal to 1. This gives rise to the following:

**Theorem 7.6.** *Let  $n \geq 2$  and  $d \geq 2$  be integers. Then there exists a semialgebraic set  $G_d \subset \mathbb{R}_d[X_1, \dots, X_n]$ , of codimension greater or equal to 1, such that for any polynomial  $f \in \mathbb{R}_d[X_1, \dots, X_n] \setminus G_d$ , the length of any trajectory of  $\nabla f$  in  $\mathbb{B}^n$  is bounded by*

$$A(n, d) = \nu(n)((3d - 4)^{n-1} + 2(3d - 3)^{n-2})$$

where  $\nu(n)$  is a constant depending only on the dimension.

*Proof.* Take  $G_d = E_d \cup F_d$ . Assume that  $f \in \mathbb{R}_d[\mathbf{X}] \setminus G_d$ , then the sets  $\Theta_1(f)$  and  $\Theta_2(f)$  are curves, by Propositions 7.2 and 7.5. Moreover their union contains the points at which the function  $|\nabla f|^2$  restricted to  $\overline{\mathbb{B}^n}$  is minimal on the fibres of  $f$ . Note that  $\Theta_1(f)$  is the zero set of a 2-form with polynomial coefficients of degree at most  $3d - 4$ . Reordering, if necessary, the variables we may assume that  $\frac{\partial f}{\partial x_1}$  vanishes on  $\Theta_1(f)$  only at a finite number of points. Then  $\tilde{\Theta}_1(f)$  - the common zeroes of the coefficients of  $dx_1 \wedge dx_i$ ,  $i = 2, \dots, n$  - contains  $\Theta_1(f)$  and possibly some other components which cut  $\Theta_1(f)$  only at isolated points. So for a generic affine hyperplane  $H$ , the intersection  $H \cap \tilde{\Theta}_1(f)$  contains all the points of  $H \cap \Theta_1(f)$  which are nondegenerate (hence isolated also in the complexification) and possible other components. By the general version of Bezout's Theorem (see e.g. [Fu]) the number of irreducible components of  $H \cap \tilde{\Theta}_1(f)$  is not greater than the product of degrees, that is  $(3d - 4)^{n-1}$ . Thus we have

**Lemma 7.7.** *For a generic affine hyperplane  $H$  the set  $H \cap \Theta_1(f)$  has at most  $(3d - 4)^{n-1}$  points.*

In the same way,  $\Theta_2(f)$  is defined by the equation of the unit sphere in  $\mathbb{R}^n$  and by the coefficients of the corresponding 3-form. The coefficients of the 3-form associated to  $\Theta_2(f)$  are polynomials of degree at most  $3(d - 1)$ . As above we can choose  $n - 2$  coefficients (and the equation of the sphere) to describe  $\Theta_2(f)$  generically. So again by the general version of Bezout's Theorem we have

**Lemma 7.8.** *For a generic affine hyperplane  $H$  the set  $H \cap \Theta_2(f)$  has at most  $2(3d - 3)^{n-2}$  points.*

Now we are in the position to finish the proof of Theorem 7.6. For a generic polynomial  $f$  of degree  $d$  we have

$$\Gamma_1 \subset \Theta_1(f) \text{ and } \Gamma_2 \subset \Theta_2(f),$$

where  $\Gamma_1$  and  $\Gamma_2$ , were defined in the beginning of this Chapter as the set of points where  $|\nabla f|^2$  has the minimum on fibres of  $f$  restricted to the open ball and respectively on fibres of  $f$  restricted to the sphere. We put  $\Gamma = \Gamma_1 \cup \Gamma_2$ . Let  $X$  be a trajectory of  $\nabla f$  in the unit ball  $\mathbb{B}^n$ . We shall prove that the length of the trajectories of  $\nabla f$  is bounded by the length of  $\Gamma$ . Actually we prove a more general result.

Let  $D$  be an open subset of  $\mathbb{R}^n$  and let  $f : D \rightarrow \mathbb{R}$  be a  $C^2$  function in some neighbourhood of  $\overline{D}$ . Let  $\Gamma \subset \overline{D}$  be a  $C^1$  curve. We assume that for each  $t \in f(D)$ , the set  $f^{-1}(t) \cap \Gamma$  consists of exactly one point and that, for all but finitely many  $t \in f(D)$ , the curve  $\Gamma$  is transverse to  $f^{-1}(t)$ . Assume that, for any  $x \in \Gamma$  and for all  $y \in f^{-1}(f(x))$ , we have  $|\nabla f(x)| \leq |\nabla f(y)|$ .

**Lemma 7.9.** *Let  $X \subset D$  be a trajectory of  $\nabla f$ , then  $\text{Length}(X) \leq \text{Length}(\Gamma)$ .*

*Proof.* Let  $x(s)$  be the arc-length parametrisation of the trajectory  $X$  of  $\nabla f$  and  $\gamma(s')$  be the arc-length parametrisation of the curve  $\Gamma$ . We fix orientations in the way that both functions  $s \mapsto (f \circ x)(s)$  and  $s' \mapsto (f \circ \gamma)(s')$  are strictly increasing. Let  $\eta : X \rightarrow \Gamma$  be the map defined as follows: for  $x \in X$ , the point  $\eta(x)$  is the unique intersection point of the fibre  $f^{-1}(f(x))$  with the curve  $\Gamma$ . In other words  $\eta = (f|_{\Gamma})^{-1} \circ (f|_X)$ . We compute now  $\eta$  in our arc-length charts, that is we consider  $h(s) = \gamma^{-1} \circ \eta \circ x(s)$ . Clearly, to prove Lemma 7.9 it is enough to show that  $h'(s) \geq 1$ . Taking derivative with respect to  $s$  in the equality  $f[x(s)] = f[\eta(x(s))]$  we obtain:

$$\langle \nabla f[x(s)], x'(s) \rangle = \langle \nabla f[\eta(x(s))], (\eta \circ x)'(s) \rangle.$$

But  $x'(s) = \frac{\nabla f[x(s)]}{|\nabla f[x(s)]|}$ , hence  $|\nabla f[x(s)]| \leq |\nabla f[\eta(x(s))]| \cdot |(\eta \circ x)'(s)|$ . Since  $\eta(x(s)) \in \Gamma$ , we have  $|\nabla f[x(s)]| \geq |\nabla f[\eta(x(s))]|$ , thus  $|(\eta \circ x)'(s)| \geq 1$ . But  $\gamma$  is an arc-length parametrisation, so

$$h'(s) = (\gamma^{-1} \circ \eta \circ x)'(s) = |(\eta \circ x)'(s)| \geq 1$$

and Lemma 7.9 follows. q.e.d.

By Lemmas 7.7 and 7.8, the Cauchy-Crofton formula applied to the curve  $\Gamma$  yields

$$\text{Length}(\Gamma) \leq \nu(n)((3d-4)^{n-1} + 2(3d-3)^{n-2}) = A(n, d),$$

where  $\nu(n)$  is the measure of the set of affine hyperplanes which meet the unit ball (see Remark 4.3 for the explicit value of  $\nu(n)$ ). Thus Theorem 7.6 follows. q.e.d.

A careful reading of the proof of Theorem 7.6 gives a more precise result. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , with  $n \geq 2$ , be a polynomial of degree  $d \geq 2$  and let  $B(r)$  be a ball of radius  $r$  in  $\mathbb{R}^n$ . Let  $D_i, i \in I$  be all connected components of  $f^{-1}(t, s) \cap B(r)$ . Let  $\Gamma = \Gamma(f)$  be the curve associated to a generic polynomial  $f$  of degree  $d$ , that is  $\Gamma$  contains all points at which  $|\nabla f|^2$  attains a local minimum on the fibres of  $f$ . Thus we have:

**Corollary 7.10.** *Let  $\lambda_i : [\alpha_i, \beta_i] \rightarrow D_i$  be a trajectory of  $\nabla f$  in  $D_i$ . Then*

$$\text{length}(\lambda_i) \leq \text{length}(\Gamma \cap D_i).$$

*In particular, if we have a trajectory  $\lambda_i$  in each  $D_i$ , then*

$$\sum_{i \in I} \text{length}(\lambda_i) \leq \text{length}(\Gamma \cap B(r)) \leq rA(n, d),$$

*where  $A(n, d) = \nu(n)((3d-4)^{n-1} + 2(3d-3)^{n-2})$  is the constant of theorem 7.11.*

We complete this Chapter by extending the bound to any polynomial function  $f$  in  $n$  variables and degree less or equal to  $d$ .

**Theorem 7.11.** *If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , with  $n \geq 2$ , is a polynomial of degree  $d \geq 2$ , then the length of any trajectory of  $\nabla f$  in a ball of radius  $r$  is bounded by*

$$\nu(n)((3d-4)^{n-1} + 2(3d-3)^{n-2}) \cdot r.$$

*Proof.* Clearly, by Corollary 6.4 it is enough to prove the estimate for the unit ball  $\mathbb{B}^n$ . Let us fix a polynomial  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  of degree  $d \geq 2$  and  $\varepsilon > 0$ . Let  $K_0(f) = \{c_1, \dots, c_k\}$  be the set of critical values of  $f$ . For any  $\eta > 0$  let us put  $K_\eta(f) = \bigcup_{i=1}^k (c_i - \eta, c_i + \eta)$  and  $C_\eta = f^{-1}(K_\eta(f))$ . We have the following

**Lemma 7.12.** *For every positive  $\varepsilon$ , there exists  $\eta > 0$  such that the length of any trajectory of  $\nabla f$  in the set  $C_\eta \cap \mathbb{B}^n$  is bounded by  $\varepsilon/2$ . (See our definition of trajectory in Chapter 2).*

By the proof of Theorem 6.1 there exists a semialgebraic curve  $\Gamma \subset \overline{\mathbb{B}^n}$  such that the length of any trajectory of  $\nabla f$  in  $\mathbb{B}^n$  is bounded by  $2 \text{Length } \Gamma$ . More precisely, as in Remark 6.3 we have: let  $I$  be finite a union of open intervals in  $\mathbb{R}$  and let  $x : I \rightarrow \mathbb{B}^n$  be a trajectory of  $\nabla f$ . Assume that  $x(I) \subset f^{-1}((t_1, t_2))$ . Then the length of  $x(I)$  is bounded by twice the length of  $\Gamma \cap f^{-1}((t_1, t_2))$ . Also recall that  $f$  is injective on  $\Gamma$ , so the length of  $\Gamma \cap f^{-1}((t_1, t_2))$  tends to 0 as  $t_1$  tends to  $t_2$  (or  $t_2$  tends to  $t_1$ ). We can take as  $t_2$  (respectively as  $t_1$ ) a critical value  $c_i$ , now it is easy to conclude since there are finitely many critical values. This ends Lemma 7.12.

Note that  $|\nabla f| > 0$  on the compact set  $B_\eta = \overline{\mathbb{B}^n} \setminus C_\eta$ . Hence

$$\inf\{|\nabla f(x)|; x \in B_\eta\} = \delta > 0.$$

Thus

**Lemma 7.13.** *For any  $\rho > 0$  there exists  $\theta > 0$  such that for any polynomial  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  of degree  $d$  we have:*

$$\text{if } \|f - g\| \leq \theta \text{ then, } \angle(\nabla f(x), \nabla g(x)) \leq \rho, \text{ for any } x \in B_\eta.$$

Here  $\angle(\cdot, \cdot)$  stands for the (unoriented) angle measure between vectors and  $\|\cdot\|$  for a norm on the space of polynomials of degree not greater than  $d$ . For instance  $\|\cdot\|$  may be the maximum of the absolute value of coefficients.

To conclude the Theorem we need to compare the length of a trajectory of  $\nabla f$  with the length a trajectory of  $\nabla g$  where  $g$  is a generic polynomial close to  $f$ .

**Lemma 7.14.** *Let  $g$  be a generic polynomial (in the sense of Theorem 7.6). Assume that  $\angle(\nabla f(x), \nabla g(x)) \leq \rho < \pi/2$ , for any  $x \in B_\eta$ . Then the length of any trajectory of  $\nabla f$  in the set  $B_\eta$  is not greater than  $\frac{1}{\cos \rho} A(n, d)$ .*

Let  $\Theta(g)$  be the restriction to  $B_\eta$  of the curves constructed in Lemmas 7.7 and 7.8 for the polynomial  $g$ . Let  $X \subset \mathbb{B}^n$  be a trajectory of  $\nabla f$ . Similarly to the proof of Lemma 7.9 we can compare the length of the trajectory  $X$  with the length of  $\Theta(g)$ . We thus obtain that the length of the trajectory  $X$  is bounded by  $\frac{1}{\cos \rho} \cdot \text{Length } \Theta(g)$ . By Theorem 7.6 we get the bound  $\frac{1}{\cos \rho} A(n, d)$ .

At this point, we can complete the proof of Theorem 7.11. For any  $\varepsilon > 0$ , by Lemma 7.12, there exists  $\eta > 0$  (and thus two sets  $C_\eta$  and  $B_\eta$ ) such that the length of  $X$  restricted to  $C_\eta \cap \mathbb{B}^n$  is bounded by  $\varepsilon/2$ . Choose  $\rho > 0$  such that

$$\frac{1}{\cos \rho} A(n, d) \leq A(n, d) + \varepsilon/2.$$

Both Lemma 7.13 and Theorem 7.6 provide a polynomial  $g \in \mathbb{R}_d[\mathbf{X}] \setminus G_d$  such that

$$\angle(\nabla f(x), \nabla g(x)) \leq \rho$$

in  $B_\eta$ . By Lemma 7.14 the length of  $X$  restricted to  $B_\eta$  is bounded by  $A(n, d) + \varepsilon/2$ . Thus finally  $\text{Length}(X) \leq A(n, d) + \varepsilon$ . Hence Theorem 7.11 follows. q.e.d.

## 8. EXAMPLES

Let us fix integers  $d$  and  $n$ , we will denote by  $D(d, n)$  the supremum of length of gradient trajectories of polynomials of degree  $d$  in the unit ball in  $\mathbb{R}^n$ . We will now show that

**Theorem 8.1.** *For any integers  $n, d \geq 2$*

$$D(2d, n) \geq 2d^{n-1}, \quad d \in \mathbb{N}.$$

*Proof.* We are going to construct a sequence of examples which will confirm the estimate from below. First we recall "sinusoidal-like" properties of Chebyshev's polynomials. Recall that the  $d$ -th Chebyshev polynomial (of the first kind)  $T_d(x)$  is determined by

$$(8.1) \quad T_d(\cos \theta) = \cos(d\theta).$$

In particular it has the following properties:

**Lemma 8.2.**

- (1)  $|T_d(x)| \leq 1$ , for  $x \in [-1, 1]$ ;
- (2)  $T_d$  has  $d+1$  extrema on  $[-1, 1]$ , and the values at each extremum is  $\pm 1$ .

Hence the length of the graph of  $T_d$  restricted to  $[-1, 1]$ , is greater than  $2d$ .

Now return to the construction and, for  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ , put

$$f_i(x) = x_{i+1} - T_d(x_i).$$

Next we define a polynomial

$$p = \sum_{i=1}^{n-1} f_i^2.$$

Note that  $p$  is a polynomial of degree  $2d$  in  $n$  variables. The zero set of  $p$  is a smooth curve which is the intersection of the hypersurfaces  $\{x_{i+1} = T_d(x_i)\}$ , for  $i = 1, \dots, n-1$ . It is not difficult to see, by Lemma 8.2, that the length of the curve  $C = p^{-1}(0) \cap [-1, 1]^n$  is at least  $2d^{n-1}$ . Now we define our polynomial by

$$f(x) = \alpha(1 - x_1^2) - p(x),$$

where  $\alpha > 0$  is small enough. We will show below that  $f$  has only one critical point  $a$  which is in fact the global maximum. Note that the set  $\{f > 0\}$  is a thin neighbourhood of the curve  $C$ . Let  $b$  be a point in  $\{f > 0\}$  very close to the end of the curve  $C$ . Here the end of  $C$  is precisely the point  $(1, T_d(1), T_d(T_d(1)), \dots)$ . Let  $x(t)$  be a trajectory of  $\nabla f$  starting from the point  $b$ , so the trajectory must stay in the set  $\{f > 0\}$  and it must end at the critical point of  $f$ . The critical point  $a$  is in fact the middle of the curve  $C$ . Hence the length of the trajectory  $x(t)$  is not less than half the length of  $C$  which is at least  $2d^{n-1}$ , by Lemma 8.2. We are almost done except that the trajectories is included in  $[-1, 1]^n$  instead of the unit ball. The trick, due to J-M. Lion, is to slightly modify the functions  $f_i$  in the

following way. Let  $\varepsilon < 1$  and for  $i = 1, \dots, n-2$  let  $f_i^\varepsilon(x) = x_{i+1} - \varepsilon T_d(x_i/\varepsilon)$  and  $f_{n-1}^\varepsilon(x) = x_n - T_d(x_{n-1}/\varepsilon)$ . Then we put

$$p_\varepsilon = \sum_{i=1}^{n-1} (f_i^\varepsilon)^2$$

and  $C = p_\varepsilon^{-1}(0) \cap [-\varepsilon, \varepsilon]^{n-1} \times [-1, 1]$ . Finally we define  $f^\varepsilon(x) = \alpha(\varepsilon^2 - x_1^2) - p_\varepsilon(x)$ . In order to stay in the unit ball, we make the substitution

$$\tilde{f}^\varepsilon(x) = f^\varepsilon((1 + (n-1)\varepsilon^2)^{-1/2}x),$$

so Theorem 8.1 holds when taking  $\varepsilon \rightarrow 0$ .

We are left with computing the critical points of  $f$ , the case of  $\tilde{f}^\varepsilon$  is similar. We have to solve the system

$$(8.2) \quad \begin{cases} \frac{\partial f}{\partial x_1} = -2f_2 \frac{\partial f_2}{\partial x_1} - 2\alpha x_1 = 0, \\ \frac{\partial f}{\partial x_i} = -2f_i - 2f_{i+1} \frac{\partial f_{i+1}}{\partial x_i} = 0, \quad i = 2, \dots, n-1, \\ \frac{\partial f}{\partial x_n} = -2f_n = 0. \end{cases}$$

Observe that if  $f_{i+1}(x) = 0$  then  $2f_{i+1} \frac{\partial f_{i+1}}{\partial x_i}(x) = \frac{\partial f_{i+1}^2}{\partial x_i}(x) = 0$ . So (8.2) reduces to a very simple system

$$(8.3) \quad x_1 = 0 \text{ and } f_i = 0, \quad i = 2, \dots, n,$$

So  $a = (0, T_d(0), T_d((T_d(0))), \dots)$  is the only critical point of  $f$ . q.e.d.

## 9. BOUNDS FOR GRADIENT TRAJECTORIES OF QUASIPOLYNOMIALS

Our method can give an explicit bound for the length of trajectories of gradients in many other cases. For instance, by Khovanskiĭ's theory [Kh], for exponential polynomials or trigonometric polynomials. Let us first recall Khovanskiĭ's definition of quasipolynomials and trigonometric quasipolynomials.

**Definition 9.1.** Let  $P$  be a polynomial of degree  $d$  in  $n+k$  variables and let  $a_1, \dots, a_k \in \mathbb{R}^n$ . Then the function  $f$  defined by  $f(x) = P(x, y_1, \dots, y_k)$  and  $y_j = \exp\langle a_j, x \rangle$  for  $j = 1, \dots, k$  is called a quasipolynomial of degree  $d$ .

Let  $a_1, \dots, a_k \in \mathbb{R}^n$  and  $P_1, \dots, P_n$  be polynomials in  $n+k$  variables of degree  $\deg P_i = d_i$ . Let  $f_1, \dots, f_n$  be the corresponding quasipolynomials. Then Khovanskiĭ proved the following

**Theorem 9.2.** ([Kh] §1.2) *The number of non-degenerate solutions of  $f_1 = \dots = f_n = 0$  is finite and at most  $d_1 \cdots d_n (\sum d_i + 1)^k 2^{\frac{k(k-1)}{2}}$ .*

Let  $f$  be a quasipolynomial of degree  $d$  in  $n$  variables and  $k$  exponentials. Assume that  $n$  and  $d$  are greater than or equal to 2. Note that the function  $f$  is definable in  $\mathbb{R}_{\text{exp}}$ , and if  $k = 0$ , then  $f$  satisfies the hypotheses of Theorem 7.11. So we may assume that  $k \geq 1$ . Using Theorem 9.2 we can compute a bound for the length of the trajectories of  $\nabla f$  inside a ball of radius  $r$ . Thus, we have the following

**Theorem 9.3.** *Let  $f$  be a quasipolynomial of degree  $d$  in  $n$  variables and  $k$  exponentials. The length of any trajectory of  $\nabla f$  in a ball of radius  $r$  is bounded by*

$$rv(n)[(3d)^{n-1}(3d(n-1)+2)^k + 2(3d+1)^{n-2}((3d+1)(n-2)+3)^k]2^{\frac{k(k-1)}{2}}.$$

*Proof.* As in Chapter 7 we define  $\Theta_1(f)$  and  $\Theta_2(f)$  as the set of points at which the level sets of the functions  $|\nabla f|^2$  and  $f$  are tangents. Again, a deformation of  $f$  by quadratic polynomials proves that for almost all  $f$  the sets  $\Theta_1(f)$  and  $\Theta_2(f)$  are either empty or consist of a finite union of definable (in  $\mathbb{R}_{\text{exp}}$ ) curves and points. Note that in this case the coefficients of  $\nabla f$  and of the Hessian  $H_f$  are exponential polynomials of degree less or equal to  $d$ . Thus the coefficients of the 2-form  $d(|\nabla f|^2) \wedge df$  are also exponential polynomials of degree less or equal to  $3d$ . Similarly, the degree of the coefficients of  $d(|\nabla f|^2) \wedge df \wedge dr$  does not exceed  $3d+1$ . Theorem 9.2 implies that the number of non-degenerate intersection points of  $\Theta_1(f)$  with a generic hyperplane  $H \subset \mathbb{R}^n$  is bounded by

$$N_1(k, n, d) = (3d)^{n-1}(3d(n-1)+2)^k 2^{\frac{k(k-1)}{2}}.$$

Similarly the number of non-degenerate intersection points of  $\Theta_2(f) \cap H$  is bounded by

$$N_2(k, n, d) = 2(3d+1)^{n-2}((n-2)(3d+1)+3)^k 2^{\frac{k(k-1)}{2}}.$$

At this point, it suffices to follow the proofs of Theorems 7.6 and 7.11 to complete the proof of Corollary 9.3. q.e.d.

Let us continue with Khovanskii's theory. We extend our computations to a more general class of function. We shall now consider trigonometric quasipolynomials. According to [Kh] we have the following definition

**Definition 9.4.** Let  $P$  be a polynomial of degree  $d$  in  $n+k+2p$  variables and let  $a_1, \dots, a_k, b_1, \dots, b_p \in \mathbb{R}^n$ . Then the function  $f$  defined by

$$f(x) = P(x, y_1, \dots, y_k, u_1, \dots, u_p, v_1, \dots, v_p)$$

where  $y_j = \exp\langle a_j, x \rangle$  for  $j = 1, \dots, k$  and  $u_q = \sin\langle b_q, x \rangle$ ,  $v_q = \cos\langle b_q, x \rangle$  for  $q = 1, \dots, p$  is called a trigonometric quasipolynomial of degree  $d$ .

If  $f$  be a trigonometric quasipolynomial of degree  $d$  in  $n+k+2p$  variables, then the restriction of  $f$  to a ball of radius  $r$  is definable in  $\mathbb{R}_{\text{an,exp}}$ , the o-minimal structure of globally subanalytic sets augmented with the exponential function. Note that in general  $f$ , as a function defined on  $\mathbb{R}^n$ , is not definable in any o-minimal structure. Let  $a_1, \dots, a_k \in \mathbb{R}^n$ ,  $b_1, \dots, b_p \in \mathbb{R}^n$  and  $P_1, \dots, P_n$  be polynomials in  $n+k+2p$  variables of degree  $\deg P_i = d_i$ . Let  $f_1, \dots, f_n$  be the corresponding trigonometric quasipolynomials. The following Theorem gives a bound on the number of roots of a system of trigonometric quasipolynomial.

**Theorem 9.5.** ([Kh] §1.3) *If  $|\langle b_q, x \rangle| < \frac{\pi}{2}$ , then the number of nonsingular solutions of  $f_1 = \dots = f_n = 0$  is finite and at most*

$$d_1 \cdots d_n \left( \sum d_i + 1 + p \right)^{p+k} 2^{p + \frac{(p+k)(p+k-1)}{2}}.$$



Note that the assumptions of Theorem 9.5 are more restrictive than those of Theorem 9.2. In order to apply our method of computation for the trigonometric quasipolynomials, we thus need to strengthen the hypotheses. Consider the semialgebraic set  $\mathcal{B} = \{b \in \mathbb{R}^n : \forall x \in \overline{\mathbb{B}^n}, |\langle b, x \rangle| < \frac{\pi}{2}\}$ . Let  $a_1, \dots, a_k \in \mathbb{R}^n$  and  $b_1, \dots, b_p \in \mathcal{B}$  and  $P$  be a polynomial in  $n + k + 2p$  variables of degree  $d$ . Let  $f$  be the corresponding trigonometric quasipolynomial then we have the following

**Theorem 9.6.** *Let  $f$  be a trigonometric quasipolynomial in  $n + k + 2p$  variables with  $b \in \mathcal{B}$ . The length of any trajectory of  $\nabla f$  in the unit ball is bounded by*

$$\nu(n) [(3d)^{n-1}M_1 + 2(3d+1)^{n-2}M_2] 2^{p+\frac{(p+k)(p+k-1)}{2}},$$

where  $M_1 = (3d(n-1) + 2 + p)^{p+k}$  and  $M_2 = ((3d+1)(n-2) + 3 + p)^{p+k}$ .

#### 10. APPLICATION I: JOINING TWO POINTS IN A CONNECTED SEMIALGEBRAIC SET

In many situations e.g. robotics or quantitative transversality (see [Do] ) it is important to find a path joining two points  $x$  and  $y$  in a given connected component  $A$  of a semialgebraic set in  $\mathbb{R}^n$ . The usual manner it is done (see Yomdin [Yo1], [Yo-C]; Canny [Ca1],[Ca2], Donaldson [Do], Basu-Pollack Roy [B-P-R]) is the following: one first constructs so called "roadmap" i.e. a connected semialgebraic curve  $C \subset A$  in such a way that it is easy to join each point of  $A$  with the curve  $C$ . Since  $C$  is arc-connected we can join  $x$  with  $y$  via  $C$ . Actually the constructed path is semialgebraic. The construction of  $C$  is in general hard and not efficient; one applies induction on  $n$  which requires several quantifiers eliminations. There are several algorithms to compute a roadmap, but they don't give a realistic estimate for the length of path in terms of  $n$  and degrees of polynomials involved in a description of  $A$ . This is due to the fact that the description of  $C$  is very complex.

We propose below a new way of joining two points in a semialgebraic set  $A$  which is a connected component of  $\{f > 0\} \cap B(r)$ , where  $f$  is a polynomial of degree  $d$  and  $B(r)$  is a ball of radius  $r$  in  $\mathbb{R}^n$ . We will show that for a generic  $f$  we can join in  $A$  any two points of  $A$  by a curve which is piecewise trajectory of  $\nabla f$  or of  $-\nabla f$ .

Examining carefully our proof of Theorem 7.6 we will show that any two points in  $A$  may be joined in  $A$  by a curve of the length bounded by  $2rA(n, d+2)$ . It seems that from the numerical viewpoint this method is very promising. In fact there are quite efficient algorithms which numerically compute trajectories of gradients.

Let  $A$  be a connected and semialgebraic subset of  $\mathbb{R}^n$ . For any pair of points  $x, y \in A$  we denote by  $d_g(x, y)$  the infimum of lengths (for the Euclidean metric) of arcs joining, in  $A$ ,  $x$  with  $y$ . Clearly  $d_g(x, y) < \infty$ , since  $A$  is semialgebraic. In fact  $d_g$  is a distance on  $A$ . We call

$$\text{diam}_g(A) = \sup_{x, y \in A} d_g(x, y)$$

the *geodesic diameter* of the set  $A$ .

We now explain the context of our work. First we recall the following result of Yomdin [Yo1],[Yo-C]

**Theorem 10.1.** *Let  $A \subset \mathbb{R}^n$  be a semialgebraic set. Then, for any ball  $B(r)$  of radius  $r$  in  $\mathbb{R}^n$ , the geodesic diameter of every connected component of  $A \cap B(r)$  is bounded by  $rK$ . The constant  $K = K(D)$  depends only on  $D$  the diagram of  $A$  that is: the dimension  $n$  and the number and degrees of polynomials describing  $A$ .*

In particular for any integers  $d, n$  there exists a constant  $K(n, d)$ , such that if  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is polynomial of degree  $d$ , then the geodesic diameter of every connected component of  $\{f > 0\} \cap B(r)$  is bounded by  $rK(n, d)$ .

The construction of arcs joining two points in a connected component of  $A \cap B(r)$  proposed by Yomdin (also by other authors in a more general setting, see references to Theorem 1.1 in [Yo-C]) gives a semialgebraic arc of "bounded complexity" or in other words of controlled diagram and its length can be estimated by Cauchy-Crofton formula. However it does not give any realistic bound for  $K$ , in particular for  $K(n, d)$ .

S. K. Donaldson in his famous paper [Do] on existence of symplectic submanifolds used in a crucial way the above special case of Theorem 10.1. In fact what he needed (and what he proved in [Do]) was the following:

**Theorem 10.2.** *For any integer  $n$  there exists  $C(n) > 0$  and  $k(n) > 0$  such that*

$$K(n, d) \leq C(n)d^{k(n)}, \text{ for any } d \in \mathbb{N}.$$

A thorough examination of his construction can give explicit estimates, but still far from realistic. In fact the two Theorems mentioned above were used by the authors to obtain a quantitative version of the Morse-Sard theorem, and next by Donaldson [Do] for a quantitative transversality. We explain this at the end of the paper. From the robotics viewpoint the main goal is to construct explicitly a path joining two points.

Our first main result is the following.

**Theorem 10.3.** *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , with  $n \geq 2$ , be a polynomial of degree  $d \geq 2$  and let  $B(r)$  be a ball of radius  $r$  in  $\mathbb{R}^n$ . Let  $D_i, i \in I$  be all connected components of  $\{f > 0\} \cap B(r)$ . Then*

$$\sum_{i \in I} \text{diam}_g(D_i) \leq 2r\nu(n)(3d+2)^{n-1},$$

*In particular  $\text{diam}_g(D_i) \leq 2r\nu(n)(3d+2)^{n-1}$  for any connected component  $D_i$ . More precisely, there is a polynomial  $g$  of degree  $d+2$  such that any two points in  $D_i$  can be joined in  $D_i$  by an arc which is a piecewise trajectory of  $\nabla g$  or  $-\nabla g$  of length not greater than  $2r\nu(n)(3d+2)^{n-1}$ .*

*Proof.* Let us fix a ball  $B(r) = B(x_0, r) = \{x \in \mathbb{R}^n; |x - x_0|^2 < r\}$  and put

$$g(x) = (r - |x - x_0|^2)f(x).$$

Then any connected component of  $\{f > 0\} \cap B(r)$  is a connected component of  $\{g > 0\}$ . Of course  $g$  is a polynomial of degree  $d+2$ .

By a small perturbation (very slightly modifying  $g^{-1}(0) \cap B(r)$ ) we may assume that  $g$  is generic in the sense of Proposition 7.2 and is a Morse function. This means  $g$  has only isolated non-degenerate critical points  $c_1, \dots, c_k$  in the ball and moreover that the corresponding critical values  $y_j = g(c_j)$  for  $j = 1, \dots, k$  are distinct.

Let  $x, y$  be two points in  $D_i$  a connected component of  $\{g > 0\}$ . We first outline the general idea of the construction of an arc joining  $x$  with  $y$  in  $D_i$ .

We may assume that  $\nabla g(x) \neq 0$ , if not we slightly move  $x$ . We take the trajectory  $\lambda_1 : [0, a_1] \rightarrow D_i$  of  $\nabla g$  starting at  $x$ , that is  $\lambda_1(0) = x$  and  $\lambda_1'(t) = \nabla g(\lambda_1(t))$ . We consider  $\lambda_1(t)$  maximal (to the right), hence

$$\lim_{t \rightarrow a_1} \lambda_1(t) = c_{j_1},$$

where  $c_{j_1}$  is one of the critical points of  $g$ . If  $c_{j_1}$  is a local maximum of  $g$  we stop at that point. If not we continue: we take a point  $x_1$  close to  $c_{j_1}$  and such that  $g(x_1) > y_{j_1} = f(c_{j_1})$ . We can start the trajectory  $\lambda_2 : [a_1, a_2] \rightarrow D_i$  of  $\nabla g$  such that  $\lambda_2(a_1) = x_1$ . (In fact we could take  $x_1 = c_{j_1}$ , since by a result of Łojasiewicz (cf. [Ło1], [Ło2]) there exists a trajectory of  $-\nabla g$  which tends to  $c_{j_1}$ ). Again  $\lim_{t \rightarrow a_2} \lambda_1(t) = c_{j_2}$  is a critical point of  $g$ . Thus, by finitely many steps we obtain a trajectory

$$\lambda_x : [0, a_x] \rightarrow D_i, \text{ such that, } \lambda_x(0) = x, \text{ and } \lambda_x(a_x) \text{ is a local maximum of } g.$$

The curve  $\lambda_x$  is continuous, analytic except at finitely many points and at each of those points has left-hand side and right-hand side derivative (by the Gradient Conjecture [K-M-P]).

In this way we have shown that any two points in  $D_j$  can be joined by a trajectory of  $\nabla g$  with a local maximum of  $g$ . Clearly, if there is only one local maximum in  $D_i$  we obtain the desired bound for the geodesic diameter of  $D_i$  directly from Theorem 7.11.

If there are several local maxima of  $g$  in a connected component  $D_i$  of  $\{g > 0\}$  we shall proceed in a more systematic way, which we describe below.

Let  $\Gamma = \Gamma(g)$  be the curve associated in Chapter 7 to our polynomial  $g$  of degree  $d + 2$ , that is  $\Gamma$  contains all points at which  $|\nabla g|^2$  attains a local minimum on the fibres of  $g$ . Note that the fibres of  $g$  are compact and thus  $\Gamma = \Gamma_1$  (cf. Chapter 7). Hence the length of  $\Gamma$  is bounded by  $r\nu(n)(3d + 2)^{n-1}$ . Let  $D_i, i \in I$  be all connected components of  $f^{-1}((t, s)) \cap B(r)$  and let  $\lambda_i : [\alpha_i, \beta_i] \rightarrow D_i$  be a trajectory of  $\nabla g$  in  $D_i$ . Then, by Corollary 7.10,  $\text{length}(\lambda_i) \leq \text{length}(\Gamma \cap D_i)$ . In particular, if we have a trajectory  $\lambda_i$  in each  $D_i$ , then

$$\sum_{i \in I} \text{length}(\lambda_i) \leq \text{length}(\Gamma \cap B(r) \cap f^{-1}((t, s))) \leq r\nu(n)(3d + 2)^{n-1}.$$

We shall also consider the connected components of sets  $\{g > t\}$ , where  $t$  is a variable. We write decomposition into connected components

$$\{g > t\} \cap B(r) = \bigcup_{i \in I_t} A_i^t.$$

Now we have

**Lemma 10.4.** *Any two points in  $A_i^t$  can be joined in  $A_i^t$  by an arc whose length is not greater than  $2\text{length}(\Gamma \cap A_i^t)$ .*

We prove the Lemma by "descending"  $t$  to 0. If  $A_i^t$  contains only one critical point of  $g$ , then, as we already explained above, we can join (in  $A_i^t$ ) any point of  $A_i^t$  to the maximum of  $g$  in  $A_i^t$ , by a trajectory of  $\nabla g$ . So by Proposition 7.10 we can join any two points in  $A_i^t$  by a curve whose length is not greater than  $2\text{length}(\Gamma \cap A_i^t)$ .

In the general case we will proceed by induction on the number of critical points of  $g$  in  $A_i^t$ .

Recall that  $g$  has only Morse singularities which we denote by  $c_j, j = 1, \dots, k$ .

Let us fix  $0 < t < s$  and assume that the interval  $(t, s)$  contains only one critical value  $y_j$ . Also assume that the component  $A_i^t$  contains the critical point  $c_j$  such that  $g(c_j) = y_j$ . Note that there are most two connected components of  $\{g > s\}$  contained in  $A_i^t$ , since  $g$  has a Morse singularity at  $c_j$ . Let us denote them by  $A_1^s$

and  $A_2^s$  and suppose that at least  $A_1^s$  is non-empty. We will take  $s > y_j$  but very close to  $y_j$ .

Take two points  $x, y \in A_i^t$ . We shall consider several cases.

*Case 1.* Assume  $g(x) < y_j$  and  $g(y) < y_j$ . We take the trajectory  $\lambda_x$  of  $\nabla g$  starting at the point  $x$  and ending at a point  $x'$  such that  $g(x') > s$ . If the trajectory passes by the critical point  $c_j$  we extend it starting from this point. Note that  $c_j$  is not a local maximum of  $g$ , so we can pass over the level  $\{g = y_j\}$ . At this point we could also take an arbitrary short segment joining a point on trajectory at which  $g < y_j$  with a point at which  $g > y_j$ .

In the same way we take the trajectory  $\lambda_y$  of  $\nabla g$  starting at the point  $y$  and ending at a point  $y'$  such that  $g(y') > s$ . Note that by Proposition 7.10 we have

$$(10.1) \quad \text{Length}(\lambda_x) + \text{Length}(\lambda_y) \leq 2\text{Length}(\Gamma \cap A_i^t \cap g^{-1}(t, s)).$$

*Case 1.1.* If  $x'$  and  $y'$  belong to the same component  $A_1^s$  or  $A_2^s$ , then by our induction hypothesis we can join  $x'$  with  $y'$  in  $A_1^s$ , respectively in  $A_2^s$ , by a piecewise trajectory of  $\nabla g$  of length at most  $2\text{Length}(\Gamma \cap A_1^s)$ , respectively at most  $2\text{Length}(\Gamma \cap A_2^s)$ . Hence, by (10.1) the total length of the curve joining  $x$  with  $y$  in  $A_i^t$  is not greater than  $2\text{Length}(\Gamma \cap A_i^t)$  as claimed in Lemma 10.4.

*Case 1.2.* Assume now that  $x' \in A_1^s$  and  $y' \in A_2^s$ . By induction hypothesis we can join, in  $A_1^s$ ,  $x'$  with a point  $x''$  very close to  $c_j$ . Respectively we can join, in  $A_2^s$ ,  $y'$  with a point  $y''$  very close to  $c_j$ . Note that the total length of both curves is not greater

$$2(\text{Length}(\Gamma \cap A_1^s) + \text{Length}(\Gamma \cap A_2^s))$$

Finally we can join, in  $A_i^t$ , the point  $x''$  with  $y''$  by a short segment (or, via  $c_j$  by a short piece of a trajectory of  $\nabla g$ ). So again, by (10.1), we deduce that the total length of the curve joining  $x$  with  $y$  in  $A_i^t$  is not greater than  $2\text{Length}(\Gamma \cap A_i^t)$  as claimed in Lemma 10.4.

The remaining cases where  $g(x) < y_j$  and  $g(y) > y_j$  or  $g(x) > y_j$  and  $g(y) > y_j$  can be handled analogously. So Lemma 10.4 follows.

Taking  $t = 0$  in Lemma 10.4 we obtain Theorem 10.3.

q.e.d.

We complete this Chapter by an example based on the one proposed in Chapter 8.

Let us fix integers  $d$  and  $n$ , we will denote by  $D_g(d, n)$  the supremum of geodesic diameters of connected components, included in a unit ball of  $\mathbb{R}^n$ , of sets  $\{f > 0\}$ , where  $f$  is a polynomial of degree  $d$ . We will now show that

**Theorem 10.5.** *For any  $n, d \in \mathbb{N}$ ,*

$$D_g(d, n) \geq 2d^{n-1}, \quad d \in \mathbb{N}.$$

*Proof.* Let  $T_d$  be the  $d$ -th Chebyshev polynomial (of the first kind) defined in Chapter 8. Recall that the length of the graph of  $T_d$  restricted to  $[-1, 1]$ , is greater than  $2d$ . Now return to the construction in  $\mathbb{R}^n$ , and as before, for  $\varepsilon \in (0, 1]$  define

$$p_{d,n}(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n-2} (x_{i+1} - \varepsilon T_d(x_i/\varepsilon))^2 + (x_n - T_d(x_{n-1}/\varepsilon))^2.$$

which is a polynomial of degree  $2d$  in  $n$  variables. The zero set of  $p_{d,n}$  is a smooth curve which is the intersection of the hypersurfaces  $\{x_{i+1} = \varepsilon T_d(x_i/\varepsilon)\}$ , for  $i =$

$1, \dots, n-2$  and  $\{x_n = T_d(x_{n-1}/\varepsilon)\}$ . Note that the length of the curve  $p_{d,n}^{-1}(0) \cap [-\varepsilon, \varepsilon]^{n-1} \times [-1, 1]$  is at least  $2d^{n-1}$ . Now choose an  $\alpha > 0$  small enough and put

$$\tilde{P}_{d,n}(x) = -p_{d,n}(x) + \alpha(x_1^2 - \varepsilon^2).$$

The set  $\{\tilde{P}_{d,n} > 0\}$  is a thin neighborhood of the curve  $p_{d,n}^{-1}(0) \cap [-\varepsilon, \varepsilon]^{n-1} \times [-1, 1]$ . So its geodesic diameter is almost equal to the length of  $p_{d,n}^{-1}(0) \cap [-\varepsilon, \varepsilon]^{n-1} \times [-1, 1]$  which is at least  $2d^{n-1}$ . Finally, to obtain the desired semialgebraic set in the unit ball we put  $s(\varepsilon) = 1 + (n-1)\varepsilon^2$  and  $P_{d,n}(x) = \tilde{P}_{d,n}(\sqrt{s(\varepsilon)}x)$ . So Theorem 10.5 holds when taking  $\varepsilon \rightarrow 0$ . q.e.d.

## 11. APPLICATION II: QUANTITATIVE MORSE-SARD THEOREM

Let us first recall the quantitative version of Morse-Sard Theorem due to Yomdin (cf. [Yo2]), which was previously mentioned.

**Theorem 11.1.** *Let  $B(r)$  be a ball of radius  $r$  in  $\mathbb{R}^n$ , and let  $f : B(r) \rightarrow \mathbb{R}$  be the restriction of a polynomial of degree  $d$ . For any  $\varepsilon > 0$  we put  $\Sigma_\varepsilon = \{x \in B(r); |\nabla f(x)| < \varepsilon\}$ . Then the set  $f(\Sigma_\varepsilon)$  can be covered by  $N(n, d)$  segments of length  $r\varepsilon$ . The number  $N(n, d)$  depends only on dimension  $n$  and degree  $d$ .*

This entropic version of Morse-Sard Theorem has the advantage of being stable under small smooth perturbations and has many applications, see [Yo2], [Yo-C]. Another important application appears in a paper of Donaldson [Do], it is so-called controlled transversality (in fact this notion goes back to Gromov). Indeed Theorem 11.1 allows to find a ball of controlled radius in the set of regular values of  $f$ . Yomdin insisted in [Yo2] that for most actual applications it is important to have an explicit and realistic estimate for the number  $N(n, d)$ . Actually he stated in [Yo2] an explicit estimate in the case  $n = 2$  which was  $N(2, d) \leq \frac{7d^2 - 15d}{4} + 3$ , but for general  $n$  no such estimate was known.

The usual way to prove Theorem 11.1 is to apply Theorem 10.1 in the following way. We consider the set

$$\Sigma_\varepsilon = \{x \in B(r); |\nabla f|^2 < \varepsilon^2\} = B(r) \cap \{p > 0\},$$

where  $p(x) = \varepsilon^2 - |\nabla f(x)|^2$ . Clearly  $p$  is of degree  $2(d-1)$ . Let  $A_i$  be a connected component of  $\Sigma_\varepsilon$ . Note that  $f(A_i)$  is an interval in  $\mathbb{R}$ , assume that  $\overline{f(A_i)} = [\alpha_i, \beta_i]$ . Let us take points  $x_i, y_i \in A_i$  such that  $f(x_i)$  is very close to  $\alpha_i$  and  $f(y_i)$  is very close to  $\beta_i$ . Let  $\lambda_i$  be a piece-wise smooth arc, constructed according to Theorem 10.3, joining  $x_i$  with  $y_i$  in  $A_i$ . So we may assume that  $f(\lambda_i) = [\alpha_i, \beta_i]$ . By the Mean Value Theorem we obtain that

$$\beta_i - \alpha_i = \text{Length}(f(\lambda_i)) \leq \varepsilon \text{Length}(\lambda_i)$$

Recall that by Theorem 10.3 the total length of all  $\lambda_i$  is not greater  $2rA(n, 2d)$ . Note that each connected component of  $\Sigma_\varepsilon$  is actually a connected component of  $\{\tilde{p} > 0\}$ , where  $\tilde{p}(x) = p(x)(r^2 - |x - x_0|^2)$ . Here  $x_0$  is the center of the ball  $B(r)$ . The classical bounds for the topology of semi-algebraic sets (cf. [Mi], see also [Yo-C]) yields that the number of bounded connected components of  $\{\tilde{p} > 0\}$  is at most

$$B(n, d) = \frac{1}{2}(\deg \tilde{p})(\deg \tilde{p} - 1)^{n-1} = d(2d-1)^{n-1}.$$

So we have at most  $B(n, d)$  segments of the total length not greater  $\varepsilon r 2A(n, 2d)$ , so they can be covered by  $B(n, d) + 2A(n, 2d)$  intervals of length  $\varepsilon r$ . Thus we obtained the following estimate

$$(11.1) \quad N(n, d) \leq d(2d-1)^{n-1} + 2\nu(n)((6d-4)^{n-1} + 2(6d-3)^{n-2}).$$

This is a first (to our knowledge) explicit bound for  $N(n, d)$ , note that  $N(n, d) \geq (d-1)^n$  since there are real polynomials of degree  $d$  which have  $(d-1)^n$  critical values.

We will give now a better estimate for  $N(n, d)$  based on a direct analysis of the curve  $\Gamma$  associated in Chapter 7 to a generic polynomial of degree  $d$ . Recall that the curve  $\Gamma$  contains all local minima of  $|\nabla f|$  on fibres of  $f$  restricted to  $\overline{B(r)}$ . By abuse of notation we denote  $\Sigma_\varepsilon = \{x \in \overline{B(r)}; |\nabla f(x)| < \varepsilon\}$  which is slightly bigger than the set in the Theorem 11.1 because we also take into account the boundary of  $B(r)$ .

Note that, if  $t \in f(\Sigma_\varepsilon)$ , then  $\inf\{|\nabla f(x)|; x \in f^{-1}(t) \cap \overline{B(r)}\} < \varepsilon$ , hence

$$f^{-1}(t) \cap \Gamma \cap \Sigma_\varepsilon \neq \emptyset.$$

Consequently

$$f(\Sigma_\varepsilon) = f(\Gamma \cap \Sigma_\varepsilon).$$

By Theorem 7.11 we know that the total length of  $\Gamma \cap \overline{B(r)}$ , hence also of  $\Gamma \cap \Sigma_\varepsilon$ , is bounded by  $rA(n, d)$ . So the total length of  $f(\Sigma_\varepsilon)$  is not greater than  $\varepsilon r A(n, d)$ . We showed above that the number of connected components of  $f(\Sigma_\varepsilon)$  is bounded by  $B(n, d) = \frac{1}{2}(2d)(2d-1)^{n-1}$ . Thus we have a better bound

**Theorem 11.2.** *Let  $N(n, d)$  be the constant in Yomdin's Theorem 11.1. Then*

$$N(n, d) \leq d(2d-1)^{n-1} + \nu(n)((3d-4)^{n-1} + 2(3d-3)^{n-2}).$$

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