BEST CONSTANT IN POINCARÉ INEQUALITIES WITH TRACES: A FREE DISCONTINUITY APPROACH

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Abstract. For $\Omega \subset \mathbb{R}^N$ open bounded and with a Lipschitz boundary, and $1 \le p < +\infty$, we consider the Poincaré inequality with trace term

$$C_p(\Omega)\|u\|_{L^p(\Omega)} \le \|\nabla u\|_{L^p(\Omega;\mathbb{R}^N)} + \|u\|_{L^p(\partial\Omega)}$$

on the Sobolev space $W^{1,p}(\Omega)$. We show that among all domains Ω with prescribed volume, the constant is minimal on balls. The proof is based on the analysis of a free discontinuity problem.

Keywords: Free discontinuity problems, functions of bounded variation, sets of finite perimeter, Robin boundary conditions.

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Contents

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be an open, bounded set with Lipschitz boundary. Given $p \in [1, +\infty[$, an equivalent norm on the Sobolev space $W^{1,p}(\Omega)$ is given by

$$\|\nabla u\|_{L^p(\Omega;\mathbb{R}^N)} + \|u\|_{L^p(\partial\Omega)}.$$

As a consequence, there exists a maximal constant $C_p(\Omega) > 0$ such that

$$C_p(\Omega)\|u\|_{L^p(\Omega)} \le \|\nabla u\|_{L^p(\Omega;\mathbb{R}^N)} + \|u\|_{L^p(\partial\Omega)} \tag{1.1}$$

for every $u \in W^{1,p}(\Omega)$. Inequality (1.1) can be seen as a Poincaré inequality with trace term.

The main result of the paper states that balls are the sets which minimize the constant in (1.1) among domains with a given volume.

Theorem 1.1 (The main result). *Let $p \in [1, +\infty[$. Then for every open, bounded set with Lipschitz boundary $\Omega \subseteq \mathbb{R}^N$ we have*

$$C_p(B) \le C_p(\Omega),$$

where $B \subseteq \mathbb{R}^N$ is a ball such that $|B| = |\Omega|$. Moreover equality holds if and only if Ω is a ball.

Essentially the whole paper is concerned with the range $1 < p < +\infty$. The case $p = 1$ is well known in the literature since $C_1(\Omega)$ is precisely the Cheeger constant of the set Ω. In this case, the proof of Theorem 1.1 comes by symmetrization. We shall comment this issue in last section of the paper, Remark 5.1. For $p > 1$, no symmetrization argument is known to work.

In order to describe our approach, let us comment the particular case $p = 2$. Moreover, we concentrate on a variant of (1.1) given by

$$\tilde{C}_2(\Omega)\|u\|^2_{L^2(\Omega)} \le \|\nabla u\|^2_{L^2(\Omega;\mathbb{R}^N)} + \|u\|^2_{L^2(\partial\Omega)}$$

for every $u \in W^{1,2}(\Omega)$. We get easily that the maximal constant is given by

$$\tilde{C}_2(\Omega) = \min_{u\in W^{1,2}(\Omega), u\neq 0} \frac{\int_\Omega |\nabla u|^2\, dx + \int_{\partial\Omega} u^2\, d\mathcal{H}^{N-1}}{\int_\Omega u^2\, dx},$$

so that $\tilde{C}_2(\Omega)$ coincides with the *first eigenvalue of the Robin-Laplace operator* on Ω with constant $\beta = 1$: more precisely we deduce that

$$\tilde{C}_2(\Omega) = \lambda^R_{1,1}(\Omega),$$

where for $\beta > 0$ the quantity $\lambda^R_{1,\beta}(\Omega)$ is characterized by the existence of a nontrivial function $u \in W^{1,2}(\Omega)$ such that

$$\begin{cases} -\Delta u = \lambda^R_{1,1}(\Omega) u & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega \\ u \ge 0 & \text{in } \Omega, \end{cases}$$

where ν denotes the outer normal to the boundary. The Robin conditions

$$\frac{\partial u}{\partial \nu} + \beta u = 0 \qquad \text{on } \partial\Omega$$

are associated to the presence of the boundary term in the Rayleigh quotient: they are somehow intermediate between the Neumann conditions ($\beta = 0$) and the Dirichlet conditions (obtained formally for $\beta \to \infty$).

The optimality of the ball for the constant $\tilde{C}_2$ is a consequence of the Faber-Krahn inequality for the Robin-Laplacian, i.e.,

$$\lambda^R_{1,\beta}(B) \le \lambda^R_{1,\beta}(\Omega), \tag{1.2}$$

where B is a ball such that $|B| = |\Omega|$ (the equality holds only if Ω is itself a ball). This inequality has been proved by Bossel [2] (for two dimensional simply connected smooth domains) and Daners [13] (in the N-dimensional setting, under Lipschitz regularity for the boundary). Their proof, by a *dearrangement procedure*, involves a direct comparison between Ω and B based not on the Rayleigh quotient representation for $\lambda^R_{1,\beta}(\Omega)$, but on a different one which involves a different quantity, named the *H-functional*.

Coming back to the kind of inequality we are interested in, if we consider

$$C_2(\Omega)\|u\|_{L^2(\Omega)} \le \|\nabla u\|_{L^2(\Omega;\mathbb{R}^N)} + \|u\|_{L^2(\partial\Omega)},$$

we see that the new constant $C_2(\Omega)$ is given now by

$$C_2(\Omega) = \min_{u\in W^{1,2}(\Omega), u\neq 0} \frac{\|\nabla u\|_{L^2(\Omega;\mathbb{R}^N)} + \|u\|_{L^2(\partial\Omega)}}{\|u\|_{L^2(\Omega)}}. \tag{1.3}$$

Even in this case we have a connection with the Robin-Laplacian: if the minimum in (1.3) is attained on a nonconstant function $u \in W^{1,2}(\Omega)$, which we may assume to be nonnegative, then by exploiting its optimality we deduce that u is the first eigenfunction for the Robin-Laplacian with constant

$$\beta_u := \frac{\|\nabla u\|_{L^2(\Omega;\mathbb{R}^N)}}{\|u\|_{L^2(\partial\Omega)}},$$

and moreover

$$C_2(\Omega) = \frac{\|u\|_{L^2(\Omega)}}{\|\nabla u\|_{L^2(\Omega;\mathbb{R}^N)}} \lambda^R_{1,\beta_u}(\Omega).$$

This representation shows that the link with the Robin-Laplacian is too weak to infer the optimality of the ball in Theorem 1.1 from the Faber-Krahn inequality (1.2). Moreover, the approach by Bossel and Daners cannot be applied to the *characteristic value* (1.3), since the Rayleigh quotient involved is now nonlinear (sums of norms are involved), and no analogue of the H-functional is known in this situation.

In order to prove Theorem 1.1 in the case $p > 1$, we follow the strategy proposed in [5] and [6] to deal with the Faber-Krahn inequality for the Robin-Laplacian, and based on the analysis of *free discontinuity functionals*. More precisely, for $p \in]1, +\infty[$, we concentrate on the free discontinuity functional

$$F(u) := \frac{\left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx\right)^{1/p} + \left(\int_{J_u} [(u^+)^p + (u^-)^p] \, d\mathcal{H}^{N-1}\right)^{1/p}}{\left(\int_{\mathbb{R}^N} u^p \, dx\right)^{1/p}}$$

defined on the set of functions

$$SBV^{\frac{1}{p}}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) \, : \, u^p \in SBV(\mathbb{R}^N), u \geq 0\}.$$

Here SBV denotes the space of *special functions of bounded variation* (see [1] and Subsection 2.2).

The basic remark, which leads to the study of the functional F, and which was the motivation for [5] and [6], is that if $u \geq 0$ is an optimal function for (1.3), then its extension to zero outside Ω is such that

$$u 1_\Omega \in SBV^{\frac{1}{p}}(\mathbb{R}^N)$$

with

$$C_p(\Omega) = F(u 1_\Omega).$$

This observation, leads to the following natural inequality. Given some constant $m > 0$,

$$\inf\{C_p(\Omega) : \Omega \text{ open, bounded, Lipschitz}, |\Omega| = m\} \geq \inf\{F(u) : u \in SBV^{\frac{1}{p}}(\mathbb{R}^N), |\{u > 0\}| = m\}.$$

Now, if we prove that the infimum in the right hand side is attained by a function u which is the extension by 0 of a minimizer for (1.3) on a ball (of volume m), then we achieve the proof of Theorem 1.1 and, even more, we provide a Poincaré inequality in $SBV^{\frac{1}{p}}(\mathbb{R}^N)$, with an optimal constant. Following the strategy of [6], the proof of Theorem 1.1 is thus obtained by showing that minimizers of F, under a volume constraint for the support, are functions supported on balls.

Our analysis, shaped after [6], can be summarized as follows.

(a) We focus on the problem

$$\inf\{F(u) : u \in SBV^{\frac{1}{p}}(\mathbb{R}^N), |\{u > 0\}| = m\},$$

and prove existence of a solution. A regularity argument of topological type à la De Giorgi, Carriero and Leaci (see Subsection 4.2) shows that *every* minimizer u of F (under a volume constraint) is such that

$$\mathcal{H}^{N-1}(J_u) < +\infty, \qquad \mathcal{H}^{N-1}(\overline{J_u} \setminus J_u) = 0,$$

and the associated support is given by an open connected set Ω with

$$\partial\Omega = \overline{J_u}.$$

In particular the boundary of Ω is an hypersurface in the weak sense of geometric measure theory, and Ω turns out to have *finite perimeter* (see Subsection 2.2).

(b) By means of a reflection technique (Proposition 4.10), it is shown that F admits minimizers of the form

$$\psi 1_\Omega,$$

where Ω is symmetric around the origin and $\psi :]0, +\infty[\to \mathbb{R}$ is smooth, radial symmetric, positive and bounded from above and below on Ω. This yields that

$$\mathcal{H}^{N-1}(\partial\Omega \setminus \partial^*\Omega) = 0,$$

where $\partial^* \Omega$ is the *reduced boundary* of Ω (see Subsection 2.2), and

$$F(\psi 1_\Omega) = \frac{\left(\int_\Omega |\psi'(|x|)|^p\, dx\right)^{1/p} + \left(\int_{\partial^*\Omega} \psi^p(|x|)\, dx\right)^{1/p}}{\left(\int_\Omega \psi^p(|x|)\, dx\right)^{1/p}}.$$

We obtain thus a candidate optimal shape, on which the functional F gains a geometrical flavor, with ψ (and its gradient) acting as volume and surface densities on Ω and its boundary.

(c) It is shown (Proposition 4.13) that the radial symmetry of ψ entails that also Ω has a circular symmetry, being either a ball or an annulus. A direct comparison shows that the annulus is not optimal, which yields that minimizers are supported on balls.

Along with the new form of the Poincaré inequality we deal with, the main technical novelties of the present paper concerning the previous analysis are the following.

(1) The circular symmetry of the optimal domain Ω in point (c) is obtained by making use of the *spherical cap symmetrization technique*, taking advantage of the radial symmetry of ψ and of the geometric flavor of the problem mentioned in point (b). This approach yields a simplified proof of the optimality of the ball also for the semilinear variants of the first eigenvalue of the Robin-Laplacian studied in [6]: in the present context, it proves to be an efficient tool to cope with the nonlinear structure of F, involving sums of norms.

(2) The structure of F and the presence of a general exponent p entail some technical difficulties, especially when dealing with the regularity analysis of point (a) (see in particular Theorem 4.5 and Theorem 4.7 where technical manipulations are needed to get rid of the norms).

The uniqueness issue is settled in Theorem 4.14, by exploiting some equality cases in a chain of inequalities which are at the core of the reflection argument mentioned in point (b).

We conclude this introduction by remarking that Sobolev inequalities with trace terms (raised at a suitable exponent) have been treated in [17] using mass transportation techniques, and showing a suitable optimality for the ball. As an extension, a Poincaré type inequality has been derived in [16], involving L^1 norms for the functions and its trace, and L^p norm for the gradient, again proving an optimality property for the ball. It is worth also to notice the result of [3], where it is proved that the ball may not be optimal, at least for some choices of norms.

The paper is organized as follows. In Section 2 we introduce the notation and recall some basic properties of functions of bounded variation and sets of finite perimeter employed in the subsequent analysis. In Section 3 we define the functional space $SBV^{\frac{1}{p}}(\mathbb{R}^N)$, recalling the associated compactness and lower semicontinuity properties. Section 4 is devoted to the analysis of the free discontinuity functional F and of its minimizers, along the lines described above in points (a), (b) and (c). Finally the proof of Theorem 1.1 is carried out in Section 5.

2. Notation and preliminaries

Throughout the paper, $B_r(x)$ will denote the open ball of center $x \in \mathbb{R}^N$ and radius $r > 0$. We say that $A \subset\subset B$ if $\bar{A}$ is compact and contained in B. If $E \subset \mathbb{R}^N$, we will denote its volume by $|E|$, its complement by E^c, and 1_E will stand for its characteristic function, i.e., $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ if $x \notin E$. We set $\omega_N := |B_1(0)|$. Moreover $\mathcal{H}^{N-1}$ will stand for the $(N-1)$-dimensional Hausdorff measure, which coincides with the usual area measure on regular hypersurfaces.

For $A \subseteq \mathbb{R}^N$ open set and $p \geq 1$, $L^p(A)$ will denote the usual Lebesgue space of p-summable functions, while $W^{1,p}(A)$ will denote the Sobolev space of functions in $L^p(A)$ whose gradient in the sense of distributions is p-summable. Moreover $\|u\|_\infty$ will stand for the sup-norm of u, while *supp*(u) will denote the set $\{u \neq 0\}$, well defined up to sets of negligible Lebesgue measure.

2.1. **A numerical inequality.** The following inequality will be fundamental for our analysis.

Lemma 2.1. *Let $p \in]1, +\infty[$. Then for every $a_1, a_2, b_1, b_2 \geq 0$ and $c_1, c_2 > 0$ we have*

$$\frac{(a_1+a_2)^{1/p}+(b_1+b_2)^{1/p}}{(c_1+c_2)^{1/p}} \geq \min\left\{\frac{a_1^{1/p}+b_1^{1/p}}{c_1^{1/p}}, \frac{a_2^{1/p}+b_2^{1/p}}{c_2^{1/p}}\right\}. \tag{2.1}$$

Moreover, if equality holds, then

$$\frac{(a_1+a_2)^{1/p}+(b_1+b_2)^{1/p}}{(c_1+c_2)^{1/p}} = \frac{a_1^{1/p}+b_1^{1/p}}{c_1^{1/p}} = \frac{a_2^{1/p}+b_2^{1/p}}{c_2^{1/p}}.$$

Proof. By contradiction, let us assume that

$$\frac{(a_1+a_2)^{1/p}+(b_1+b_2)^{1/p}}{(c_1+c_2)^{1/p}} < \min\left\{\frac{a_1^{1/p}+b_1^{1/p}}{c_1^{1/p}}, \frac{a_2^{1/p}+b_2^{1/p}}{c_2^{1/p}}\right\}.$$

In particular we get

$$\frac{(a_1+a_2)^{1/p}+(b_1+b_2)^{1/p}}{(c_1+c_2)^{1/p}} < \frac{a_1^{1/p}+b_1^{1/p}}{c_1^{1/p}},$$

which gives

$$c_1\left((a_1+a_2)^{1/p}+(b_1+b_2)^{1/p}\right)^p < (c_1+c_2)\left(a_1^{1/p}+b_1^{1/p}\right)^p. \tag{2.2}$$

Similarly we get

$$c_2\left((a_1+a_2)^{1/p}+(b_1+b_2)^{1/p}\right)^p < (c_1+c_2)\left(a_2^{1/p}+b_2^{1/p}\right)^p. \tag{2.3}$$

Summing (2.2) and (2.3) we get

$$\left((a_1+a_2)^{1/p}+(b_1+b_2)^{1/p}\right)^p < \left(a_1^{1/p}+b_1^{1/p}\right)^p + \left(a_2^{1/p}+b_2^{1/p}\right)^p.$$

Choosing

$$a_i := A_i^p, \quad b_i := B_i^p, \qquad i = 1, 2,$$

we get

$$(A_1^p + A_2^p)^{1/p} + (B_1^p + B_2^p)^{1/p} < ((A_1+B_1)^p + (A_2+B_2)^p)^{1/p}, \tag{2.4}$$

which is against the triangle inequality of the p-norm on $\mathbb{R}^2$.

Let us assume now that

$$\frac{(a_1+a_2)^{1/p}+(b_1+b_2)^{1/p}}{(c_1+c_2)^{1/p}} = \min\left\{\frac{a_1^{1/p}+b_1^{1/p}}{c_1^{1/p}}, \frac{a_2^{1/p}+b_2^{1/p}}{c_2^{1/p}}\right\}.$$

Then we necessarily have

$$\min\left\{\frac{a_1^{1/p}+b_1^{1/p}}{c_1^{1/p}}, \frac{a_2^{1/p}+b_2^{1/p}}{c_2^{1/p}}\right\} = \frac{a_1^{1/p}+b_1^{1/p}}{c_1^{1/p}} = \frac{a_2^{1/p}+b_2^{1/p}}{c_2^{1/p}},$$

because otherwise one of the relations (2.2) and (2.3) would become an equality, the other remaining a strict inequality, which yields again to (2.4), a contradiction. The proof is thus concluded. □

2.2. **Functions of bounded variation and sets of finite perimeter.** Let $\Omega \subseteq \mathbb{R}^N$ be an open set. We say that $u \in BV(\Omega)$ if $u \in L^1(\Omega)$ and its derivative in the sense of distributions is a finite Radon measure on Ω, i.e., $Du \in \mathcal{M}_b(\Omega; \mathbb{R}^N)$. $BV(\Omega)$ is called the space of *functions of bounded variation* on Ω. $BV(\Omega)$ is a Banach space under the norm $\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + \|Du\|_{\mathcal{M}_b(\Omega;\mathbb{R}^N)}$. We refer the reader to [1] for an exhaustive treatment of the space BV.

Concerning the fine properties, a function $u \in BV(\Omega)$ (or better every representative of u) is a.e. approximately differentiable on Ω (see [1, Definition 3.70]), with approximate gradient $\nabla u \in L^1(\Omega; \mathbb{R}^N)$. Moreover, the jump set J_u is a $\mathcal{H}^{N-1}$-countably rectifiable set, i.e., $J_u \subseteq \cup_{i\in\mathbb{N}} M_i$

up to a $\mathcal{H}^{N-1}$-negligible set, with M_i a C^1-hypersurface in $\mathbb{R}^N$. The measure Du admits the following representation for every Borel set $B \subseteq \Omega$:

$$Du(B) = \int_B \nabla u \, dx + \int_{J_u \cap B} (u^+ - u^-)\nu_u \, d\mathcal{H}^{N-1} + D^c u(B),$$

where $\nu_u(x)$ is the normal to J_u at x, and $D^c u$ is singular with respect to the Lebesgue measure and concentrated outside J_u. $D^c u$ is usually referred to as the *Cantor part* of Du. $u^\pm$ are the upper and lower approximate limits of u at x. The normal ν_u coincides $\mathcal{H}^{N-1}$-a.e. on J_u with the normal to the hypersurfaces M_i. The direction of $\nu_u(x)$ is chosen in such a way that $u^\pm(x)$ is the approximate limit of u at x on the sets $\{y \in \mathbb{R}^N : \nu_u(x) \cdot (y - x) \gtrless 0\}$. Moreover, $u^\pm$ coincide $\mathcal{H}^{N-1}$-almost everywhere on J_u with the traces $\gamma^\pm(u)$ of u on J_u which are defined by the following Lebesgue-type limit quotient relation

$$\lim_{r \to 0} \frac{1}{r^N} \int_{B_r^\pm(x)} |u(x) - \gamma^\pm(u)(x)| \, dx = 0$$

where $B_r^\pm(x) := \{y \in B_r(x) : \nu_u(x) \cdot (y - x) \gtrless 0\}$ (see [1, Remark 3.79]).

The space of *special functions of bounded variation* on Ω is defined as

$$SBV(\Omega) := \{u \in BV(\Omega) \, : \, D^c u = 0\}.$$

Such a space proved to be very useful in the study of free discontinuity problems arising in different contexts, like for example image segmentation or fracture mechanics.

Given $E \subseteq \mathbb{R}^N$ measurable, we say that E has finite perimeter if

$$Per(E; \mathbb{R}^N) := \sup \left\{ \int_E div(\varphi) \, dx \, : \, \varphi \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N), \|\varphi\|_\infty \le 1 \right\} < +\infty.$$

If $|E| < +\infty$, then E has finite perimeter if and only if $1_E \in BV(\mathbb{R}^N)$. It turns out that

$$D1_E = \nu_E \mathcal{H}^{N-1} \lfloor \partial^* E, \qquad Per(E; \mathbb{R}^N) = \mathcal{H}^{N-1}(\partial^* E),$$

where $\partial^* E$ is called the *reduced boundary* of E, and ν_E is the associated inner approximate normal (see [1, Section 3.5]). We have that $\partial^* E$ is $\mathcal{H}^{N-1}$-countably rectifiable and it is contained in the topological boundary ∂E. Moreover, the points in $\partial^* E$ have density $1/2$ with respect to E, with

$$\mathcal{H}^{N-1}(\partial^e E \setminus \partial^* E) = 0,$$

where $\partial^e E$ (the *essential boundary*) is the set of points whose density with respect to E is neither zero nor one.

2.3. **Almost quasi-minimizer of the Mumford-Shah functional.** In Section 4, we will use the notion of almost quasi minimality for SBV functions with respect to Mumford-Shah type functionals.

Definition 2.2 (**Almost quasi-mimimality**). *Let $\Omega \subseteq \mathbb{R}^N$ be open and $1 < p < +\infty$. We say that $u \in SBV(\Omega)$ is an almost quasi-minimizer for the Mumford-Shah functional with exponent p if there exist $r_0, c_1, c_2, c_3 > 0$ such that for every $r < r_0$, $x_0 \in \Omega$ and $v \in SBV_{loc}(\Omega)$ with $\{v \neq u\} \subset \bar{B}_r(x_0) \subset\subset \Omega$ we have*

$$\int_{B_r(x_0)} |\nabla u|^p \, dx + c_1 \mathcal{H}^{N-1}(J_u \cap \bar{B}_r(x_0)) \le \int_{B_r(x_0)} |\nabla v|^p \, dx + c_2 \mathcal{H}^{N-1}(J_v \cap \bar{B}_r(x_0)) + c_3 r^N.$$

The previous notion is a variant of the minimality property employed by De Giorgi, Carriero and Leaci [14] to study regularity properties of minimizers of the Mumford-Shah functional, the main difference lying in the fact that different constants appear in front of the surface terms.

The analysis of [14] can be extended to cover also this (slightly) more general setting (see [20]), yielding the following result for which we refer to [7, Theorem 2.3].

Theorem 2.3. *Let $\Omega \subseteq \mathbb{R}^N$ be open and let $u \in SBV_{loc}(\Omega)$ satisfy the minimality property of Definition 2.2. Then the jump set of u is essentially closed in Ω, i.e.,*

$$\mathcal{H}^{N-1}\left((\overline{J_u} \setminus J_u) \cap \Omega\right) = 0.$$

2.4. **Symmetrization techniques.** In Section 4.3 we will employ some basic properties of the *radial symmetric decreasing rearrangement* for functions and of the *spherical cap symmetrization* of sets. We recall here their definitions and the basic properties we will employ: we refer the reader to e.g. [21] and [8, Section 9.2] for further details.

(a) *Radial symmetric decreasing rearrangement of a function.* Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let u be a measurable nonnegative function defined on Ω. If B is a ball centered at the origin with $|B| = |\Omega|$, we define the radial symmetric decreasing rearrangement of u as the radial function u^* defined on B such that for every $c > 0$

$$|\{u \geq c\}| = |\{u^* \geq c\}|.$$

It turns out that

$$\int_B (u^*)^p \, dx = \int_\Omega u^p \, dx$$

for every $p \in [1, +\infty[$. Moreover, if in addition $u \in W_0^{1,p}(\Omega)$, then $u^* \in W_0^{1,p}(\Omega)$ with

$$\int_B |\nabla u^*|^p \, dx \leq \int_\Omega |\nabla u|^p \, dx. \tag{2.5}$$

(b) *Spherical cap symmetrization of a set.* Let $E \subseteq \mathbb{R}^N$ be a measurable set. For every sphere $B_r(0)$, let C_r be the spherical cap centered at $(0, 0, \ldots, r)$ such that

$$\mathcal{H}^{N-1}(C_r) = \mathcal{H}^{N-1}(E \cap \partial B_r(0)).$$

The *spherical cap symmetrization* of E is given by $\tilde{E} \subseteq \mathbb{R}^N$ such that $\partial B_r(0) \cap \tilde{E} = C_r$ for every $r > 0$. It turns out that (see e.g. [8, Section 9.2], [18, Remark 4] or [19, Section 6]) if E has finite perimeter, also $\tilde{E}$ is of finite perimeter, and for every radial positive measurable function $g(r)$

$$\int_{\partial^* \tilde{E}} g(r) \, d\mathcal{H}^{N-1} \leq \int_{\partial^* E} g(r) \, d\mathcal{H}^{N-1}. \tag{2.6}$$

The previous inequality states that any perimeter with radial density decreases by spherical cap symmetrization: this property is reminiscent of the more usual one regarding the Schwartz symmetrization across an hyperplane.

Remark 2.4. Following [21, Lemma 1], if u is a Lipschitz continuous function on Ω open and bounded, it turns out that u^* is also Lipschitz continuous on B. Moreover, the inequality (2.5), based on the use of the coarea formula and of the isoperimetric inequality, still holds true provided that for almost every $c > 0$

$$\mathcal{H}^{N-1}(\{u = c\} \cap \Omega) \geq \mathcal{H}^{N-1}(\{u^* = c\}),$$

i.e., an isoperimetric control is available for the *inner* boundaries of upper levels. We will use this property in the final step of the proof of Proposition 4.13 for a smooth function defined on an annulus.

3. The space $SBV^{\frac{1}{p}}(\mathbb{R}^N)$

In this section we introduce a suitable space of functions of bounded variation type which will be fundamental for our analysis.

Given $p \in]1, +\infty[$ we set

$$SBV^{\frac{1}{p}}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) \, : \, u^p \in SBV(\mathbb{R}^N) \, , \, u \geq 0\}.$$

In the case $p = 2$, the space has been introduced in [5] to study the Faber-Krahn inequality for the first eigenvalue of the Robin-Laplacian, and it has used in [6] to address some related semilinear variants including the case of the torsional rigidity.

In the following lemma we detail some basic properties of elements in $SBV^{\frac{1}{p}}(\mathbb{R}^N)$: the proof follows closely [5, Lemma 1] and will not be given.

Lemma 3.1. *Let $u \in SBV^{\frac{1}{p}}(\mathbb{R}^N)$. Then the following items hold true.*

(a) *u is a.e. approximately differentiable (see* [1, Definition 3.70]*) with approximate gradient ∇u such that*

$$\nabla(u^p) = pu^{p-1}\nabla u \qquad \text{a.e. in } \mathbb{R}^N.$$

(b) *The jump set J_u is $\mathcal{H}^{N-1}$-countably rectifiable and a normal ν_u can be chosen in such a way that the jump part of the derivative is given by*

$$D^j(u^p) = [(u^+)^p - (u^-)^p]\,\nu_u\,\mathcal{H}^{N-1} \llcorner J_u.$$

(c) *For every $\varepsilon > 0$ and $\Omega \subset \mathbb{R}^N$ open and bounded we have $(u-\varepsilon)^+ \in SBV(\Omega)$.*

The following compactness and lower semicontinuity properties are a straightforward variant of [5, Theorem 2].

Theorem 3.2. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBV^{\frac{1}{p}}(\mathbb{R}^N)$ and let $C > 0$ be such that for every $n \in \mathbb{N}$*

$$\int_{\mathbb{R}^N} |\nabla u_n|^p\,dx + \int_{J_{u_n}} [(u_n^+)^p + (u_n^-)^p]\,d\mathcal{H}^{N-1} + \int_{\mathbb{R}^N} u_n^p\,dx \le C.$$

Then there exist $u \in SBV^{\frac{1}{p}}(\mathbb{R}^N)$ and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that the following items hold true.

(a) *Compactness: $u_{n_k} \to u$ strongly in $L^p_{loc}(\mathbb{R}^N)$.*
(b) *Lower semicontinuity: for every open set $A \subseteq \mathbb{R}^N$ we have*

$$\int_A |\nabla u|^p\,dx \le \liminf_{k\to\infty} \int_A |\nabla u_{n_k}|^p\,dx,$$

and

$$\int_{J_u\cap A} [(u^+)^p + (u^-)^p]\,d\mathcal{H}^{N-1} \le \liminf_{k\to\infty} \int_{J_{u_{n_k}}\cap A} [(u_{n_k}^+)^p + (u_{n_k}^-)^p]\,d\mathcal{H}^{N-1}.$$

In the subsequent sections, we will make use of the following inequality.

Proposition 3.3. *Given $m > 0$, there exists $\lambda_p(m) > 0$ such that for every $u \in SBV^{\frac{1}{p}}(\mathbb{R}^N)$ with $|supp(u)| \le m$*

$$\int_{\mathbb{R}^N} |\nabla u|^p\,dx + \int_{J_u} [(u^+)^p + (u^-)^p]\,d\mathcal{H}^{N-1} \ge \lambda_p(m) \int_{\mathbb{R}^N} u^p\,dx.$$

Moreover, for $m \le 1$ we have

$$\lambda_p(m) \ge \frac{1}{m^{1/N}}\lambda_p(1). \tag{3.1}$$

Proof. Let us set

$$\lambda_p(m) := \inf_{\substack{u\in SBV^{\frac{1}{p}}(\mathbb{R}^N),\, u\neq 0\\ |supp(u)|\le m}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p\,dx + \int_{J_u} [(u^+)^p + (u^-)^p]\,d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} u^p\,dx},$$

and let us check that $\lambda_p(m) > 0$.

By contradiction assume that there exists $u_n \in SBV^{\frac{1}{p}}(\mathbb{R}^N)$, $u_n \neq 0$, with $|supp(u_n)| \le m$,

$$\int_{\mathbb{R}^N} u_n^p dx = 1, \tag{3.2}$$

and such that

$$\int_{\mathbb{R}^N} |\nabla u_n|^p\,dx + \int_{J_{u_n}} [(u_n^+)^p + (u_n^-)^p]\,d\mathcal{H}^{N-1} \to 0.$$

By assumption we have that $u_n^p \in SBV(\mathbb{R}^N)$: by employing the embedding $BV(\mathbb{R}^N) \hookrightarrow L^{\frac{N}{N-1}}(\mathbb{R}^N)$, Hölder's and Young's inequalities, for every $\varepsilon > 0$ we can find $c_\varepsilon > 0$ such that

$$\begin{aligned}
C_N \left(\int_{\mathbb{R}^N} u_n^{\frac{pN}{N-1}}\, dx \right)^{\frac{N-1}{N}} &\le |Du_n^p|(\mathbb{R}^N) \le \int_{\mathbb{R}^N} p u_n^{p-1} |\nabla u_n|\, dx + \int_{J_{u_n}} [(u_n^+)^p + (u_n^-)^p]\, d\mathcal{H}^{N-1} \\
&\le p \left(\int_{\mathbb{R}^N} u_n^p\, dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N} |\nabla u_n|^p\, dx \right)^{\frac{1}{p}} + \int_{J_{u_n}} [(u_n^+)^p + (u_n^-)^p]\, d\mathcal{H}^{N-1} \\
&\le \left\{ \varepsilon \int_{\mathbb{R}^N} u_n^p\, dx + c_\varepsilon \int_{\mathbb{R}^N} |\nabla u_n|^p\, dx \right\} + \int_{J_{u_n}} [(u_n^+)^p + (u_n^-)^p]\, d\mathcal{H}^{N-1} \\
&\le \varepsilon \left(\int_{\mathbb{R}^N} u_n^{\frac{pN}{N-1}} \right)^{\frac{N-1}{N}} |supp\,(u_n)|^{\frac{1}{N}} + c_\varepsilon \int_{\mathbb{R}^N} |\nabla u_n|^p\, dx + \int_{J_{u_n}} [(u_n^+)^p + (u_n^-)^p]\, d\mathcal{H}^{N-1}.
\end{aligned}$$

Letting ε be sufficiently small we can absorb the first integral of the right-hand side in the left-hand side to get for some $C_\varepsilon > 0$

$$\left(\int_{\mathbb{R}^N} u_n^{\frac{pN}{N-1}}\, dx \right)^{\frac{N-1}{N}} \le C_\varepsilon \left(\int_{\mathbb{R}^N} |\nabla u_n|^p\, dx + \int_{J_{u_n}} [(u_n^+)^p + (u_n^-)^p]\, d\mathcal{H}^{N-1} \right) \to 0.$$

But then

$$\int_{\mathbb{R}^N} u_n^p\, dx \le \left(\int_{\mathbb{R}^N} u_n^{\frac{pN}{N-1}}\, dx \right)^{\frac{N-1}{N}} |supp(u_n)|^{\frac{1}{N}} \to 0$$

which is against (3.2). We conclude thus that $\lambda_p(m) > 0$.

Let us prove inequality (3.1). Let $u \in SBV^{\frac{1}{p}}(\mathbb{R}^N)$ with $|supp(u)| \le m$. Setting

$$v(x) := u(tx)$$

we obtain that $|supp(v)| = |supp(u)|/t^N$. If we choose $t := m^{1/N}$, v is an admissible function to compute $\lambda_p(1)$ so that

$$\lambda_p(1) \le \frac{t^p \int_{\mathbb{R}^N} |\nabla u|^p dx + t \int_{J_u} [(u^+)^p + (u^-)^p] d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} u^p dx}.$$

If $m \le 1$ we get

$$\lambda_p(1) \le m^{1/N} \frac{\int_{\mathbb{R}^N} |\nabla u|^p dx + \int_{J_u} [(u^+)^p + (u^-)^p] d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} u^p dx},$$

so that inequality (3.1) easily follows. $\square$

Remark 3.4. It turns out that $\lambda_p(m)$ is equal to the first eigenvalue of the p-Laplace operator under Robin boundary conditions with constant $\beta = 1$ on a ball B with $|B| = m$. For details, we refer the reader to Remark 5.2 at the end of the paper.

We conclude the section recalling the following result (see [7, Theorem 3.5 and Remark 3.7]), which is crucial for our analysis, which will be used in the proof of Theorem 4.5.

Proposition 3.5. *Given $u \in SBV^{\frac{1}{p}}(\mathbb{R}^N)$, assume that there exist $\varepsilon_0, c_1, c_2 > 0$ such that for a.e. $0 < \delta < \varepsilon < \varepsilon_0$ we have*

$$\int_{\{u \le \varepsilon\}} |\nabla u|^p\, dx + c_1 \delta^p \mathcal{H}^{N-1}(\partial^* \{\delta < u < \varepsilon\} \cap J_u) \le c_2 \varepsilon^p \mathcal{H}^{N-1}(\partial^* \{u > \varepsilon\} \setminus J_u).$$

Then there exists $\alpha > 0$ such that

$$u \ge \alpha \qquad \text{a.e. on } supp(u).$$

4. The free discontinuity problem

Given $p \in]1,+\infty[$ and $m > 0$, we concentrate in this section on the free discontinuity problem

$$\min_{\substack{u \in SBV^{\frac{1}{p}}(\mathbb{R}^N), u \neq 0 \\ |supp(u)| \leq m}} F(u) \tag{4.1}$$

where $SBV^{\frac{1}{p}}(\mathbb{R}^N)$ is the space introduced in Section 3, while F is the free discontinuity functional

$$F(u) := \frac{\|\nabla u\|_p + \mathcal{E}_s(u)^{1/p}}{\|u\|_p}$$

with

$$\mathcal{E}_s(u) := \int_{J_u} [(u^+)^p + (u^-)^p] \, d\mathcal{H}^{N-1}.$$

Existence of minimizers can be proved by performing a concentration compactness alternative as in [5, Theorem 4].

Theorem 4.1 (**Existence of minimizers**). *The minimum problem* (4.1) *admits a solution.*

Proof. It is sufficient to adapt the proof of [5, Theorem 4] using the compactness and lower semicontinuity properties given in Theorem 3.2 and employing the numerical inequality of Lemma 2.1 (to exclude the dichotomy case). □

Remark 4.2. Notice that if u is a minimizer of problem (4.1), then $|supp(u)| = m$. This is a consequence of the following simple rescaling property: for every $t \geq 1$

$$\min_{\substack{v \in SBV^{\frac{1}{p}}(\mathbb{R}^N), v \neq 0 \\ |supp(v)| \leq tm}} F(v) \leq t^{-\frac{1}{Np}} \min_{\substack{u \in SBV^{\frac{1}{p}}(\mathbb{R}^N), u \neq 0 \\ |supp(u)| \leq m}} F(u).$$

4.1. **First properties of minimizers.** This subsection is devoted to the proof of some pivotal properties of minimizers. In particular we are interested in bounds from above and below (on the support).

Let us start with the bound from above.

Theorem 4.3 (L^∞**-bound**). *Let u be a minimizer of* (4.1). *Then $u \in L^\infty(\mathbb{R}^N)$.*

Proof. It is not restrictive to assume

$$\|u\|_{L^p(\mathbb{R}^N)} = 1.$$

Let us assume by contradiction that $u \notin L^\infty(\mathbb{R}^N)$. We divide the proof in several steps.

Step 1. Assume that $\nabla u \neq 0$ and $J_u \neq \emptyset$. By exploiting the Euler-Lagrange equation satisfied by u, for every $\varphi \in SBV^{\frac{1}{p}}(\mathbb{R}^N)$ such that $J_\varphi \subseteq J_u$ we have

$$\frac{\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx}{\left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx\right)^{1-\frac{1}{p}}} + \frac{\int_{J_u} [(u^+)^{p-1}\gamma_1(\varphi) + (u^-)^{p-1}\gamma_2(\varphi)] \, d\mathcal{H}^{N-1}}{\left(\int_{J_u} [(u^+)^p + (u^-)^p] \, d\mathcal{H}^{N-1}\right)^{1-\frac{1}{p}}} = F(u) \int_{\mathbb{R}^N} u^{p-1} \varphi \, dx, \tag{4.2}$$

where $\gamma_1(\varphi)$ and $\gamma_2(\varphi)$ are the traces of φ on the rectifiable set J_u oriented by the normal ν_u.

By multiplying equation (4.2) with

$$\left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx\right)^{1-\frac{1}{p}} + \left(\int_{J_u} [(u^+)^p + (u^-)^p] \, d\mathcal{H}^{N-1}\right)^{1-\frac{1}{p}},$$

we obtain the inequality

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx + \int_{J_u} [(u^+)^{p-1}\gamma_1(\varphi) + (u^-)^{p-1}\gamma_2(\varphi)] \, d\mathcal{H}^{N-1} \leq 2F(u)^p \int_{\mathbb{R}^N} u^{p-1} \varphi \, dx. \tag{4.3}$$

Notice that inequality (4.3) still holds true if either $J_u = \emptyset$, $\nabla u \neq 0$ or $J_u \neq \emptyset$, $\nabla u \equiv 0$, in the last case requiring that also $\nabla \varphi \equiv 0$.

Step 2. Let us consider for every $M > 0$ the function $u_M = (u - M)_+ \in SBV^{\frac{1}{p}}(\mathbb{R}^N)$. Recalling that

$$\nabla u_M = \nabla u 1_{\{u \geq M\}} \qquad \text{and} \qquad J_{u_M} \subseteq J_u,$$

u_M turns out to be an admissible test function for inequality (4.3), so that we infer

$$\int_{\mathbb{R}^N} |\nabla u_M|^p \, dx + \int_{J_u} [(u^+)^{p-1} u_M^+ + (u^-)^{p-1} u_M^-] \, d\mathcal{H}^{N-1} \leq 2F(u)^p \int_{\mathbb{R}^N} u^{p-1} u_M \, dx. \tag{4.4}$$

Notice that

$$\int_{J_u} [(u^+)^{p-1} u_M^+ + (u^-)^{p-1} u_M^-] \, d\mathcal{H}^{N-1} \geq \int_{J_{u_M}} [(u_M^+)^p + (u_M^-)^p] d\mathcal{H}^{N-1}$$

and

$$\int_{\mathbb{R}^N} u^{p-1} u_M \, dx = \int_{\mathbb{R}^N \cap \{u \geq M\}} (u_M + M)^{p-1} u_M \, dx \leq C_1 \int_{\mathbb{R}^N} (u_M^p + M^{p-1} u_M) \, dx$$

for some constant $C_1 > 0$. From (4.4) we conclude

$$\int_{\mathbb{R}^N} |\nabla u_M|^p \, dx + \int_{J_{u_M}} [(u_M^+)^p + (u_M^-)^p] d\mathcal{H}^{N-1} \leq C_2 \int_{\mathbb{R}^N} (u_M^p + M^{p-1} u_M) dx \tag{4.5}$$

for some $C_2 > 0$.

Step 3. Let

$$\alpha(M) := |\{u \geq M\}|.$$

Since we are assuming $u \notin L^\infty(\mathbb{R}^N)$, we have

$$\forall M > 0 \, : \, \alpha(M) > 0 \qquad \text{and} \qquad \lim_{M \to +\infty} \alpha(M) = 0.$$

In view of Proposition 3.3, inequality (4.5) entails

$$\lambda_p(\alpha(M)) \int_{\mathbb{R}^N} u_M^p \, dx \leq C_2 \int_{\mathbb{R}^N} \left(u_M^p + M^{p-1} u_M \right) \, dx.$$

By (3.1), for M sufficiently large we infer

$$\frac{1}{\alpha(M)^{\frac{1}{N}}} \int_{\mathbb{R}^N} u_M^p \, dx \leq C_3 \int_{\mathbb{R}^N} M^{p-1} u_M \, dx \tag{4.6}$$

for some $C_3 > 0$.

By Holder inequality and (4.6) we deduce

$$\int_{\mathbb{R}^N} u_M \, dx \leq \left(\int_{\mathbb{R}^N} u_M^p \, dx \right)^{\frac{1}{p}} \alpha(M)^{\frac{p-1}{p}} \leq \left(C_3 M^{p-1} \alpha(M)^{\frac{1}{N}} \int_{\mathbb{R}^N} u_M \, dx \right)^{\frac{1}{p}} \alpha(M)^{\frac{p-1}{p}}$$

so that

$$\int_{\mathbb{R}^N} u_M \, dx \leq C_4 M \alpha(M)^{1 + \frac{1}{N(p-1)}}. \tag{4.7}$$

for some $C_4 > 0$.

Setting now

$$g(M) = \int_{\mathbb{R}^N} u_M dx,$$

and recalling that $g'(M) = -\alpha(M)$, we can rewrite inequality (4.7) as

$$g(M) \leq C_4 M (-g'(M))^{1 + \frac{1}{N(p-1)}}.$$

It follows that there exists $M_0 > 0$ such that for a.e. $M \geq M_0$

$$\frac{1}{M^\gamma} \leq -C_5 \frac{g'(M)}{g(M)^\gamma}$$

with $\gamma < 1$ and $C_5 > 0$. Integrating between M_0 and M we get

$$g(M)^{1-\gamma} - g(M_0)^{1-\gamma} \le -\frac{1}{C_5}\left(M^{1-\gamma} - M_0^{1-\gamma}\right).$$

Being $g(M)^{1-\gamma} > 0$, the above inequality entails in particular

$$-g(M_0)^{1-\gamma} \le -\frac{1}{C_5}\left(M^{1-\gamma} - M_0^{1-\gamma}\right),$$

which yields a contradiction letting $M \to +\infty$.

We thus conclude that $u \in L^\infty(\mathbb{R}^N)$, and the proof is concluded. □

Let us now come to the bound from below on the support of minimizers. We need the following perturbation lemma.

Lemma 4.4. *Let u be a minimizer of problem* (4.1). *Then the following items hold true.*

(a) *There exist $\varepsilon > 0$ and $\tilde{k} > 0$ such that for every $v \in SBV^{\frac{1}{p}}(\mathbb{R}^N)$ with*

$$|supp(u)| < |supp(v)| < |supp(u)| + \varepsilon, \tag{4.8}$$

then

$$F(u) + \tilde{k}|supp(u)| \le F(v) + \tilde{k}|supp(v)|. \tag{4.9}$$

(b) *There exist $\varepsilon > 0$ and $\hat{k} > 0$ such that for every $v \in SBV^{\frac{1}{p}}(\mathbb{R}^N)$ with*

$$|supp(u)| - \varepsilon < |supp(v)| < |supp(u)|, \tag{4.10}$$

then

$$F(u) + \hat{k}|supp(u)| \le F(v) + \hat{k}|supp(v)|. \tag{4.11}$$

Proof. The proof is very similar to that of [6, Lemma 6.12].

Let us start with point (a). By contradiction, let us assume that for every $\varepsilon > 0$ and $\tilde{k} > 0$ there exists $v \in SBV^{\frac{1}{p}}(\mathbb{R}^N)$ satisfying (4.8) but for which (4.9) is violated. Let us consider $\varepsilon_n \to 0$ and $\tilde{k}_n \to +\infty$ and let us denote by v_n the associate function such that

$$|supp(u)| < |supp(v_n)| < |supp(u)| + \varepsilon_n,$$

and

$$F(u) + \tilde{k}_n|supp(u)| > F(v_n) + \tilde{k}_n|supp(v_n)|. \tag{4.12}$$

Let us set

$$t_n := \left(\frac{|supp(v_n)|}{|supp(u)|}\right)^{\frac{1}{N}}.$$

Then $t_n > 1$ and $t_n \to 1$. If we set

$$w_n(x) := v_n(t_n x)$$

we obtain

$$|supp(w_n)| = \frac{|supp(v_n)|}{t_n^N} = |supp(u)|,$$

which implies (being w_n admissible for problem (4.1))

$$F(u) \le F(w_n) = \frac{t_n\|\nabla v_n\|_p + t_n^{1/p}\mathcal{E}_s(v_n)^{1/p}}{\|v_n\|_p} \le t_n F(v_n). \tag{4.13}$$

Since $|supp(v_n)| = t_n^N|supp(u)|$, by using (4.12) and (4.13), we get

$$F(u) + \tilde{k}_n|supp(u)| > F(v_n) + \tilde{k}_n|supp(v_n)| \ge t_n^{-1}F(u) + \tilde{k}_n t_n^N|supp(u)|,$$

so that

$$\tilde{k}_n \le \frac{1 - t_n^{-1}}{t_n^N - 1}\frac{F(u)}{|supp(u)|}.$$

But the right hand side is bounded as $n \to +\infty$, against $\tilde{k}_n \to +\infty$. The proof of point (a) is thus concluded.

Let us pass to point (b). We proceed again by contradiction by considering $\varepsilon_n \to 0$, $\hat{k}_n \to 0$ and the associated $v_n \in SBV^{\frac{1}{p}}(\mathbb{R}^N)$ satisfying (4.10) but violating (4.11). Reasoning as above we get

$$F(u) \le t_n^{1/p} F(v_n)$$

for $t_n \nearrow 1$, so that

$$\hat{k}_n \ge \frac{t_n^{-1/p} - 1}{1 - t_n^N} \frac{F(u)}{|supp(u)|}, \tag{4.14}$$

against $\hat{k}_n \to 0$. Point (b) thus follows, and the proof is now complete. □

We are now in a position to prove the following bound from below which is essential for our analysis.

Theorem 4.5 (Bound from below). *Let u be a minimizer for* (4.1)*. Then there exists $\alpha > 0$ such that*

$$u \ge \alpha \qquad \text{a.e. on } supp(u).$$

Proof. We can assume $\|u\|_{L^p(\mathbb{R}^N)} = 1$. Moreover, thanks to Proposition 4.3, we have $u \in L^\infty(\mathbb{R}^N)$. Assume by contradiction that for every ε small enough

$$|\{u < \varepsilon\}| > 0. \tag{4.15}$$

Notice that for a.e. $\varepsilon > 0$ we have

$$u 1_{\{u \ge \varepsilon\}} \in SBV^{\frac{1}{p}}(\mathbb{R}^N).$$

In view of Lemma 4.4, there exists $\hat{k} > 0$ such that comparing u and $u1_{\{u\ge\varepsilon\}}$ (with ε small enough) we get

$$\|\nabla u\|_p + \mathcal{E}_s(u)^{\frac{1}{p}} + \hat{k}|\{u < \varepsilon\}| \le \frac{\|\nabla u 1_{\{u\ge\varepsilon\}}\|_p + \mathcal{E}_s(u1_{\{u\ge\varepsilon\}})^{\frac{1}{p}}}{\|u1_{\{u\ge\varepsilon\}}\|_p}. \tag{4.16}$$

Moreover we may write

$$\frac{1}{\|u1_{\{u\ge\varepsilon\}}\|_p} = \left(\frac{1}{\int_{\{u\ge\varepsilon\}} u^p\,dx} \right)^{\frac{1}{p}} = \left(\frac{1}{1 - \int_{\{u<\varepsilon\}} u^p\,dx} \right)^{\frac{1}{p}} < 1 + C\varepsilon^p |\{u < \varepsilon\}|$$

for some $C > 0$. As a consequence, for ε small enough inequality (4.16) entails

$$\|\nabla u\|_p + \mathcal{E}_s(u)^{\frac{1}{p}} \le \|\nabla u 1_{\{u\ge\varepsilon\}}\|_p + \mathcal{E}_s(u1_{\{u\ge\varepsilon\}})^{\frac{1}{p}}. \tag{4.17}$$

Assume that $\nabla u \neq 0$. Recalling that for every $a > b > 0$

$$\frac{1}{p} a^{\frac{1}{p}-1}(a - b) \le a^{1/p} - b^{1/p} \le \frac{1}{p} b^{\frac{1}{p}-1}(a-b), \tag{4.18}$$

we get

$$\|\nabla u\|_p - \|\nabla u 1_{\{u\ge\varepsilon\}}\|_p \ge \frac{\int_{\mathbb{R}^N} |\nabla u|^p\,dx - \int_{\{u\ge\varepsilon\}} |\nabla u|^p\,dx}{p\|\nabla u\|_p^{p-1}} = \frac{\int_{\{u<\varepsilon\}} |\nabla u|^p\,dx}{p\|\nabla u\|_p^{p-1}},$$

so that from (4.17) we deduce

$$\frac{\int_{\{u<\varepsilon\}} |\nabla u|^p\,dx}{p\|\nabla u\|_p^{p-1}} + \mathcal{E}_s(u)^{\frac{1}{p}} \le \mathcal{E}_s(u1_{\{u\ge\varepsilon\}})^{\frac{1}{p}}.$$

Notice that in particular $\mathcal{E}_s(u1_{\{u\ge\varepsilon\}}) \ge \mathcal{E}_s(u)$, so that using again (4.18) we obtain

$$\frac{\int_{\{u<\varepsilon\}} |\nabla u|^p\,dx}{\|\nabla u\|_p^{p-1}} \le \frac{\mathcal{E}_s(u1_{\{u\ge\varepsilon\}}) - \mathcal{E}_s(u)}{\mathcal{E}_s(u)^{\frac{p-1}{p}}}.$$

By setting

$$\beta_u := \frac{\|\nabla u\|_p^{p-1}}{\mathcal{E}_s(u)^{\frac{p-1}{p}}}$$

we obtain, writing explicitly $\mathcal{E}_s(u)$ and $\mathcal{E}_s(u1_{\{u\geq\varepsilon\}})$,

$$\begin{aligned}(4.19)\qquad &\int_{\{u<\varepsilon\}} |\nabla u|^p\,dx + \beta_u \int_{J_u} [(u^+)^p + (u^-)^p]\,d\mathcal{H}^{N-1} \\ &\leq \beta_u \int_{J_u\cap\{u^-<\varepsilon\leq u^+\}} (u^+)^p\,d\mathcal{H}^{N-1} + \beta_u \int_{J_u\cap\{\varepsilon\leq u^-<u^+\}} [(u^+)^p + (u^-)^p]\,d\mathcal{H}^{N-1} \\ &\qquad + \beta_u\varepsilon^p\mathcal{H}^{N-1}(\partial^*\{u>\varepsilon\}\setminus J_u),\end{aligned}$$

which yields

$$\int_{\{u<\varepsilon\}} |\nabla u|^p\,dx + \beta_u \int_{J_u\cap\{u^-<u^+\leq\varepsilon\}} [(u^-)^p + (u^+)^p]\,d\mathcal{H}^{N-1} \leq \beta_u\varepsilon^p\mathcal{H}^{N-1}(\partial^*\{u>\varepsilon\}\setminus J_u).$$

We deduce in particular that for a.e. $0<\delta<\varepsilon$

$$\int_{\{u\leq\varepsilon\}} |\nabla u|^p\,dx + \beta_u\delta^p\mathcal{H}^{N-1}(\partial^*\{\delta<u<\varepsilon\}\cap J_u) \leq \beta_u\varepsilon^p\mathcal{H}^{N-1}(\partial^*\{u>\varepsilon\}\setminus J_u). \tag{4.20}$$

From Proposition 3.5 we obtain that (4.15) cannot hold.

In the case $\nabla u \equiv 0$, inequality (4.20) still holds provided that we choose $\beta_u := 1$: again we reach a contradiction, so that the proof is concluded. □

Remark 4.6. The bound from below can be established also by adapting the arguments proposed in [9, Theorem 3.2], where a free discontinuity approach similar to the present one has been proposed to deal with free boundary problems arising in thermal insulation.

4.2. **Regularity properties of the minimizers.** In this subsection we show that a minimizer of problem (4.1) satisfies a local minimality property for a Mumford-Shah functional with exponent p: this yields some regularity properties for the jump set and consequently for the support.

Theorem 4.7 (**Essential closedness of the jump set**). *Let u be a minimizer of problem* (4.1). *Then $u\in SBV(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, $\mathcal{H}^{N-1}(J_u)<+\infty$ and J_u is essentially closed, i.e.,*

$$\mathcal{H}^{N-1}(\overline{J_u}\setminus J_u)=0.$$

Proof. By Proposition 4.3 we know that $u\in L^\infty(\mathbb{R}^N)$, while Theorem 4.5 entails

$$u\geq\alpha>0 \quad \text{a.e. on } supp(u). \tag{4.21}$$

Since $u^p\in SBV(\mathbb{R}^N)$, the chain rule in BV (see [1, Theorem 3.96] entails $u\in SBV(\mathbb{R}^N)$. Finally we have

$$\alpha^p\mathcal{H}^{N-1}(J_u)\leq\int_{J_u}[(u^+)^p+(u^-)^p)\,d\mathcal{H}^{N-1}<+\infty,$$

so that $\mathcal{H}^{N-1}(J_u)<+\infty$. Notice moreover that $\mathcal{H}^{N-1}(J_u)>0$ (since otherwise $u\in W^{1,1}(\mathbb{R}^N)$ and (4.21) cannot hold).

In order to conclude the proof, we need to show that J_u is essentially closed. We will show that u is an *almost quasi-minimizer* of the Mumford-Shah functional with exponent p according to Definition 2.2, and then we conclude using Theorem 2.3. We divide the proof in two steps.

Step 1. Given $r_0>0$, let us consider $v\in SBV_{loc}(\mathbb{R}^N)$ such that $\{v\neq u\}\subset\bar{B}_r(x_0)$, where $r<r_0$. Let us assume that

$$\int_{B_r(x_0)}|\nabla v|^p\,dx+2||u||_\infty^p\mathcal{H}^{N-1}(J_v\cap\bar{B}_r(x_0))\leq\int_{B_r(x_0)}|\nabla u|^p\,dx+\mathcal{H}^{N-1}(J_u\cap\bar{B}_r(x_0)), \tag{4.22}$$

and let us set

$$\tilde{v}:=\min\{|v|,||u||_\infty\}.$$

Since

$$|supp(\tilde{v})|\leq|supp(u)|+\omega_N r^N,$$

either by the minimality of u or by the perturbed minimality given by Lemma 4.4, we deduce that if r_0 is small enough, there exists $k > 0$ such that

$$\frac{\|\nabla u\|_p + \mathcal{E}_s(u)^{\frac{1}{p}}}{\|u\|_p} \le \frac{\|\nabla \tilde{v}\|_p + \mathcal{E}_s(\tilde{v})^{\frac{1}{p}}}{\|\tilde{v}\|_p} + k\omega_N r^N. \tag{4.23}$$

In view of (4.22) and the definition of $\tilde{v}$ we have

$$\|\nabla \tilde{v}\|_p + \mathcal{E}_s(\tilde{v})^{\frac{1}{p}} \le C_1, \tag{4.24}$$

where C_1 depends only on u.

Assuming as usual $\|u\|_p = 1$, and since

$$\left| \int_{B_r(x_0)} (|\tilde{v}|^p - |u|^p)\, dx \right| \le 2\|u\|_\infty^p \omega_N r^N,$$

up to reducing r_0 we have

$$\frac{1}{\|\tilde{v}\|_p} = \frac{1}{\left(1 + \int_{B_r(x_0)} (|\tilde{v}|^p - |u|^p)\, dx\right)^{\frac{1}{p}}} < 1 + \frac{4\|u\|_\infty^p \omega_N r^N}{p}.$$

Then, from (4.23), we infer

$$\|\nabla u\|_p + \mathcal{E}_s(u)^{\frac{1}{p}} \le \|\nabla \tilde{v}\|_p + \mathcal{E}_s(\tilde{v})^{\frac{1}{p}} + C_2 r^N, \tag{4.25}$$

where C_2 depends only on u.

We claim that there exist $k_1, k_2, k_3 > 0$ depending only on u such that

$$\begin{aligned} \int_{B_r(x_0)} |\nabla u|^p dx &+ k_1 \int_{J_u \cap \overline{B}_r(x_0)} [(u^+)^p + (u^-)^p]\, d\mathcal{H}^{N-1} \\ &\le \int_{B_r(x_0)} |\nabla \tilde{v}|^p dx + k_2 \int_{J_v \cap \overline{B}_r(x_0)} [(\tilde{v}^+)^p + (\tilde{v}^-)^p]\, d\mathcal{H}^{N-1} + k_3 r^N. \end{aligned} \tag{4.26}$$

Thanks to (4.21) and in view of the very definition of $\tilde{v}$ we obtain

$$\begin{aligned} \int_{B_r(x_0)} |\nabla u|^p dx &+ k_1 \alpha^p \mathcal{H}^{N-1}(J_u \cap \overline{B}_r(x_0)) \\ &\le \int_{B_r(x_0)} |\nabla v|^p dx + 2k_2 \|u\|_\infty^p \mathcal{H}^{N-1}(J_v \cap \overline{B}_r(x_0)) + k_3 r^N. \end{aligned} \tag{4.27}$$

Up to reducing the constant k_1 and increasing the constants k_2, k_3 (if necessary), we see that inequality (4.27) still holds even if v does not satisfy assumption (4.22). We conclude that u is an *almost quasi-minimizer* of the Mumford-Shah functional with exponent p according to Definition 2.2. In view of Theorem 2.3, the essential closedness of J_u follows.

Step 2. In order to conclude the proof, we need to show that claim (4.26) holds true. Up to reducing r_0, we can assume that for every $x_0 \in \mathbb{R}^N$

$$\begin{aligned} &\int_{J_u \cap B^c_{r_0}(x_0)} [(u^+)^p + (u^-)^p]\, d\mathcal{H}^{N-1} \ge \frac{1}{2} \mathcal{E}_s(u) > 0, \\ &\int_{B^c_{r_0}(x_0)} |\nabla u|^p dx \ge \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^p dx. \end{aligned} \tag{4.28}$$

Assume $\nabla u \neq 0$. We may write

$$\begin{aligned} \|\nabla u\|_p - \|\nabla \tilde{v}\|_p &= c_1(u, \tilde{v}) \left[\int_{\mathbb{R}^N} |\nabla u|^p dx - \int_{\mathbb{R}^N} |\nabla \tilde{v}|^p dx \right] \\ &= c_1(u, \tilde{v}) \left[\int_{B_r(x_0)} |\nabla u|^p dx - \int_{B_r(x_0)} |\nabla \tilde{v}|^p dx \right], \end{aligned} \tag{4.29}$$

where for a suitable $0 < \theta_1 < 1$

$$c_1(u, \tilde{v}) := \frac{1}{p} \left(\theta_1 \int_{\mathbb{R}^N} |\nabla u|^p dx + (1 - \theta_1) \int_{\mathbb{R}^N} |\nabla \tilde{v}|^p dx \right)^{\frac{1-p}{p}}.$$

Analogously

$$\mathcal{E}_s(\tilde{v})^{\frac{1}{p}} - \mathcal{E}_s(u)^{\frac{1}{p}} = c_2(u, \tilde{v}) \left(\mathcal{E}_s(\tilde{v}) - \mathcal{E}_s(u) \right), \tag{4.30}$$

with

$$c_2(u, \tilde{v}) := \frac{1}{p} \left(\theta_2 \mathcal{E}_s(u) + (1 - \theta_2) \mathcal{E}_s(\tilde{v}) \right)^{\frac{1-p}{p}}$$

and $0 < \theta_2 < 1$.

Substituting (4.29) and (4.30) into (4.25) we get

$$\int_{B_r(x_0)} |\nabla u|^p dx + \frac{c_2(u, \tilde{v})}{c_1(u, \tilde{v})} \mathcal{E}_s(u) \leq \int_{B_r(x_0)} |\nabla \tilde{v}|^p dx + \frac{c_2(u, \tilde{v})}{c_1(u, \tilde{v})} \mathcal{E}_s(\tilde{v}) + \frac{c r^N}{c_1(u, \tilde{v})}. \tag{4.31}$$

In view of (4.28), (4.22) and (4.24), and recalling the very definition of $\tilde{v}$, we may write

$$\vartheta_1 \int_{\mathbb{R}^N} |\nabla u|^p \, dx + (1 - \vartheta_1) \int_{\mathbb{R}^N} |\nabla \tilde{v}|^p \, dx \geq \int_{B_r^c(x_0)} |\nabla u|^p \, dx \geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^p \, dx > 0$$

and

$$\begin{aligned} \vartheta_1 \int_{\mathbb{R}^N} |\nabla u|^p \, dx + (1 - \vartheta_1) \int_{\mathbb{R}^N} |\nabla \tilde{v}|^p \, dx &\leq \vartheta_1 \int_{\mathbb{R}^N} |\nabla u|^p \, dx + (1 - \vartheta_1) \int_{\mathbb{R}^N} |\nabla v|^p \, dx \\ &= \int_{\mathbb{R}^N} |\nabla u|^p \, dx + \int_{B_r(x_0)} |\nabla v|^p \, dx \leq C_3, \end{aligned}$$

where C_3 depends only on u.

Similarly we have

$$\theta_2 \mathcal{E}_s(u) + (1 - \theta_2) \mathcal{E}_s(\tilde{v}) \geq \int_{J_u \cap B_r^c(x_0)} [(u^+)^p + (u^-)^p] \, d\mathcal{H}^{N-1} \geq \frac{1}{2} \int_{J_u} [(u^+)^p + (u^-)^p] \, d\mathcal{H}^{N-1} > 0,$$

and

$$\begin{aligned} \theta_2 \mathcal{E}_s(u) + (1 - \theta_2) \mathcal{E}_s(\tilde{v}) &\leq \int_{J_u} [(u^+)^p + (u^-)^p] \, d\mathcal{H}^{N-1} + \int_{J_{\tilde{v}} \cap B_r(x_0)} [(\tilde{v}^+)^p + (\tilde{v}^-)^p] \, d\mathcal{H}^{N-1} \\ &\leq 2\|u\|_\infty^p (\mathcal{H}^{N-1}(J_u) + \mathcal{H}^{N-1}(J_v \cap \overline{B}_r(x_0))) \leq C_4, \end{aligned}$$

where C_4 depends only on u.

Combining the previous inequalities, we conclude that

$$k_1 \leq \frac{c_2(u, \tilde{v})}{c_1(u, \tilde{v})} \leq k_2 \qquad \text{and} \qquad \frac{c}{c_1(u, \tilde{v})} \leq k_3$$

for suitable constants k_i depending only on u, so that inequality (4.31) entails claim (4.26).

Assume now $\nabla u \equiv 0$. Coming back to (4.25), we can raise the inequality to the power p getting for some constants $\hat{k}_1, \hat{k}_2$ depending only on p

$$\mathcal{E}_s(u) \leq \hat{k}_1 \int_{B_r(x_0)} |\nabla \tilde{v}|^p \, dx + \hat{k}_1 \mathcal{E}_s(\tilde{v}) + \hat{k}_2 r^{Np},$$

from which we deduce again claim (4.26) (up to reducing r_0 if necessary). $\square$

We are now in a position to draw the main regularity properties we need for minimizers of our free discontinuity problem.

Theorem 4.8 (**Regularity properties of minimizers).** *Let u be a minimizer of problem* (4.1). *The following items hold true.*

(a) $u \in SBV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *with* $\mathcal{H}^{N-1}(J_u) < +\infty$ *and*

$$\mathcal{H}^{N-1}(\overline{J_u} \setminus J_u) = 0.$$

(b) *The support of u is an open and connected set Ω such that*

$$\partial\Omega = \overline{J_u}.$$

In particular Ω has finite perimeter in $\mathbb{R}^N$.

(c) *The restriction of u to Ω is an element of $W^{1,p}(\Omega)$ such that, either it is constant or*

$$-\Delta_p u = \lambda u^{p-1} \qquad \text{in } \Omega, \tag{4.32}$$

for a suitable $\lambda > 0$. In particular $u \in C^{1,\gamma}(\Omega)$ for some $0 < \gamma < 1$, and there exists $\alpha > 0$ such that

$$u \geq \alpha \qquad \text{on } \Omega. \tag{4.33}$$

Proof. Point (a) follows by Theorem 4.7. Let us come to point (b).

Notice that $u \in W^{1,p}(\mathbb{R}^N \setminus \overline{J_u})$. Assume $\nabla u \neq 0$. By exploiting the Euler-Lagrange equation for the functional F on $\mathbb{R}^N \setminus \overline{J_u}$, we get for every $\varphi \in C_c^\infty(\mathbb{R}^N \setminus \overline{J_u})$

$$\int_{\mathbb{R}^N \setminus \overline{J_u}} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx = \lambda \int_{\mathbb{R}^N \setminus \overline{J_u}} u^{p-1} \varphi \, dx,$$

where

$$\lambda := \frac{\|\nabla u\|_p^{p-1} F(u)}{\|u\|_p^{p-1}}.$$

By regularity (see e.g. [15]) we infer

$$u \in C^{1,\gamma}(\mathbb{R}^N \setminus \overline{J_u})$$

for some $0 < \gamma < 1$.

Let us decompose the open set $\mathbb{R}^N \setminus \overline{J}_u$ into its connected components, and select those on which u is not identically zero. If we denote by Ω their union, it turns out $\partial\Omega = \overline{J_u}$ so that

$$\mathcal{H}^{N-1}(\partial\Omega) = \mathcal{H}^{N-1}(\overline{J}_u) = \mathcal{H}^{N-1}(J_u) < +\infty.$$

In particular Ω has finite perimeter and equation (4.32) holds true. Moreover, in view of the regularity of u and of the bound from below given by Theorem 4.5, we get easily that Ω is the support of u, and that (4.33) is satisfied (indeed Ω cannot contain strictly the support of u, otherwise the function should approach continuously zero).

Let us show that Ω is connected. By contradiction, let us assume that

$$\Omega = \Omega_1 \cup \Omega_2$$

with Ω_1, Ω_2 open sets such that $\Omega_1 \neq \emptyset$, $\Omega_2 \neq \emptyset$ and $\Omega_1 \cap \Omega_2 = \emptyset$. Note that Ω_i has finite perimeter for $i = 1, 2$ since $\partial\Omega_i \subseteq \partial\Omega$. Let us set

$$u_i := u 1_{\Omega_i} \quad i = 1, 2.$$

Since $u \in L^\infty(\mathbb{R}^N)$, by [1, Theorem 3.84] we get $u_i \in SBV(R^N)$ with

$$Du_i = Du \lfloor \Omega_i^{(1)} - u_{\partial^*\Omega_i} \nu_{\Omega_i} \mathcal{H}^{N-1} \lfloor \partial^*\Omega_i,$$

where $\Omega_i^{(1)}$ denotes the point of density 1 of Ω_i, ν_{Ω_i} stands for the exterior normal to Ω_i, while $u_{\partial^*\Omega_i}$ denotes the trace on $\partial^*\Omega_i$ of u coming from Ω_i. Notice that $supp(u_i) = \Omega_i$. Moreover by construction the following additivity relation concerning the surface energy holds true:

$$\int_{J_u} [(u^+)^p + (u^-)^p] \, d\mathcal{H}^{N-1} = \int_{J_{u_1}} [(u_1^+)^p + (u_1^-)^p] \, d\mathcal{H}^{N-1} + \int_{J_{u_2}} [(u_2^+)^p + (u_2^-)^p] \, d\mathcal{H}^{N-1}. \tag{4.34}$$

By employing the numerical inequality of Lemma 2.1, we may assume

$$\frac{\left(\int_{\mathbb{R}^N}|\nabla u|^p\,dx\right)^{1/p}+\left(\int_{J_u}[(u^+)^p+(u^-)^p]\,d\mathcal{H}^{N-1}\right)^{1/p}}{\left(\int_{\mathbb{R}^N}u^p\,dx\right)^{\frac{1}{p}}}$$
$$=\frac{\left(\int_{\mathbb{R}^N}|\nabla u_1|^p\,dx+\int_{\mathbb{R}^N}|\nabla u_2|^p\,dx\right)^{1/p}+\left(\int_{J_{u_1}}[(u_1^+)^p+(u_1^-)^p]\,d\mathcal{H}^{N-1}+\int_{J_{u_2}}[(u_2^+)^p+(u_2^-)^p]\,d\mathcal{H}^{N-1}\right)^{1/p}}{\left(\int_{\mathbb{R}^N}u_1^p\,dx+\int_{\mathbb{R}^N}u_2^p\,dx\right)^{\frac{1}{p}}}$$
$$\geq\frac{\left(\int_{\mathbb{R}^N}|\nabla u_1|^p\,dx\right)^{1/p}+\left(\int_{J_{u_1}}[(u_1^+)^p+(u_1^-)^p]\,d\mathcal{H}^{N-1}\right)^{1/p}}{\left(\int_{\mathbb{R}^N}u_1^p\,dx\right)^{\frac{1}{p}}},$$

so that u_1 is a minimizer of the free discontinuity problem (4.1) with $|supp(u_1)|<|supp(u)|$. This is in contradiction with Remark 4.2.

If $\nabla u\equiv 0$, we can follow the previous arguments and conclude that u is constant on Ω, so that the proof is concluded. □

Remark 4.9. The proof of the connectedness of Ω depends heavily on the additivity relation (4.34): the proof for $p=2$, carried out in detail in [6, Theorem 6.15], extends readily to this case.

4.3. **Minimizers are supported on balls.** In this subsection we want to show that minimizers of (4.1) are supported on balls, and more precisely that they are of the form

$$\psi 1_B,$$

where B is a ball of volume m, and ψ is a radial function with respect to the center of B.

Let us start with the following result.

Proposition 4.10. *Let v be a minimizer of problem (4.1). We can associate to v, by means of successive reflections across N orthogonal hyperplanes, a new minimizer u such that the following items hold true.*

(a) Properties of the support. *The support Ω of u is open, connected, and such that*

$$\mathcal{H}^{N-1}(\partial\Omega\setminus\partial^*\Omega)=0. \tag{4.35}$$

Up to a translation, we may assume that Ω is symmetric with respect to the origin.

(b) Radiality of the function. *There exists a function $\psi: I\to]0,+\infty[$ of class C^1, where $I=[0,+\infty[$ or $I=]0,+\infty[$, such that either ψ is constant or*

$$-\left(r^{N-1}|\psi'|^{p-2}\psi'\right)'=\lambda|\psi|^{p-2}\psi r^{N-1}, \tag{4.36}$$

and, up to a translation, for every $x\in\Omega$

$$u(x)=\psi(|x|). \tag{4.37}$$

Here $\lambda>0$ is the constant appearing in equation (4.32) satisfied by v on its support.

(c) *We have*

$$F(u)=\frac{\left(\int_\Omega|\psi'(|x|)|^p\,dx\right)^{1/p}+\left(\int_{\partial^*\Omega}\psi^p(|x|)\,dx\right)^{1/p}}{\left(\int_\Omega\psi^p(|x|)\,dx\right)^{1/p}}. \tag{4.38}$$

Proof. We divide the proof in several steps.

Step 1: existence of a symmetric minimizer. Let us consider an hyperplane π_1 parallel to $x_1=0$ which splits the support of v in two parts of equal measure, and let us set

$$v_1:=v1_{\pi_1^+}\qquad\text{and}\qquad v_2:=v1_{\pi_1^-},$$

where $\pi_1^\pm$ are the two half spaces determined by π_1.

Note that the term

$$\int_{J_v\cap\pi_1}[(v^+)^p+(v^-)^p]\,d\mathcal{H}^{N-1}$$

which (eventually) appears in the surface part of the free discontinuity functional F can be reinterpreted as

$$\int_{J_v\cap\pi_1}[\gamma(v_1)^p+\gamma(v_2)^p]\,d\mathcal{H}^{N-1},$$

where $\gamma(v_1),\gamma(v_2)$ are the traces of v_1 and v_2 on π_1.

By using the numerical inequality of Lemma 2.1 we may assume

$$F(v)\geq\frac{\left(\int_{\pi_1^+}|\nabla v|^2\,dx\right)^{1/p}+\left(\int_{J_v\cap\pi_1^+}[(v^+)^p+(v^-)^p]\,d\mathcal{H}^{N-1}+\int_{J_v\cap\pi_1}\gamma(v_1)^p\,d\mathcal{H}^{N-1}\right)^{\frac{1}{p}}}{\left(\int_{\pi_1^+}v^p\,dx\right)^{1/p}},$$

so that by reflecting v_1 across π_1 we obtain a new minimizer of problem (4.1) symmetric with respect to π_1, still denoted v_1.

We operate in the same way on v_1 by employing an hyperplane π_2 parallel to $x_2=0$, and obtaining a new minimizer v_2 symmetric with respect to both π_1 and π_2. Proceeding in this by considering hyperplanes parallel to $x_i=0$ for $i=3,\dots,N$, we end up with a minimizer $v_N=:u$ which turns out to be symmetric with respect to N orthogonal hyperplanes, whose intersection we may take as the new origin of $\mathbb{R}^N$.

Step 2: radiality of the minimizer. Let $\Omega\subseteq\mathbb{R}^N$ open and connected be the support of u according to Theorem 4.8: we have that u is $C^{1,\gamma}$ on Ω for some $\gamma>0$. If $x\in\Omega$, and π is a hyperplane through x and the origin, we can reflect again u across π obtaining a new minimizer $\hat{u}$: indeed the arguments of Step 1 can be applied since by the symmetry properties of u, the hyperplane π splits Ω in two parts of equal measure. If $\hat{\Omega}$ is the associated support, we have that $x\in\hat{\Omega}$, and by the symmetry and the regularity of $\hat{u}$ we conclude that

$$D_\nu u(x)=0,$$

where ν is orthogonal to π. This means that u is locally radial.

Assume that u is not constant on Ω, and let $x_0\in\Omega$ with $r_0:=|x_0|$. Let $\psi:I\to\mathbb{R}$ be the solution of

$$-\left(|\psi'|^{p-2}\psi' r^{N-1}\right)'=\lambda|\psi|^{p-2}\psi r^{N-1}$$

with

$$\psi(r_0)=u(x_0)\qquad\text{and}\qquad\psi'(r_0)=\partial_r u(x_0),$$

where λ is the constant appearing in the equation (4.32) satisfied by the nonconstant function u on Ω. In view of [22, Section 3], there exists a unique solution for this problem, for which either $I=[0,+\infty[$ or $I=]0,+\infty[$. Clearly

$$u(x)=\psi(|x|)$$

locally near x_0: but since Ω is connected, the equality extends to the entire Ω. Point (b) is thus proved.

Step 3: conclusion. In order to complete the proof of point (a), we need to show that (4.35) holds true. Let $x\in\partial\Omega$ be a point of density zero or one for Ω: since the function ψ is of class C^1, we deduce that $x\notin J_u$. Since $\partial\Omega=\overline{J_u}$ (thanks to Theorem 4.8), we deduce (recall that $\partial^e\Omega$ is the essential boundary of Ω, see Subsection 2.2)

$$\partial\Omega\setminus\partial^*\Omega\subseteq\left(\overline{J_u}\setminus J_u\right)\cup(\partial^e\Omega\setminus\partial^*\Omega),$$

so that

$$\mathcal{H}^{N-1}(\partial\Omega\setminus\partial^*\Omega)\leq\mathcal{H}^{N-1}\left(\overline{J_u}\setminus J_u\right)+\mathcal{H}^{N-1}(\partial^e\Omega\setminus\partial^*\Omega)=0.$$

Point (a) is thus completely proved. Finally, point (c) is an immediate consequence of the fact that $u=\psi 1_\Omega$, $\partial\Omega=\overline{J_u}$ and $\mathcal{H}^{N-1}(\partial\Omega\setminus\partial^*\Omega)=0$, so that the proof is concluded. $\square$

Remark 4.11. The previous proof shows that every minimizer v of problem (4.1) generates new minimizers $v_1, v_2, \dots, v_N := u$, such that v_{i+1} is obtained from v_i through a reflection across the hyperplane $x_i = k$ which splits the support of v_i in two parts with the same volume: since the inequalities of type (2.1) on which the argument is based are indeed equalities, both $v_i 1_{\{x_i<k\}}$ and $v_i 1_{\{x_i>k\}}$ generate an admissible v_{i+1}.

Remark 4.12. The proof of Proposition 4.10 shows that given any minimizer v of (4.1) and any hyperplane π which splits the associated support in two parts with the same volume, both the restrictions $v1_{\pi^\pm}$ generate by reflection across π a new minimizer of the functional.

We now show that the support of the symmetric minimizer constructed in Proposition 4.10 is indeed a ball.

Proposition 4.13. *Let u be the minimizer of* (4.1) *given by Proposition 4.10. Then the associated support Ω is a ball.*

Proof. If $u = \psi 1_\Omega$ with ψ constant, then by (4.38) we deduce

$$F(u) = \frac{\mathcal{H}^{N-1}(\partial^*\Omega)^{1/p}}{|\Omega|^{1/p}}.$$

If B is a ball such that $|B| = |\Omega|$, by considering the admissible function 1_B we obtain

$$\frac{\mathcal{H}^{N-1}(\partial^*\Omega)^{1/p}}{|\Omega|^{1/p}} = F(u) \le F(1_B) = \frac{\mathcal{H}^{N-1}(\partial B)^{1/p}}{|B|^{1/p}},$$

which yields, in view of the isoperimetric property of the ball, that Ω coincides up to negligible sets with a ball.

Assume that ψ is not constant on Ω. We divide the proof in several steps.

Step 1. Let $\tilde\Omega$ be the spherical cap symmetrization of Ω (see Subsection 2.4 for the definition): such a set is open and with finite perimeter. By passing to polar coordinates, and using the fact that $\mathcal{H}^{N-1}(\tilde\Omega \cap \partial B_r(0)) = \mathcal{H}^{N-1}(\Omega \cap \partial B_r(0))$, we deduce that

$$\int_{\tilde\Omega} |\psi'(|x|)|^p\,dx = \int_\Omega |\psi'(|x|)|^p\,dx \qquad \text{and} \qquad \int_{\tilde\Omega} \psi^p(|x|)\,dx = \int_\Omega \psi^p(|x|)\,dx.$$

Thanks to inequality (2.6) on perimeters with radial densities applied to $g = \psi^p$, we get in view of (4.38)

$$\frac{\left(\int_{\tilde\Omega} |\psi'(|x|)|^p\,dx\right)^{1/p} + \left(\int_{\partial^*\tilde\Omega} \psi^p(|x|)\,dx\right)^{1/p}}{\left(\int_{\tilde\Omega} \psi^p(|x|)\,dx\right)^{1/p}} \le \frac{\left(\int_{\Omega} |\psi'(|x|)|^p\,dx\right)^{1/p} + \left(\int_{\partial^*\Omega} \psi^p(|x|)\,dx\right)^{1/p}}{\left(\int_{\Omega} \psi^p(|x|)\,dx\right)^{1/p}} = F(u). \tag{4.39}$$

Notice that $\psi \in W^{1,p}(\tilde\Omega) \cap L^\infty(\tilde\Omega)$ with (recall (4.33) and (4.37))

$$\psi \ge \alpha > 0 \qquad \text{on } \tilde\Omega,$$

for some $\alpha > 0$. Thanks to [1, Theorem 3.84] we get that the function $\psi 1_{\tilde\Omega}$ belongs to $SBV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and in particular it is thus an element of $SBV^{1/p}(\mathbb{R}^N)$. Since

$$F(\psi 1_{\tilde\Omega}) = \frac{\left(\int_{\tilde\Omega} |\psi'(|x|)|^p\,dx\right)^{1/p} + \left(\int_{\partial^*\tilde\Omega} \psi^p(|x|)\,dx\right)^{1/p}}{\left(\int_{\tilde\Omega} \psi^p(|x|)\,dx\right)^{1/p}},$$

we deduce from (4.39) that also $\psi 1_{\tilde\Omega}$ is a minimizer of (4.1). In particular $\tilde\Omega$ (which is the associated support) is also connected.

Step 2. We claim that $\tilde{\Omega}$ has a spherical symmetry, that is, $\tilde{\Omega}$ is either a ball or an annulus. In order to see this, in view of the structure of $\tilde{\Omega}$ (which was constructed through a spherical cap symmetrization), it suffices to check that

$$|\tilde{\Omega} \cap \{x_N > 0\}| = |\tilde{\Omega} \cap \{x_N < 0\}|.$$

Indeed, if this holds true, we get that for a.e. $r > 0$ either $\partial B_r(0) \subseteq \tilde{\Omega}$ or $\partial B_r(0) \cap \tilde{\Omega} = \emptyset$. In view of its connectedness, we deduce then that $\tilde{\Omega}$ is either a ball or an annulus.

Assume by contradiction that

$$|\tilde{\Omega} \cap \{x_N > 0\}| > |\tilde{\Omega} \cap \{x_N < 0\}|.$$

Then we can find an hyperplane $x_N = k$ with $k > 0$ which splits $\tilde{\Omega}$ in two parts with the same volume. According to Remark 4.12, one of the two parts generates by reflection a new minimizer $\hat{\psi} 1_{\hat{\Omega}}$ for the free discontinuity problem (4.1). By construction, the support $\hat{\Omega}$ is already symmetric with respect to the coordinate hyperplanes $x_i = 0$ for $i = 1, \ldots, N-1$. If we apply Proposition 4.10 to $\hat{\psi} 1_{\hat{\Omega}}$, we deduce that $\hat{\psi}$ is radial with respect to the point $(0, 0, \ldots, 0, k)$. But this is impossible, since $\hat{\psi}$ coincides with a reflection of ψ which is radial about the origin and it is not constant thanks to equation (4.32).

Step 3. We claim that $\tilde{\Omega}$ is a ball. In order to see this, we need to exclude that

$$\tilde{\Omega} = A_{r_1,r_2} := \{x \in \mathbb{R}^N \,:\, r_1 < |x| < r_2\}.$$

Assume by contradiction that $u = \psi 1_{A_{r_1,r_2}}$ is a minimizer for problem (4.1). Then, by exploiting the associated Euler-Lagrange equation, it turns out that ψ satisfies on $\partial A_{r_1,r_2}$ the boundary conditions of Robin type

$$-|\psi'|^{p-2}(r_1)\psi'(r_1) + \beta\psi^{p-1}(r_1) = 0 \qquad \text{and} \qquad |\psi'|^{p-2}(r_2)\psi'(r_2) + \beta\psi^{p-1}(r_2) = 0$$

for a suitable constant $\beta > 0$. This entails that

$$\psi'(r_1) > 0 \qquad \text{and} \qquad \psi'(r_2) < 0.$$

Since from the differential equation (4.36) we get that

$$r \mapsto r^{N-1}|\psi'(r)|^{p-2}\psi'(r)$$

is decreasing on $[r_1, r_2]$, we deduce that there exists $r_3 \in]r_1, r_2[$ such that

$$\psi \text{ is increasing on } [r_1, r_3] \text{ and decreasing on } [r_3, r_2].$$

Two situations may happen, leading both to a contradiction.

(a) Assume

$$\psi(r_1) \leq \psi(r_2).$$

Let $r_4 \in]r_1, r_3[$ be such that $\psi(r_4) = \psi(r_2)$. Let us consider the radial symmetric decreasing rearrangement ψ^* of ψ restricted to A_{r_4,r_2} supported on the ball $B_{\bar{r}}(0)$ (see Subsection 2.4 for the definition), and let us extend it with the value $\psi(r_2)$ on the larger ball $B_{\hat{r}}(0)$, where

$$|B_{\bar{r}}(0)| = |A_{r_4,r_2}| \qquad \text{and} \qquad |B_{\hat{r}}(0)| = |A_{r_1,r_2}|.$$

Let us denote this function with φ. We have

$$\int_{B_{\hat{r}}(0)} |\nabla\varphi|^p \, dx \leq \int_{A_{r_1,r_2}} |\nabla\psi|^p \, dx, \qquad \int_{B_{\hat{r}}(0)} \varphi^p \, dx > \int_{A_{r_1,r_2}} \psi^p \, dx,$$

and

$$\int_{\partial B_{\hat{r}}(0)} \varphi^p \, d\mathcal{H}^{N-1} < \int_{\partial A_{r_1,r_2}} \psi^p \, d\mathcal{H}^{N-1}.$$

The inequality on the gradient comes from the general properties of the radial symmetric decreasing rearrangement, together with the fact that φ is constant on $B_{\hat{r}}(0) \setminus B_{\bar{r}}(0)$. The second inequality comes instead from the equality

$$\int_{B_{\bar{r}}(0)} \varphi^p \, dx = \int_{A_{r_4,r_2}} |\psi|^p \, dx$$

together with the fact that $\psi \le \psi(r_2)$ on $[r_1, r_4]$. Finally the inequality for the surface integral is a consequence of the fact that

$$\int_{\partial A_{r_1,r_2}} \psi^p \, d\mathcal{H}^{N-1} > \int_{\partial B_{r_2}(0)} \psi^p \, d\mathcal{H}^{N-1} > \psi^p(r_2)\mathcal{H}^{N-1}(\partial B_{\hat{r}}(0)) = \int_{\partial B_{\hat{r}}(0)} \varphi^p \, d\mathcal{H}^{N-1}.$$

We conclude that

$$F(\psi 1_{A_{r_1,r_2}}) = \frac{\left(\int_{A_{r_1,r_2}} |\nabla\psi|^p \, dx\right)^{1/p} + \left(\int_{\partial A_{r_1,r_2}} \psi^p \, d\mathcal{H}^{N-1}\right)^{1/p}}{\left(\int_{A_{r_1,r_2}} \psi^p \, dx\right)^{1/p}}$$
$$> \frac{\left(\int_{B_{\hat{r}}(0)} |\nabla\varphi|^p \, dx\right)^{1/p} + \left(\int_{\partial B_{\hat{r}}(0)} \varphi^p \, d\mathcal{H}^{N-1}\right)^{1/p}}{\left(\int_{B_{\hat{r}}(0)} \varphi^p \, dx\right)^{1/p}} = F(\varphi 1_{B_{\hat{r}}(0)}),$$

so that $\psi 1_{A_{r_1,r_2}}$ is not a minimizer for problem (4.1).

(b) Let us assume that

$$\psi(r_1) > \psi(r_2).$$

In this case we proceed directly with a radial symmetric decreasing rearrangement of $\psi 1_{A_{r_1,r_2}}$. We obtain a new function ψ^* supported on a ball $B_r(0)$ with $|B_r(0)| = |A_{r_1,r_2}|$. In view of the regularity of ψ on A_{r_1,r_2}, by Remark 2.4 we get that ψ^* is Lipschitz continuous on $B_r(0)$. Moreover, the geometry of ψ entails that for every $a > \psi(r_2)$

$$\mathcal{H}^{N-1}(\{\psi = a\} \cap A_{r_1,r_2}) > \mathcal{H}^{N-1}(\{\psi^* = a\})$$

(notice that $\{\psi > a\}$ is an annulus and $\{\psi^* > a\}$ is a ball with equal volume, whose radius is therefore strictly less than the external radius of the former). Again by Remark 2.4, we deduce

$$\int_{B_r(0)} |\nabla\psi^*|^p \, dx \le \int_{A_{r_1,r_2}} |\nabla\psi|^p \, dx.$$

Finally, since

$$\int_{B_r(0)} (\psi^*)^p \, dx = \int_{A_{r_1,r_2}} \psi^p \, dx \qquad \text{and} \qquad \int_{\partial B_r(0)} (\psi^*)^p \, d\mathcal{H}^{N-1} < \int_{\partial A_{r_1,r_2}} \psi^p \, d\mathcal{H}^{N-1},$$

we obtain as above

$$F(\psi 1_{A_{r_1,r_2}}) > F(\psi^* 1_{B_r}(0)),$$

so that $\psi 1_{A_{r_1,r_2}}$ is not a minimizer for problem (4.1).

□

We conclude the section with the following uniqueness result.

Theorem 4.14. *Any minimizer of problem* (4.1) *is of the form*

$$\psi 1_B$$

where B is a ball of volume m, and $\psi \in W^{1,p}(B)$ is radial with respect to the center of B.

Proof. Let v be a minimizer of problem (4.1) with associated support Ω. Let us apply Proposition 4.10 and Remark 4.11 to this minimizer, generating the family of minimizers

$$v_1, v_2, \dots, v_N$$

with associated supports $\Omega_1, \dots, \Omega_N$.

By Proposition 4.13, we know that v_N is supported on a ball B, and radially symmetric with respect to the center inside. We may assume that B is centered at the origin. The function v_N has been obtained from v_{N-1} by means of a reflection across the hyperplane $x_N = 0$ which splits Ω_{N-1} in two parts $\Omega^{\pm}_{N-1}$ of the same volume. Moreover we know that both the restrictions of v_{N-1} to $\Omega^{\pm}_{N-1}$ generate an admissible v_N. We thus conclude that $\Omega_{N-1} = B$, and the associated function which is radial.

In turn v_{N-1} is obtained from v_{N-2} by means of a reflection across the hyperplane $x_{N-1}=0$ which splits Ω_{N-2} in two parts $\Omega^{\pm}_{N-2}$ of the same volume. Again, both the restrictions of v_{N-2} to $\Omega^{\pm}_{N-2}$ generate an admissible v_{N-1}. Recalling that Ω_{N-2} is connected and symmetric with respect to the hyperplanes

$$x_1=0, \quad x_2=0, \quad \dots \quad x_{N-2}=0,$$

we conclude for example that

$$\Omega^{+}_{N-2}=B\cap\{x_{N-1}<0\} \qquad \text{and} \qquad \Omega^{-}_{N-2}=\tilde{B}\cap\{x_{N-1}>0\},$$

where $\tilde{B}$ is a ball of volume m with center on the line $x_1=x_2=\cdots=x_{N-1}=0$. But in the procedure of Proposition 4.10 we could exchange the role of the coordinates x_{N-1} and x_N. If we symmetrize firstly with respect to a hyperplane parallel to $x_N=0$, we should get a minimizer supported on a ball, and this is possible only if $B=\tilde{B}$. We conclude that $\Omega_{N-2}=B$ with associated function which is radial. Proceeding in this way, we come back to Ω which is itself equal to B with associated function which is radial. The proof is thus concluded.

□

5. Proof of the main result

We are now in a position to prove our main theorem. We consider the case $1<p<+\infty$ which relies on the free discontinuity analysis of Section 4. The case $p=1$ is discussed separately and will be dealt directly through rearrangements of BV functions.

The case $p>1$. Let $\Omega\subseteq\mathbb{R}^N$ be open, bounded and with a Lipschitz boundary. Let $u\in W^{1,p}(\Omega)$ on which the Poincaré constant in (1.1) is attained, that is

$$C_p(\Omega)=\frac{\|\nabla u\|_{L^p(\Omega;\mathbb{R}^N)}+\|u\|_{L^p(\partial\Omega)}}{\|u\|_{L^p(\Omega)}}.$$

It is not restrictive to assume that $u\geq 0$ on Ω. Notice that thanks to [1, Theorem 3.84]

$$u^p 1_\Omega\in SBV(\mathbb{R}^N).$$

We deduce $v:=u1_\Omega\in SBV^{\frac{1}{p}}(\mathbb{R}^N)$. Notice that

$$\|\nabla v\|_p=\|\nabla u\|_{L^p(\Omega;\mathbb{R}^N)} \qquad \text{and} \qquad \|v\|_p=\|u\|_{L^p(\Omega)}.$$

Moreover we have $J_v\subseteq\partial\Omega$ with

$$\int_{J_v}[(v^+)^p+(v^-)^p]\,d\mathcal{H}^{N-1}=\int_{J_v}u^p\,d\mathcal{H}^{N-1}=\int_{\partial\Omega}u^p\,dx$$

since $u=0$ on $\partial\Omega\setminus J_v$. We conclude that

$$F(v)=\frac{\|\nabla u\|_{L^p(\Omega;\mathbb{R}^N)}+\|u\|_{L^p(\partial\Omega)}}{\|u\|_{L^p(\Omega)}}=C_p(\Omega).$$

Thanks to Theorem 4.14, we have that minimizers of the free discontinuity problem (4.1) for $m=|\Omega|$ are of the form

$$\psi 1_B,$$

where B is a ball of volume m, and $\psi\in W^{1,p}(B)$ is radial with respect to its center. In particular we get

$$C_p(\Omega)=F(v)\geq F(\psi 1_B)=\frac{\|\nabla\psi\|_{L^p(B;\mathbb{R}^N)}+\|\psi\|_{L^p(\partial B)}}{\|v\|_{L^p(B)}}\geq C_p(B), \tag{5.1}$$

which shows the optimality of the ball for the constant in inequality (1.1).

Let us come to the uniqueness issue. If $C_p(\Omega)=C_p(B)$, then inequalities in (5.1) become equalities, which yields that $v=u1_\Omega$ is a minimizer for the free discontinuity problem (4.1). By Theorem 4.14, we get that Ω is a ball, and the proof is concluded.

The case $p = 1$. In this case, the Poincaré inequality is related to the Cheeger constant. We give below a direct proof working with rearrangements in BV, but one can also use an equivalent definition for the Cheeger constant and rely on classical symmetrization of sets (see the remark below).

Let $\Omega \subseteq \mathbb{R}^N$ be open, bounded and with a Lipschitz boundary. Notice that by using a standard approximation argument, the inequality

$$C_1(\Omega)\|u\|_{L^1(\Omega)} \le \|\nabla u\|_{L^1(\Omega;\mathbb{R}^N)} + \|u\|_{L^1(\partial\Omega)}, \qquad u \in W^{1,1}(\Omega),$$

can be rewritten

$$C_1(\Omega)\|u\|_{L^1(\Omega)} \le |Du|(\mathbb{R}^N), \qquad u \in BV(\mathbb{R}^N), \quad supp(u) \subseteq \Omega. \tag{5.2}$$

Equality is attained on a nonnegative function $\tilde{u} \neq 0$. If we pass to its radial symmetric decreasing rearrangement $\tilde{u}^*$ supported on a ball $\tilde{B}$ centered at the origin with $|\tilde{B}| = |supp(\tilde{u})|$ (see Subsection 2.4), as a consequence of the coarea formula together with the isoperimetric inequality we get (see e.g. [11, Theorem 1.3])

$$|D\tilde{u}^*|(\mathbb{R}^N) \le |D\tilde{u}|(\mathbb{R}^N),$$

so that

$$C_1(\Omega) = \frac{|D\bar{u}|(\mathbb{R}^N)}{\|\bar{u}\|_{L^1(\Omega)}} \ge \frac{|D\bar{u}^*|(\mathbb{R}^N)}{\|\bar{u}^*\|_{L^1(\tilde{B})}} \ge C_1(\tilde{B}).$$

If B is a ball with $|B| = |\Omega|$, we get (monotonicity of C_1 under dilations)

$$C_1(\Omega) \ge C_1(\tilde{B}) \ge C_1(B), \tag{5.3}$$

i.e., the ball is an optimal domain.

If $C_1(\Omega) = C_1(B)$, then equality holds in (5.3), so that we infer in particular

$$|D\tilde{u}^*|(\mathbb{R}^N) = |D\tilde{u}|(\mathbb{R}^N) \qquad \text{and} \qquad |supp(\tilde{u})| = |\tilde{B}| = |\Omega|.$$

By [11, Theorem 1.5], we deduce that u agrees a.e. with a function whose level sets are open balls. This entails that the support of $\tilde{u}$ is a ball contained in Ω with volume equal to $|\Omega|$, i.e., Ω is itself a ball. The proof is now concluded.

Remark 5.1 (**Relationship between $C_1(\Omega)$ and the Cheeger constant).** Using the co-area formula and Cavalieri's principle (see for instance [10]), the following equality occurs

$$C_1(\Omega) = \min\left\{ \frac{Per(E;\mathbb{R}^N)}{|E|} : E \text{ measurable}, E \subset \Omega \right\}.$$

In other words, $C_1(\Omega)$ is precisley the Cheeger constant of the set Ω. Above, $Per(E;\mathbb{R}^N)$ is the perimeter of the set E in the sense of geometric measure theory (see Subsection 2.2).

Then, inequality $C_1(\Omega) \ge C_1(B)$ comes directly by symmetrization and the isoperimetric inequality. If $C_1(\Omega) = C_1(B)$, then there exists a Cheeger set E in Ω which is a ball of the size of B. Since $|\Omega| = |B|$, we get $\Omega = B$.

Remark 5.2 (**Faber-Krahn inequality for the Robin p-Laplacian).** Let us consider for every $1 < p < +\infty$ and $\beta > 0$ the Poincaré inequality with trace term

$$\forall u \in W^{1,p}(\Omega) \,:\, \tilde{C}_p(\Omega,\beta) \int_\Omega |u|^p\,dx \le \int_\Omega |\nabla u|^p\,dx + \beta \int_{\partial\Omega} |u|^p\,d\mathcal{H}^{N-1}.$$

We see that

$$\tilde{C}_p(\Omega,\beta) = \min_{u \in W^{1,p}(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^p\,dx + \beta \int_{\partial\Omega} |u|^p\,d\mathcal{H}^{N-1}}{\int_\Omega |u|^p\,dx},$$

so that $\tilde{C}_p(\Omega,\beta)$ is the first eigenvalue of the p-Laplace operator on Ω with Robin boundary condition.

The analysis of the previous section can be easily adapted to deal with the free discontinuity functional

$$\tilde{F}(u) := \frac{\int_{\mathbb{R}^N} |\nabla u|^p\,dx + \beta \int_{J_u} [(u^+)^p + (u^-)^p]\,d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} u^p\,dx},$$

showing that

$$\tilde{C}_p(B,\beta) \le \tilde{C}_p(\Omega,\beta), \qquad |B| = |\Omega|,$$

with equality if and only if Ω is a ball. This provides thus an alternative proof of the Faber-Krahn inequality for the Robin-p-Laplace operator established in [12] (see also [4]).

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