Stability and uniqueness for the crack identification problem

Zakaria BELHACHMI & Dorin BUCUR
Département de Mathématiques, UMR-CNRS 7122
Université de Metz
Ile du Saulcy, 57045 Metz Cedex 01, France
belhach@math.univ-metz.fr & bucur@math.univ-metz.fr

Abstract

The paper deals with the identifiability of non-smooth defects by boundary measurements, and the stability of their detection. We introduce and analyse a new pointwise regularity concept at the boundary of an open set which turns out to play a crucial role in the identifiabilty of defects by two boundary measurements. As a consequence, we prove the unique identifiability for a large class of closed sets, including sets with infinite number of connected components of positive capacity and totally disconnected sets. In order to rigorously justify numerical approximation results of defects by optimal design methods, we prove a geometric stability result of the defect identification process, without any a priori smoothness assumptions.

1 Introduction

The paper deals with the defect identification problem by boundary measurements. Roughly speaking the problem can be formulated as follows: given a smooth bounded domain $\Omega \subseteq \mathbb{R}^2$, find a closed set $K \subseteq \Omega$ knowing the traces on the boundary $w_{i|\partial\Omega}$ of the solutions of

$$\begin{cases} -\Delta w_i & = 0 \text{ in } \Omega \setminus K \\ \frac{\partial w_i}{\partial n} & = 0 \text{ on } \partial K \\ \frac{\partial w_i}{\partial n} & = \psi_i \text{ on } \partial\Omega \end{cases} \tag{1}$$

for *several inputs* ψ_i. We refer the reader to the paper [4] for a complete review of the most important and up to date results concerning this problem.

There are three main challenges when dealing with such a problem:

- the uniqueness of the defect for a given number of measures: *may different defects give the same measures ?*
- stability with respect to the measurements: *do close measures give "close" cracks?* There is a subsequent question. What is the *right* sense of closeness for defects: *close in geometry, or in behavior (like γ-convergence)*?

- (numerical) reconstruction of the defects and rigorous convergence results.

A way to tackle this geometric inverse problem is to use optimal design methods. From this view point, one needs to understand the three items above in the context of minimal regularity assumptions for defects. Dropping *a priori* regularity assumptions for the stability purpose allows, for example, to give a formal justification to the convergence of the approximation process by a shape optimization approach.

The purpose of this paper is twofold. In a first step we introduce and analyse a new pointwise regularity concept at the boundary of an open set, called *conductivity*, which plays a crucial role in the identifiabilty of defects by two boundary measurements. It is known that one measure can not uniquely determine even a smooth curve K, and, following Alessandrini and Diaz Valenzuela [1], two suitably chosen inputs can uniquely determine closed sets K which can be decomposed in a finite union of disjoint continua (see also [17]). Roughly speaking, we prove that unique identification holds for the family of defects which are conductive at quasi-every point of their boundaries (see Theorem 3.9). As a consequence, we prove unique identifiability by two boundary measurements for a large class of closed sets, including sets with infinite number of connected components of positive capacity and totally disconnected sets. Our proof uses the scheme of Alessandrini and Diaz Valenzuela based on non existence of critical points for suitable holomorphic functions. The construction of critical points relies on the conductivity regularity concept. With respect to the proof of Alessandrini and Diaz Valenzuela several new technical difficulties appear, which are related to the fact that harmonic conjugates of solutions are not Hölder continuous up to the boundary and information given by the unique continuation principle can not be propagated "across" the defects. The conductivity regularity concept has several common features with the Dirichlet regularity related to the Wiener criterion, but we are not able to prove or disprove their equivalence. Nevertheless, our result associated to the Kellogg property, also shows that the equivalence of the two regularity concepts (conductivity and Dirichlet regularity) would straightly forward imply the conjecture that **all** closed sets are uniquely identifiable up to a set of zero capacity, by two boundary measurements.

The second purpose of the paper is to investigate the stability of the detection from the shape optimization point of view. Precisely, we prove that asymptotic geometric stability holds in the class of defects having a uniform bound on the number of connected components (see Theorem 4.3). Roughly speaking, convergence of the measures in the space of traces implies geometric convergence of the defects (this is the frame of the so called Tikhonov principle [20]). All previous stability results (see [2, 4, 11, 18] and references therein) require to know *a priori* a quantitative estimate of the smoothness of the defects, and provide quantitative estimates for the stability. By dropping the *a priori* smoothness hypotheses we lose any quantitative estimate but, and here is the main interest of such a result, we can rigorously justify that suitable numerical approximations of the defects are convergent (see Theorems 5.1 and 5.3). This result is to be compared to the one obtained in [8] for shape optimization problems associated to the Dirichlet-Laplacian and relies deeply on the shape stability result of [6] and on the elimination of the smoothness hypotheses. Stability results based on *à priori* smoothness cannot be used to achieve shape convergence for numerical approximations in the optimal design framework.

2 Setting the problem

In all the paper, Ω denotes a bounded simply connected open set in $\mathbb{R}^2$ with smooth boundary. By $|E|$ we denote the Lebesgue measure of the set E and by cap (E) its capacity, i.e.

$$\text{cap }(E) = \inf\{\int_{\mathbb{R}^2} |\nabla u|^2 + |u|^2 dx, \quad u \in \mathcal{U}_E\}$$

where $\mathcal{U}_E$ is the class of all functions $u \in C_c^\infty(\mathbb{R}^2)$ such that $u \geq 1$ a.e. in a neighbourhood of E. It is said that a property $p(x)$ holds quasi everywhere on E (shortly q.e. on E) if the set of all points $x \in E$ for which $p(x)$ does not hold has capacity zero. We refer to [14] for details concerning capacity.

A function u is said *quasi continuous* if for every $\epsilon > 0$ there exists an open set A_ϵ such that cap $(A_\epsilon) < \epsilon$ and $u|_{\Omega \setminus A_\epsilon}$ is continuous in $\Omega \setminus A_\epsilon$. In all the paper, every time we refer to pointwise properties of Sobolev functions, we implicitly consider quasi continuous representatives.

The usual Sobolev space is denoted by $H^1(\Omega)$. Recall that every function $u \in H^1(\Omega)$ has a quasi-continuous representative, unique up to a set of zero capacity. Considering quasi-continuous representatives, one can define the trace (as restriction) of a function $u \in H^1(\Omega)$ on every continuum of positive diameter. We recall the following result (see [6]).

Lemma 2.1 *Let $u \in H^1(\Omega)$ and K_1, K_2 two compact connected sets in Ω with positive diameter. If there exist two different constants $c_1, c_2 \in \mathbb{R}$ such that $u(x) = c_1$ q.e. on K_1 and $u(x) = c_2$ q.e. on K_2 then $K_1 \cap K_2 = \emptyset$.*

We also recall the definition of the following functional space. Let $U \subseteq \mathbb{R}^2$ be an open set; the Dirichlet space $\mathcal{L}^{1,2}(U)$ is defined as [14].

$$\mathcal{L}^{1,2}(U) = \{u \in L^2_{loc}(U) : \nabla u \in [L^2(U)]^2\}, \tag{2}$$

where the gradient of u is taken in the sense of distributions. Introducing the equivalence relation

$$u\mathcal{R}v \text{ if } \int_U |\nabla(u - v)|^2 dx = 0,$$

the quotient space $\mathcal{L}^{1,2}(U)_{/\mathcal{R}} := L^{1,2}(U)$ is a Hilbert space for the scalar product

$$(u, v)_{L^{1,2}(U)} = \int_U \nabla u \nabla v dx.$$

Let C be a connected component of U and let $u, v \in \mathcal{L}^{1,2}(U)$ such that $u\mathcal{R}v$. Then $u - v$ is constant a.e. on C.

Following [12, Corollary 2.2] if U is smooth enough (e.g. with Lipschitz continuous boundary) then $\mathcal{L}^{1,2}(U) = H^1(U)$. If U is not smooth, then $H^1(U)$ might be strictly contained in $\mathcal{L}^{1,2}(U)$. Observe also that if U is not smooth enough, several "well known" properties of H^1-spaces fail to be true, as for example the Poincaré-Wirtinger inequality.

For an arbitrary set $F \subseteq \mathbb{R}^2$ and for $\varepsilon > 0$ let us denote the dilation of F by ε, $F^\varepsilon = \cup_{x \in F} B(x, \varepsilon)$ being the union of all open balls centred in points of F with radius ε, and by $\overline{F}^\varepsilon$ its closure. Clearly the following holds for $\varepsilon < \nu$: $F^\varepsilon = (\overline{F})^\varepsilon \subseteq \overline{(F^\varepsilon)} \subseteq F^\nu$.

Definition 2.2 *The Hausdorff distance between two compact sets $K_1, K_2 \subseteq \mathbb{R}^2$ is defined by*

$$d_H(K_1, K_2) = \inf\{\varepsilon > 0 : K_1 \subseteq K_2^\varepsilon\ , K_2 \subseteq K_1^\varepsilon\}.$$

Note that the family of closed subsets of a given compact of $\mathbb{R}^2$ is compact for the Hausdorff metric. We refer to [6] and [19] for more details on the Hausdorff metric and on the following.

Lemma 2.3 *Let $\{u_n\}_{n\in\mathbb{N}} \subseteq H^1(\Omega)$, $\{K_n\}_{n\in\mathbb{N}}$ a sequence of compact connected sets in Ω and $\{c_n\}_{n\in\mathbb{N}}$ a sequence of constants such that $u_n(x) = c_n$ q.e. on K_n. If $K_n \overset{H}{\longrightarrow} K$ then K is connected. Suppose that $u_n \overset{H^1(\Omega)}{\rightharpoonup} u$. Then there exists a constant $c \in \mathbb{R}$ such that $c_n \longrightarrow c$ and $u(x) = c$ q.e. on $K \cap \Omega$.*

Let $K \subset \Omega$ be compact and $\psi \in L^2(\partial\Omega)$ be such that $\int_{\partial\Omega} \psi = 0$. We consider in the sequel the **the perfectly insulating problem**

$$\begin{cases} -\Delta w & = 0 \text{ in } \Omega \setminus K \\ \frac{\partial w}{\partial n} & = 0 \text{ on } \partial K \\ \frac{\partial w}{\partial n} & = \psi \text{ on } \partial\Omega \end{cases} \tag{3}$$

Since K is the unknown of the problem and may vary, if ambiguity occurs on the choice of K, this solution will be denoted $w_{\psi,K}$. It is clear that using the usual tools (e.g. Lax-Milgram Theorem [3], see also [6]) one has the following.

Proposition 2.4 *There exists a unique solution $u \in L^{1,2}(\Omega \setminus K)$ obtained by solving the following minimisation problem*

$$\min_{\phi \in L^{1,2}(\Omega\setminus K)} \frac{1}{2} \int_{\Omega\setminus K} |\nabla \phi|^2 dx - \int_{\partial\Omega} \phi\psi d\sigma.$$

Let us denote by 1_U the characteristic function of the set U. The following result has a simple proof (see Section 4 and [6, 10]).

Proposition 2.5 *The following holds for $\varepsilon \to 0$*

$$\nabla w_{\psi,\overline{K}^\varepsilon} 1_{\Omega\setminus\overline{K}^\varepsilon} \overset{L^2(\Omega,\mathbb{R}^2)}{\longrightarrow} \nabla w_{\psi,K} 1_{\Omega\setminus K}.$$

Note also, that for $\varepsilon > 0$, the function $w_{\psi,\overline{K}^\varepsilon}$ has a harmonic conjugate in $\Omega \setminus \overline{K}^\varepsilon$ (see [1], for example), i.e. is the real part of a holomorphic function. Following Proposition 2.5 and the usual properties of holomorphic functions, the function $w_{\psi,K}$ has also a harmonic conjugate. The problem which is solved by the conjugate functions will be clarified by studying the following.

The perfectly conducting problem. A dual problem, called "the perfectly conducting case" has been introduced in [11] for one connected crack and extended in [1] for a finite number of connected cracks, say $K = \cup_{i=1}^k K_i$. In this case, the problem is formulated as follows (see for instance [1]):

$$\begin{cases} -\Delta u = 0 \text{ in } \ \Omega \setminus K \\ u = c_i \text{ q.e. on } K_i \\ \frac{\partial u}{\partial n} = \psi \text{ on } \partial\Omega \end{cases} \tag{4}$$

The constants c_i are uniquely determined by the no-flux condition that the solution u has to satisfy: for every smooth Jordan curve $\gamma \subseteq \Omega \setminus K$ $\int_\gamma \frac{\partial u}{\partial n} d\sigma = 0$.

Moreover, the solution of this problem is given by the minimisation of the following energy functional

$$\min\{\frac{1}{2}\int_{\Omega\setminus K} |\nabla u|^2 dx - \int_{\partial\Omega} u\psi d\sigma : u \in H^1(\Omega), u \text{ q.e. constant on } K_i\}. \tag{5}$$

Details concerning the equivalence of the formulations (4) and (5) can be found in [1] (see also [6] for more details concerning the formulation via quasi-continuous functions).

Here is the main key for understanding the uniqueness of the inverse problem for arbitrary compact sets. We manage the perfectly conductive case for arbitrary K by introducing the following Sobolev like space. Let Ω be a bounded open set, and $F \subseteq \overline{\Omega}$ an arbitrary set. We define

$$H^1_{cond,F}(\Omega) = cl_{H^1(\Omega)}\{u \in H^1(\Omega) : \exists \varepsilon > 0, \nabla u = 0 \ a.e. \text{ on } F^\varepsilon \cap \Omega\}. \tag{6}$$

Let us denote by $u(F)$ the image of the set F by u. For sets F which have a certain regularity (e.g. finite number of Lipschitz connected components), the previous definition coincides with

$$cl_{H^1(\Omega)}\{u \in H^1(\Omega) : u(F) \text{ is finite}\} \tag{7}$$

and

$$cl_{H^1(\Omega)}\{u \in C^\infty(\Omega) \cap H^1(\Omega) : u(F) \text{ finite}\},$$

but it is not clear whether this holds for arbitrary F. Of course, in (7) we consider quasi-continuous representatives. Observe also that if $u \in H^1_{cond,F}(\Omega)$ then $|u| \in H^1_{cond,F}(\Omega)$.

Note that the following inequality holds for every function $u \in H^1(\Omega)$ such that $\int_{\partial\Omega} u d\sigma = 0$

$$\int_{\partial\Omega} u^2 dx \leq C \int_\Omega |\nabla u|^2 dx, \tag{8}$$

where C is a constant depending only on Ω. This is a consequence of the trace theorem and the Poincaré inequality in $H^1(\Omega)$. Note also that $u \mapsto \int_\Omega |\nabla u|^2 dx$ is a norm on $\{u \in H^1(\Omega), \int_{\partial\Omega} u d\sigma = 0\}$ and that for every $u \in H^1_{cond,K}(\Omega)$ we have $\nabla u = 0$ a.e. on K.

We see problem (4) for an arbitrary K only by its variational formulation

$$\min\{\frac{1}{2}\int_{\Omega\setminus K} |\nabla u|^2 dx - \int_{\partial\Omega} u\psi d\sigma : u \in H^1_{cond,K}(\Omega)\}. \tag{9}$$

Proposition 2.6 *Problem (9) has a unique solution such that* $\int_{\partial\Omega} u d\sigma = 0$.

Note that the gradient of the solution is unique. We can fix a representative such that $\int_{\partial\Omega} u d\sigma = 0$.

Proof To prove the existence of a solution for problem (9), the Lax-Milgram theorem can be directly used. Nevertheless, in order to familiarise the reader with the space $H^1_{cond,K}(\Omega)$, we show this by using the direct methods of the calculus of variations.

Let $u_n \in H^1_{cond,K}(\Omega)$ be a minimizing sequence. We can assume that $\int_{\partial\Omega} u_n d\sigma = 0$, if not we simply add suitable constants. Since, $0 \in H^1_{cond,K}(\Omega)$, we can also assume

$$\frac{1}{2}\int_{\Omega\setminus K} |\nabla u_n|^2 dx - \int_{\partial\Omega} u_n \psi d\sigma \leq 0.$$

Using the Cauchy inequality together with (8) there exists a constant M depending only on Ω such that

$$\int_{\Omega\setminus K} |\nabla u_n|^2 dx \leq M.$$

There exists $u \in H^1_{cond,K}(\Omega)$ such that $\nabla u_n \overset{L^2(\Omega,\mathbb{R}^2)}{\rightharpoonup} \nabla u$ and $u_{n|\partial\Omega} \overset{L^2(\partial\Omega)}{\longrightarrow} u_{|\partial\Omega}$. Consequently

$$\frac{1}{2}\int_{\Omega\setminus K} |\nabla u|^2 dx - \int_{\partial\Omega} u\psi d\sigma \leq \liminf_{n\to\infty} \frac{1}{2}\int_{\Omega\setminus K} |\nabla u_n|^2 dx - \int_{\partial\Omega} u_n \psi d\sigma,$$

hence u is a solution of (9).

The uniqueness of the solution comes from the convexity of the energy functional. □

Note the following facts: the solution given by Proposition 2.6 satisfies $-\Delta u = 0$ on $\Omega \setminus K$ in the sense of distributions and $\frac{\partial u}{\partial n} = \psi$ on $\partial\Omega$ in the weak sense of traces; for every $x \in K$ such that $x \in U_x \subseteq K$, where U_x is a continuum, the solution of (9) is constant q.e. on U_x.

We give the following proposition which has a simple direct proof, and refer to Section 4 for a more detailed discussion of the stability question.

Proposition 2.7 *The following holds for* $\varepsilon \to 0$

$$\nabla u_{\psi,\overline{K}^\varepsilon} \overset{L^2(\Omega,\mathbb{R}^2)}{\longrightarrow} \nabla u_{\psi,K}.$$

Proof For simplicity, let us denote $u_\varepsilon = u_{\psi,\overline{K}^\varepsilon}$. As in Proposition 2.6, there exists a constant M independent on ε such that

$$\int_{\Omega} |\nabla u_\varepsilon|^2 dx \leq M.$$

There exists $u \in H^1(\Omega)$ such that $\nabla u_\varepsilon \overset{L^2(\Omega,\mathbb{R}^2)}{\rightharpoonup} \nabla u$ and $u_{\varepsilon|\partial\Omega} \overset{L^2(\partial\Omega)}{\longrightarrow} u_{|\partial\Omega}$. We get

$$\frac{1}{2}\int_{\Omega} |\nabla u|^2 dx - \int_{\partial\Omega} u\psi d\sigma \leq \liminf_{\varepsilon\to 0} \frac{1}{2}\int_{\Omega} |\nabla u_\varepsilon|^2 dx - \int_{\partial\Omega} u_\varepsilon \psi d\sigma. \tag{10}$$

Note that $u \in H^1_{cond,K}(\Omega)$ since $u_\varepsilon \in H^1_{cond,\overline{K}^\varepsilon}(\Omega) \subseteq H^1_{cond,K}(\Omega)$. Let u^* be the solution of (9) in $H^1_{cond,K}(\Omega)$. Then

$$\frac{1}{2}\int_{\Omega} |\nabla u^*|^2 dx - \int_{\partial\Omega} u^*\psi d\sigma \leq \frac{1}{2}\int_{\Omega} |\nabla u|^2 dx - \int_{\partial\Omega} u\psi d\sigma.$$

From the definition of $H^1_{cond,K}(\Omega)$, there exists a sequence $\phi_n \in H^1(\Omega)$, such that $\nabla\phi_n = 0$ a.e. on $K^{1/n}$, such that $\int_{\partial\Omega} \phi_n d\sigma = 0$ and $\phi_n \longrightarrow u^*$ in $H^1(\Omega)$-strong. Choosing suitable couples (ε, n) such that $\varepsilon < 1/n$, we get $\phi_n \in H^1_{cond,\overline{K}^\varepsilon}(\Omega)$. Consequently,

$$\begin{aligned} \limsup_{\varepsilon\to 0} \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 dx - \int_{\partial\Omega} u_\varepsilon \psi d\sigma &\le \lim_{n\to\infty} \frac{1}{2}\int_\Omega |\nabla\phi_n|^2 dx - \int_{\partial\Omega} \phi_n \psi d\sigma \\ &= \frac{1}{2}\int_\Omega |\nabla u^*|^2 dx - \int_{\partial\Omega} u^* \psi d\sigma. \end{aligned} \tag{11}$$

From (10) and (11) we get $u = u^* = u_{\psi,K}$, and from the convergence of the L^2-norms of the gradients we get that

$$\nabla u_\varepsilon \longrightarrow \nabla u_{\psi,K} \ - \ strong\ L^2.$$

□

The result of Proposition 2.7 is still true if on $\partial\Omega$ one considers Dirichlet boundary conditions.

Proposition 2.8 [Existence of stream functions] *Let w and u be the solutions of (3) and (4) respectively. There exists holomorphic functions W and U in $\Omega \setminus K$ such that $w = \operatorname{Re} W$ and $u = \operatorname{Re} U$. Moreover, $\operatorname{Im} W$ and $\operatorname{Im} U$ solve problems (12) and (13) below, respectively. Let Ψ a primitive of ψ on $\partial\Omega$. The problem solved by $\operatorname{Im} W$ is*

$$\min\{\int_\Omega |\nabla\phi|^2 dx : \phi \in H^1_{cond,K}(\Omega), \phi = \Psi \ on\ \partial\Omega\} \tag{12}$$

and the problem solved by $\operatorname{Im} U$ is

$$\begin{cases} -\Delta\phi &= 0 \ in\ \ \Omega \setminus K \\ \frac{\partial\phi}{\partial n} &= 0 \ on\ \partial K \\ \phi &= \Psi \ on\ \partial\Omega \end{cases} \tag{13}$$

Proof For $\varepsilon > 0$, the results is true in $\Omega \setminus \overline{K}^\varepsilon$ by [1, 6]. Making $\varepsilon \to 0$, the result is true in $\Omega \setminus K$ as consequence of Propositions 2.5 and 2.7. □

3 Unique identifiability by two boundary measurements

In a first step we introduce a regularity notion, called *conductivity*, for a point of the boundary of an open set U; this kind of regularity should be rather related to the notion of Wiener regular point, than to the usual smoothness of the boundary.

In a second step, we prove that all sets which q.e. satisfy this regularity assumption on their boundaries are uniquely (up to a set of zero capacity) identifiable by two boundary measurements. The proof follows the same steps as in [1], and essentially is obtained by approximating these sets with sets with a finite number of connected components.

Definition 3.1 *Let K be a compact subset of Ω. A point $x \in K$ is a capacity point for K if $\forall r > 0$, $\operatorname{cap}(B_{x,r} \cap K) > 0$.*

Proposition 3.2 *The set K^* of capacity points of a compact set K is compact and* cap $(K \setminus K^*) = 0$.

Proof The compactness comes directly and the relation cap $(K \setminus K^*) = 0$ follows from the Lindelöf property and the sub-additivity of the capacity. □

Remark 3.3 From now on, every time we consider a compact set K, we replace it implicitly with K^*. Since cap $(K \setminus K^*) = 0$, problems (3) and (4) have the same solutions on $\Omega \setminus K$ and $\Omega \setminus K^*$, respectively. From a practical point of view, every time when an open set $U \subseteq \Omega$ is considered, it is replaced with $\Omega \setminus (\overline{\Omega} \setminus U)^*$.

Definition 3.4 *Let U be an open subset of Ω and $x \in \partial U$. We say that x is conductive for U if for every $r > 0$ and for every $\varphi \in C(\overline{U}) \cap H^1_{cond,\partial U \cap B_{x,r}}(\Omega)$*

$$\liminf_{\substack{y \to x \\ y \in \partial U}} \frac{|\varphi(y) - \varphi(x)|}{|y - x|} = 0. \tag{14}$$

Roughly speaking, x is a conductivity point if there exits a "path" of conductivity on ∂U passing through x and having locally positive capacity. Note that every conductivity point is a capacity point for U^c.

Proposition 3.5 *Let K be a compact subset of Ω such that $\Omega \setminus K$ is connected and F a continuum of positive diameter such that $x \in F \subseteq \partial(\Omega \setminus K)$. Then x is a conductivity point for $\Omega \setminus K$.*

Proof Let $\varphi \in C(\overline{\Omega \setminus K}) \cap H^1_{cond,\partial(\Omega \setminus K) \cap B_{x,r}}(\Omega)$. Then, φ is quasi everywhere constant on the continuum of positive diameter $\tilde{F}$ passing through x and contained in $F \cap B_{x,r}$. Indeed, if ϕ is a quasi-continuous representative of φ on Ω, then ϕ is finely continuous q.e. (see [16]) and coincides q.e. with φ on $\Omega \setminus K$. Since every point of $\partial(\Omega \setminus K)$ is thick with respect to $\Omega \setminus K$ (which is connected), we conclude that $\phi(x) = \varphi(x)$ q.e. on $\partial(\Omega \setminus K)$. Therefore φ is quasi-everywhere constant on $\tilde{F}$ and relation (14) holds. □

In the sequel we give two examples of the form $U = \Omega \setminus K$ with points $x \in \partial K$ which are disconnected from the rest of the set K; in one of them we show that such a point may be conductive despite the fact it is not contained in any continuum of positive diameter (subset of K).

Example 3.6 Let $\Omega = [-2, 2] \times [-2, 2]$ and

$$K = \{(0,0)\} \bigcup_{n \in \mathbb{N}^*} \frac{1}{n} \times \left[0, \frac{1}{n}\right].$$

Then $(0,0)$ is not a conductivity point for $\Omega \setminus K$; consider for example $u(x,y) = x$. Obviously $u \in C(\overline{\Omega})$ and it is easy to see (e.g. [5, 6]) that $u \in H^1_{cond,K}(\Omega)$ by approaching u strongly in $H^1(\Omega)$ by the sequence u_n of solutions of the following equations. Let $K_n = \bigcup_{k=1}^{n} \left[\frac{1}{k} -\right.$

$\frac{1}{n^2}, \frac{1}{k} + \frac{1}{n^2}\Big] \times \Big[0, \frac{1}{k}\Big]$ and let u_n solve $-\Delta u_n = 0$ in $\Omega \setminus K_n$, $u_n = u$ on $\partial\Omega$ and $u_n = \frac{1}{k}$ on $\Big[\frac{1}{k} - \frac{1}{n^2}, \frac{1}{k} + \frac{1}{n^2}\Big] \times \Big[0, \frac{1}{k}\Big]$.

On the other hand, (14) does not hold, since $\frac{|u(x,y)-u(0)|}{|(x,y)|} \geq \sqrt{2}^{-1}$, for every $(x, y) \in K$.

Example 3.7 Let now

$$K = \{(0,0)\} \bigcup_{n\in\mathbb{N}^*} \{b_n\} \times \Big[0, r_n\Big],$$

where

$$b_n = \sum_{k=n}^{\infty} \frac{1}{k^2(k+1)^3},$$

and

$$r_n = \frac{1}{n}.$$

Then $(0, 0)$ is a conductivity point for $\Omega \setminus K$ which is not contained in a continuum of positive capacity of K.

The proof needs some computation. We give it in the Appendix, at the end of the paper.

We also prove in the Appendix the following proposition, which is an extension of Proposition 3.5.

Proposition 3.8 *Let K be a compact subset of Ω such that $\Omega \setminus K$ is connected. Every $x \in \partial(\Omega \setminus K)$ for which there exists a continuum of positive diameter U_x such that $x \in U_x \subseteq K$, is a conductivity point for $\Omega \setminus K$.*

In particular, if K is a continuum of positive diameter, then $\Omega \setminus K$ is conductive at every point of $\partial(\Omega \setminus K)$. As well, if K is a compact set having a finite number of connected components, then $\Omega \setminus K$ is conductive at quasi every point of its boundary (except the isolated points).

Note that if K is a compact subset of Ω, the only detectable part of ∂K is the one contained in the boundary of the connected component of $\Omega \setminus K$ which touches $\partial\Omega$. For this reason, we shall assume (only) in the following theorem that $\Omega \setminus K$ is connected.

The fluxes we consider are defined as in [1]. Consider a division of $\partial\Omega$ into three connected disjoint parts $\Gamma_0, \Gamma_1, \Gamma_2$. For $i = 0, 1, 2$ we consider on $\partial\Omega$ a nonnegative function η_i such that $\text{supp}\, \eta_{\text{i}} \subseteq \Gamma_{\text{i}}$, $\eta_i \in L^2(\partial\Omega)$, $\int_{\partial\Omega} \eta_i = 0$. We take for $k = 1, 2$, $\psi_k = \eta_0 - \eta_k$.

Theorem 3.9 *Let $K, \tilde{K}$ be two compact subsets of Ω such that $\Omega \setminus K$, $\Omega \setminus \tilde{K}$ are connected. Let ψ_1, ψ_2 be two fluxes on $\partial\Omega$ chosen as above. Suppose that for $k = 1, 2$ either $w_{\psi_k,K|\partial\Omega} = w_{\psi_k,\tilde{K}|\partial\Omega}$, or $u_{\psi_k,K|\partial\Omega} = u_{\psi_k,\tilde{K}|\partial\Omega}$. If $\Omega \setminus K$ and $\Omega \setminus \tilde{K}$ are conductive at quasi every point of their boundaries, then $K = \tilde{K}$ q.e.*

Proof The proof relies on the non-existence of geometrical critical points for particular holomorphic functions. Let us first prove that if $K \neq \tilde{K}$ q.e., then we may find a geometrical critical point for the solution of (3) (or (4)) with a boundary data of the form $\alpha\psi_1 + \beta\psi_2$, for certain couple α, β which satisfies $\alpha^2 + \beta^2 = 1$. The proof follows the same lines as in [1], in the new hypotheses on the conductivity of the sets K and $\tilde{K}$. A new kind of difficulty appears, since the unique continuation property does not give information over all $\Omega \setminus (K \cup \tilde{K})$.

We shall consider only problem (3) (the case (4) follows the same ideas). For $k = 1, 2$, let w_k^*, $\tilde{w}_k^*$ be the conjugate functions of $w_k = w_{\psi_k, K}$, $\tilde{w}_k = w_{\psi_k, \tilde{K}}$, such that $w_k + i w_k^*$ are holomorphic in $\Omega \setminus K$ and $\tilde{w}_k + i\tilde{w}_k^*$ are holomorphic in $\Omega \setminus \tilde{K}$, respectively. Note that for the boundary condition $\alpha\psi_1 + \beta\psi_2$ the solution of (3) on $\Omega \setminus K$ is $\alpha w_1 + \beta w_2$ and that the harmonic conjugate of this function in $\Omega \setminus K$ is $\alpha w_1^* + \beta w_2^*$.

From the unique continuation property, we get as in [1] that $w_k = \tilde{w}_k$ on G, where G is the connected component of $\Omega \setminus (K \cup \tilde{K})$ satisfying $\partial\Omega \subseteq \partial G$ (the functions w_k and $\tilde{w}_k$ have the same Cauchy data on $\partial\Omega$). The main difficulty is that the information is not obtained over all $\Omega \setminus (K \cup \tilde{K})$. In [1], using the particular structure of K and $\tilde{K}$, the information could be extended in $\Omega \setminus (K \cup \tilde{K})$.

Let us suppose that $\Omega \setminus K \neq \Omega \setminus \tilde{K}$, say $\Omega \setminus K \not\subseteq \Omega \setminus \tilde{K}$. There exists $x \in \Omega \setminus K$ such that $x \notin \Omega \setminus \tilde{K}$ (i.e. $x \in \tilde{K}$). Since $\Omega \setminus K$ is connected and $\partial\Omega$ is smooth, there exists a smooth curve $\gamma : [0, 1] \to \overline{\Omega} \setminus K$ such that $\gamma(0) = x$, $\gamma((0, 1)) \subseteq \Omega \setminus K$, $\gamma(1) \in \partial\Omega$. Let $x_0 = \gamma(t_0)$, where

$$t_0 = \sup\{t \in [0, 1] : \gamma(t) \in \tilde{K}\}.$$

Obviously, $x_0 \in \partial\tilde{K}$ and also $x_0 \in \partial G$. Since $d(x_0, K) > 0$, there exists a ball $B_{x_0,r}$ such that $B_{x_0,r} \cap K = \emptyset$.

We prove in the sequel the following.

Lemma 3.10 *For every $\delta > 0$, G has a conductivity point on $\partial G \cap B_{x_0,\delta}$.*

Proof of Lemma 3.10. For every $\varepsilon > 0$ we consider the open set $\tilde{K}^\varepsilon$. There exists an open polygonal set V_ε such that

$$\tilde{K}^{\varepsilon/2} \subseteq V_\varepsilon \subseteq \tilde{K}^\varepsilon.$$

Let U_ε be the connected component of V_ε which contains x_0. Choosing a sequence (ε_n) such that $\varepsilon_n \to 0$, $\varepsilon_{n+1} < \varepsilon_n/2$ we get

$$U_{\varepsilon_{n+1}} \subseteq U_{\varepsilon_n}.$$

There are two possibilities:

1. diam $(U_{\varepsilon_n}) \to 0$;
2. diam $(U_{\varepsilon_n}) \to \eta > 0$.

The first case. Suppose that diam $(U_{\varepsilon_n}) \to 0$. For n large enough we have $U_{\varepsilon_n} \subseteq B_{x_0,r/2}$. Let A_n be the connected component of $\Omega \setminus \overline{U}_{\varepsilon_n}$ such that $\partial\Omega \subseteq \partial A_n$. Let $P_n = \partial A_n \setminus \partial\Omega$. Then P_n is a closed polygonal Jordan curve, which separates Ω in two regions. We observe that $P_n \subseteq \Omega \setminus (K \cup \tilde{K})$ because $P_n \cap K = \emptyset$ (since $P_n \subseteq \overline{B}_{x_0,r/2}$) and $P_n \cap \tilde{K} = \emptyset$ since ($P_n \subseteq \partial U_{\varepsilon_n}$ and $d(\partial U_{\varepsilon_n}, \tilde{K}) = \varepsilon_n/2$).

Since P_n intersects γ and γ lies in G, the connectedness of G implies that P_n is entirely in G. Therefore, for ξ small enough, we have that

$$G \cap B_{x_0,\xi} = (\Omega \setminus \tilde{K}) \cap B_{x_0,\xi}. \tag{15}$$

$$\partial G \cap B_{x_0,\xi} = \partial(\Omega \setminus \tilde{K}) \cap B_{x_0,\xi}. \tag{16}$$

In this case two possibilities may hold: either x_0 is a conductivity point or not. If it is not a conductivity point, we replace it by a close point of ∂G which is conductive. Such a point exists, since following [5, Lemma 4.5] x_0 is a capacity point also for ∂G, and the family of points of ∂G which are not conductive is, by hypothesis, of zero capacity (note that ∂G coincides locally with $\partial(\Omega \setminus \tilde{K})$).

The second case. Suppose that diam $(U_{\varepsilon_n}) \to \eta > 0$. We observe that $\bigcap_n U_{\varepsilon_n} = C$ where C is a continuum such that $x \in C \subseteq \tilde{K}$, diam $C = \eta$. Let $0 < \xi < \eta/2$. Then $C \cap \partial B_{x_0,\xi} \neq \emptyset$.

Denoting again by A_n the connected component of $\Omega \setminus \overline{U}_{\varepsilon_n}$ such that $\partial\Omega \subseteq \partial A_n$, let us set again $P_n = \partial A_n \setminus \partial\Omega$. Then P_n is polygonal Jordan curve satisfying $P_n \cap \tilde{K} = \emptyset$. Let us denote $z_n = \gamma(t_n)$, where

$$t_n = \min\{t \in [0,1], \gamma(t) \in P_n\}. \tag{17}$$

We observe that z_n is well defined and $z_n \to x_0$, for $n \to \infty$.

Since P_n contains in its "interior" region the continuum C and $P_n \cap \tilde{K} = \emptyset$ and $(P_n \cap B_{x_0,\xi}) \cap K = \emptyset$, there exists a connected component F_n of P_n passing through z_n which is contained in G and cuts $\partial B_{x_0,\xi}$ at least in two points. For $n \to \infty$, we have that F_n converges in the Hausdorff sense to a continuum F which contains x_0 and lies in the boundary of G. From Proposition 3.5 x_0 is a conductivity point for G.

□

Proof of Theorem 3.9 (continuation) Let x_0 be the conductive point given by Lemma 3.10. Up to translation by constants, we can assume that for $k = 1, 2$ $w_k(x_0) = w_k^*(x_0) = 0$. Note that the function $|\tilde{w}_1^*| + |\tilde{w}_2^*|$ belongs to $H^1_{cond,\tilde{K}}(\Omega)$ and equals $|w_1^*| + |w_2^*|$ on G. This last function is continuous in a neighbourhood of x_0, hence we can apply the conductivity property to $|w_1^*| + |w_2^*|$ in x_0.

There exists a sequence of points x_n such that $x_n \in \partial G$, $x_n \to x_0$ and

$$\frac{|w_1^*(x_n)| + |w_2^*(x_n)|}{|x_n - x_0|} \to 0. \tag{18}$$

hence, for for $k = 1, 2$

$$\frac{w_k^*(x_n)}{|x_n - x_0|} \to 0. \tag{19}$$

We suitably chose values α_n, β_n, such that $\alpha_n^2 + \beta_n^2 = 1$ and $\alpha_n w_1(x_n) + \beta_n w_2(x_n) = 0$. Choosing a subsequence of $(\alpha_n)_n, (\beta_n)_n$ such that $\alpha_n \to \alpha_0$, $\beta_n \to \beta_0$ and using relations (19) we have that the sequence of holomorphic functions

$$f_n = (\alpha_n w_1 + \beta_n w_2) + i(\alpha_n w_1^* + \beta_n w_2^*),$$

satisfy

$$\frac{f_n(x_n) - f_n(x_0)}{|x_n - x_0|} \to 0.$$

Consequently, x_0 is a geometrical critical point for $f_0 = (\alpha_0 w_1 + \beta_0 w_2) + i(\alpha_0 w_1^* + \beta_0 w_2^*)$. Indeed, we have for $n \to \infty$

$$\frac{f_0(x_n) - f_0(x_0)}{|x_n - x_0|}$$
$$= \frac{f_n(x_n) - f_n(x_0)}{|x_n - x_0|} - (\alpha_n - \alpha_0)\frac{w_1(x_n) + iw_1^*(x_n)}{|x_n - x_0|} - (\beta_n - \beta_0)\frac{w_2(x_n) + iw_2^*(x_n)}{|x_n - x_0|} \to 0.$$

The last two terms converge to zero thanks to the holomorphy in x_0 of the functions $w_k + iw_k^*$.

To get the contradiction we observe that f_0 can not have geometrical critical points in $\Omega \setminus K$. Indeed, this is a consequence of the result of [1] applied on $\Omega \setminus \overline{K}^{\varepsilon}$, by passing to the limit $K^{\varepsilon} \to K$ and using the continuity property of critical points.

□

The main difficulty in the proof of this theorem is the fact that the unique continuation property gives information only in the connected component of $\Omega \setminus (K \cup \tilde{K})$ touching $\partial\Omega$. In [1], this information is extended over all $\Omega \setminus (K \cup \tilde{K})$ by using the connectedness of the cracks. Here we are not able to do that, and for this reason we can use information only "on one side" of the crack $\tilde{K}$, namely on G. Since the conductivity hypothesis is formulated in $\Omega \setminus \tilde{K}$ and not in G, we are brought to discuss the two cases of Lemma 3.10.

Remark 3.11 The conductivity property is somehow related to the thickness property relying on the Wiener criterion. It would be of interest to characterise all sets which are q.e. conductive at the boundary, and in this way to characterise all detectable sets by two boundary measurements. Even an example of a compact set which is not conductive q.e. would be of interest.

Remark 3.12 Notice from relations (15)-(16) that the density of the conductive points of ∂G into ∂G is sufficient for carrying the proof. In the Appendix, we give an example of a Cantor set which is conductive in a dense set of its boundary points and consequently the unique identifiability for this totally disconnected set holds true (see Example 6.2).

4 Sequential stability of the inverse problems

Let ψ_1, ψ_2 be two fluxes on $\partial\Omega$ which uniquely identify q.e. conductive sets (Theorem 3.9) as well for problem (3) as for (4). For a compact set $K \subseteq \Omega$ and a sequence of compacts $(K_n)_n$ such that

$$w_{\psi_i, K_n|\partial\Omega} \overset{L^2(\partial\Omega)}{\rightharpoonup} w_{\psi_i, K|\partial\Omega}, \quad i = 1, 2$$

or

$$u_{\psi_i, K_n|\partial\Omega} \overset{L^2(\partial\Omega)}{\rightharpoonup} u_{\psi_i, K|\partial\Omega}, \quad i = 1, 2$$

we wonder if $K_n \overset{H}{\to} K$.

This assertion is in general false. First, the convergence in the Hausdorff metric does not have much in common with the behavior of the PDE on "moving" cracks. For this reason, it is not senseless to think stability in terms of *behavior*, i.e. two close measurements should give cracks such that *all* measurements are close. This approach is to be compared to the γ-convergence of sets (see [5]) which has a certain relation with the geometric convergence, but is not at all equivalent. This kind of approach seems necessary as soon as one deals with "wild" cracks without any *a priori* structure. Nevertheless, we restrict ourself to the Hausdorff metric because it seems quite difficult to describe the general behavior of sets. Note that for homogeneous Neumann boundary conditions the general behavior of the direct problem for moving domains is not, up to our knowledge, known.

Second, uniqueness holds for sets $\Omega \setminus K$ which are q.e. conductive with the convention that $\Omega \setminus K$ is connected. If $\Omega \setminus K$ is disconnected, the only identifiable part is the connected component "touching" $\partial\Omega$. From a purely geometric point of view, this means that different geometries for K may give similar measures. Here we explain what can happen, from the information we have, namely the coincidence of the identifiable connected components. Under mild assumptions on K, our result becomes a standard stability result.

In order to understand the sequential stability for the crack identification problem, the usual tool relies on the stability of the direct problem associated to compactness and uniqueness of the identification. Compactness is a geometric property of the Hausdorff convergence and for uniqueness we rely on Theorem 3.9. The geometric stability of the direct problems (3) and (4) relies on the Mosco convergence of the Sobolev spaces $H^1(\Omega \setminus K_n)$ for problem (3) and $H^1_{cond,K_n}(\Omega)$ for problem (4). Ultimately, because of the existence of harmonic conjugates for solutions of problem (3), the only important case to be studied is the Mosco convergence of $H^1_{cond,K_n}(\Omega)$ (see [6]).

Let X be a Hilbert space and $\{G_n\}_{n\in\mathbb{N}}$ a sequence of subsets of X. The weak upper and the strong lower limits in the sense of Kuratowski are defined as follows:

$$w - \limsup_{n\to\infty} G_n = \{u \in X : \exists \{n_k\}_k, \exists u_{n_k} \in G_{n_k} \text{ such that } u_{n_k} \stackrel{w-X}{\rightharpoonup} u\}$$

$$s - \liminf_{n\to\infty} G_n = \{u \in X : \exists u_n \in G_n \text{ such that } u_n \stackrel{s-X}{\longrightarrow} u\}$$

If $\{G_n\}_{n\in\mathbb{N}}$ are closed subspaces in X, it is said that G_n converges in the sense of Mosco to G if

$M_1)$ $G \subseteq s - \liminf_{n\to\infty} G_n$,

$M_2)$ $w - \limsup_{n\to\infty} G_n \subseteq G$.

Note that in general $s - \liminf_{n\to\infty} G_n \subseteq w - \limsup_{n\to\infty} G_n$. Therefore, if G_n converges in the sense of Mosco to G, then

$$s - \liminf_{n\to\infty} G_n = G = w - \limsup_{n\to\infty} G_n.$$

For our purposes, we consider the compact sets $K_n, K \subseteq \Omega$ and wonder if $G_n := H^1_{cond,K_n}(\Omega)$ converges in the sense of Mosco to $H^1_{cond,K}(\Omega)$ into the space $X := H^1(\Omega)$.

Assume in the sequel that $K_n \stackrel{H}{\to} K$. Then condition M_1 is immediately satisfied. Indeed, using density, it is enough to consider $u \in H^1_{cond,K}(\Omega)$ such that $\nabla u = 0$ a.e. on K^ε, for some

$\varepsilon > 0$. Following the Hausdorff convergence, for n large enough we have $K_n \subseteq K^{\frac{\varepsilon}{2}}$, hence $K_n^{\frac{\varepsilon}{2}} \subseteq K^\varepsilon$ and so $\nabla u = 0$ a.e. on $K_n^{\frac{\varepsilon}{2}}$, therefore $u \in H^1_{cond,K_n}(\Omega)$.

In general, condition M_2 is not true. Take for example $\Omega = (-2,2) \times (-2,2)$, $K_n = \cup_{k=0}^n \{\frac{k}{n}\} \times [0,1]$ and $u_n(x,y) = x$. A second example which typically impeaches M_2 to hold is when K_n consists on many small disconnected sets e.g. $K_n = \cup_{k,p=0}^n \overline{B}((\frac{k}{n}, \frac{p}{n}), \varepsilon_n)$, $\varepsilon_n > 0$, $\varepsilon_n \to 0$. Then

$$K_n \xrightarrow{H} [0,1] \times [0,1],$$

but for a suitable choice of ε_n every function of $H^1(\Omega)$ can be written as limit of a sequence of $H^1_{cond,K_n}(\Omega)$ (choose ε_n such that cap $(K_n) \to 0$).

Theorem 4.1 *Let $K_n, K \subseteq \Omega$, $K_n \xrightarrow{H} K$. If M_2 occurs, then for every $\psi \in L^2(\partial\Omega)$ such that $\int_{\partial\Omega} \frac{\partial\psi}{\partial n} d\sigma = 0$ we have*

1. $u_{K_n,\psi|\partial\Omega} \xrightarrow{L^2(\partial\Omega)} u_{K,\psi|\partial\Omega}$,

2. $w_{K_n,\psi|\partial\Omega} \xrightarrow{L^2(\partial\Omega)} w_{K,\psi|\partial\Omega}$.

Proof Let us prove assertion 1. For simplicity, we set $u_n = u_{K_n,\psi}$ and $u = u_{K,\psi}$.

As in Propositions 2.6, and 2.7 there exists a uniform bound M such that $\int_\Omega |\nabla u_n|^2 dx \leq M$. For a subsequence (still denoted using the same index), we can write $u_n \rightharpoonup \tilde{u}$ weakly - $H^1(\Omega)$. From the second Mosco condition, which is assumed by hypothesis, we get $\tilde{u} \in H^1_{cond,K}(\Omega)$. In order to prove that $\tilde{u} = u$, we observe that

$$\frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_{\partial\Omega} u\psi d\sigma \leq \frac{1}{2}\int_\Omega |\nabla \tilde{u}|^2 dx - \int_{\partial\Omega} \tilde{u}\psi d\sigma \leq \liminf_{n\to\infty} \frac{1}{2}\int_\Omega |\nabla u_n|^2 dx - \int_{\partial\Omega} u_n\psi d\sigma. \tag{20}$$

We also note that the first Mosco condition is a direct consequence of the geometric convergence $K_n \xrightarrow{H} K$. Indeed, for proving M_1 it is enough to consider $\phi \in H^1_{cond,K}(\Omega)$ such that $\nabla\phi = 0$ a.e. on K^ε, for some $\varepsilon > 0$ (this set is dense in $H^1_{cond,K}(\Omega)$). Indeed, for n large enough, such that $K_n \subseteq K^{\varepsilon/2}$ we get that $\nabla\phi = 0$ a.e. on $K_n^{\varepsilon/2}$, hence $\phi \in H^1_{cond,K_n}(\Omega)$.

Let $\phi_n \in H^1_{cond,K_n}(\Omega)$ such that $\phi_n \longrightarrow u$ in $H^1(\Omega)$-strong. We get

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{2}\int_\Omega |\nabla u_n|^2 dx - \int_{\partial\Omega} u_n\psi d\sigma &\leq \lim_{n\to\infty} \frac{1}{2}\int_\Omega |\nabla \phi_n|^2 dx - \int_{\partial\Omega} \phi_n\psi d\sigma \\ &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_{\partial\Omega} u\psi d\sigma. \end{aligned} \tag{21}$$

From (20) and (21) we get $\tilde{u} = u$ and the strong H^1 convergence $u_n \longrightarrow u$. The convergence $u_{K_n,\psi|\partial\Omega} \xrightarrow{L^2(\partial\Omega)} u_{K,\psi|\partial\Omega}$ follows from the trace theorem.

To prove the assertion 2. of the theorem, namely $w_{K_n,\psi|\partial\Omega} \xrightarrow{L^2(\partial\Omega)} w_{K,\psi|\partial\Omega}$, one has to use the following duality argument, which was already been applied in [6].

Following Proposition 2.8, let v_n be the conjugate function of $w_{K_n,\psi|\partial\Omega}$ in $\Omega \setminus K_n$. Then v_n solves the following problem (set as an energy minimization)

$$\min\{\int_\Omega |\nabla v|^2 dx : v \in H^1_{cond,K_n}(\Omega), v = \Psi \text{ } on \text{ } \partial\Omega\},$$

where Ψ is a primitive of ψ on $\partial\Omega$.

Note that v_n solves a problem very similar to (9), with the only difference that on the fixed boundary $\partial\Omega$ the Neumann condition is replaced with a Dirichlet one. The same proof as for the first point of this theorem can be repeated. We get

$$\nabla v_n \overset{L^2(\Omega,\mathbb{R}^2)}{\longrightarrow} \nabla v$$

or

$$\nabla v_n 1_{\Omega\setminus K_n} \overset{L^2(\Omega,\mathbb{R}^2)}{\longrightarrow} \nabla v 1_{\Omega\setminus K},$$

since $\nabla v_n = 0$ a.e. on K_n. In terms of conjugate functions, this gives

$$\nabla w_{K_n,\psi|\partial\Omega} 1_{\Omega\setminus K_n} \overset{L^2(\Omega,\mathbb{R}^2)}{\longrightarrow} w_{K,\psi|\partial\Omega} 1_{\Omega\setminus K}.$$

Applying the trace theorem for $w_{K_n,\psi|\partial\Omega}$ into a smooth neighbourhood U of $\partial\Omega$ we get $w_{K_n,\psi|\partial\Omega} \overset{L^2(\partial\Omega)}{\longrightarrow} w_{K,\psi|\partial\Omega}$. □

In the next proposition we prove that condition M_2 is satisfied, provided that the number of connected components of K_n and K are uniformly bounded (we denote by $\sharp K$ the number of connected compoenents of K).

Proposition 4.2 *Let $K \subseteq \Omega, K_n \overset{H}{\longrightarrow} K$, $\sharp K_n \leq M$. Then the second Mosco condition holds for $H^1_{cond,K_n}(\Omega)$ and $H^1_{cond,K}(\Omega)$.*

Proof Let $\phi_n \in H^1_{cond,K_n}(\Omega)$ such that $\phi_n \rightharpoonup \phi$ in $H^1(\Omega)$-weak. Let K_α be a connected component of K. We observe first that ϕ is q.e.-constant on K_α. Indeed, from the Hausdorff convergence, K_α can be written $K_\alpha = \bigcup_{i=1}^{M} K_\alpha^i$, where (up to subsequences) $K_\alpha^i = H - \lim_{n\to\infty} K_n^i$, where K_n^i are connected components of K_n. In our notation, some of these components can be chosen empty sets. Following Lemma 2.3 (see also [19] and [6]) we get ϕ q.e.-constant on K_α^i and following Lemma 2.1 we get ϕ q.e.-constant on K_α.

In order to prove that $\phi \in H^1_{cond,K}(\Omega)$ we use Hedberg's result [13], which asserts that ϕ can be approached strongly in $H^1(\Omega)$ by a sequence of functions which, for every α are constant q.e. (hence a.e.) on a neighbourhood of K_α. □

In the sequel we give several situations when stability occurs. To simplify the notation, for every compact $K \subseteq \Omega$ we denote G_K the connected component of $\Omega \setminus K$ which "touches" $\partial\Omega$.

Theorem 4.3 *Let ψ_1, ψ_2 be as in Theorem 3.9. Suppose F is a compact subset of Ω and that (K_n) is a family of compact subsets of F such that*

$$\exists M > 0 \quad \forall n \in \mathbb{N} \quad \sharp K_n \leq M.$$

If either

$$u_{K_n,\psi_i|\partial\Omega} \overset{L^2(\partial\Omega)}{\longrightarrow} u_i \quad i = 1, 2,$$

or

$$w_{K_n,\psi|\partial\Omega} \overset{L^2(\partial\Omega)}{\longrightarrow} w_i \quad i = 1, 2$$

holds, then there exists a compact set $K \subseteq \Omega$ such that $\sharp K \leq M$ and a subsequence $K_{n_k} \overset{H}{\to} K$ and for $i = 1, 2$ $u_i = u_{\psi_i,K|\partial\Omega}$ (respectively $w_i = w_{\psi_i,K|\partial\Omega}$).

If for another subsequence, we have $K_{n'_k} \overset{H}{\to} \tilde{K}$, then $G_K = G_{\tilde{K}}$ q.e.

Proof By the compactness of the Hausdorff convergence we can write $K_{n_k} \overset{H}{\longrightarrow} K$, with $K \subseteq F$, and $\sharp K \leq M$. Using Theorem 4.1 and Proposition 4.2 we get $u_{K_{n_k},\psi_i|\partial\Omega} \overset{L^2(\partial\Omega)}{\longrightarrow} u_{K,\psi_i|\partial\Omega}$ (and the same for w). Hence $u_i = u_{K,\psi_i|\partial\Omega}$ (and the same for w).

If for another subsequence, we have $K_{n'_k} \overset{H}{\to} \tilde{K}$, we use the uniqueness Theorem 3.9 and get the conclusion. □

Remark 4.4 Theorem 4.3 is not a standard stability result, as one might expect. It is rather a description of possible situations regarding stability.

Nevertheless, under mild assumptions on K, this becomes a usual sequential stability result.

In the following, K, K_n and $\tilde{K}$ are as in Theorem 4.3.

Corollary 4.5 *Let K be such that $\sharp K^* = M$, $\Omega \setminus K$ is connected and $\overset{\circ}{K} = \emptyset$. Then $\tilde{K} = K$ and the hole sequence K_n converges into the Hausdorff metric to K.*

Proof From Theorem 4.3

$$G_{\tilde{K}} = G_K = \Omega \setminus K \quad q.e. \tag{22}$$

Moreover, thanks to the hypothesis $\sharp K^* = M$, K has M connected components and each one has positive diameter. Consequently, equality (22) holds everywhere. From the definition of $G_{\tilde{K}}$, relation (22) implies $\tilde{K} \subseteq K$. The converse is also true since $K = \partial K = \partial G_{\tilde{K}} \setminus \partial\Omega \subseteq \partial\tilde{K} \subseteq \tilde{K}$. □

Example 4.6 In order to give geometric intuition on the sense in which Theorem 4.3 should be understood, we give in Figures 1 and 2 two examples of cracks and cavities which give close measurements.

We give an example where stability comes from the structure of K and requires that all K_n satisfy a uniform identifiability assumption. From a practical point of view this result might be helpful if all cracks do not have interior points, and locally their diameter are beyond a detectability level.

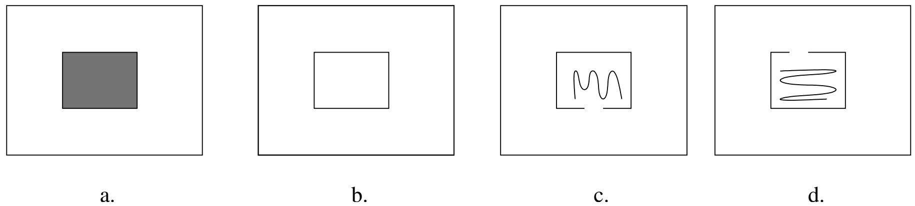

Figure 1: Four compacts giving “close” measures: cavity a. gives the same measure as crack b.; asymptotically, cracks c. and d. give the same measures (as soon as the apertures of the rectangular cracks go to zero).

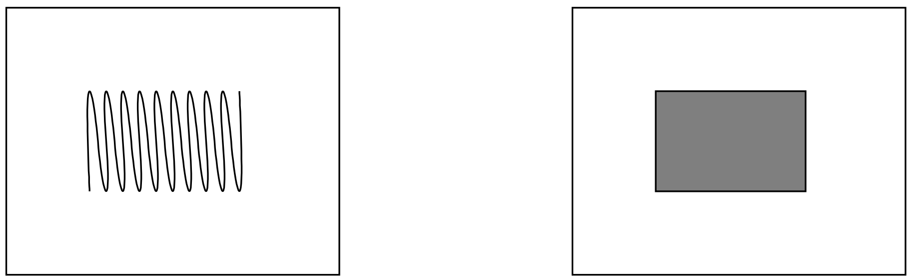

Figure 2: Two compacts giving “close” measures; on the left a “long and dense” curve and on the right a cavity

Definition 4.7 *Let $\varepsilon > 0$. A compact set K is ε-detectable if for every $x \in K$, the diameter of the connected component of K containing x is greater than or equal to ε. A compact set $K \subseteq \Omega$ is called ε-stable if*

$$H^1_{cond,K}(\Omega) = \{u \in H^1(\Omega), \forall x \in K, \exists U_x \text{ continuum, diam } (U_x) \geq \varepsilon \text{ s. t. } u = c_x \text{ q.e. on } U_x\}. \tag{23}$$

To simplify, let us denote the space on the right hand side $H^1_\varepsilon(\Omega)$. Notice that if K is arbitrary equality (23) does not occur; e.g. if K has an interior point. Take for example $\Omega = [-2, 2] \times [-2, 2]$, $K = [0, 1] \times [0, 1]$ and $u(x, y) = x$.

An example of ε-stable K is $K = \left\{ \bigcup_{n=1}^{\infty} \{\frac{1}{n}\} \times [0, 1] \right\} \bigcup \{0\} \times [0, 1]$. Indeed, continua on lines are intervals. Hence on each vertical segment a function $u \in H^1_\varepsilon(\Omega)$ can take only a finite number of values. Using Lemma 2.1 it follows that u is q.e. constant on each vertical segment. Using the same argument as in Example 3.6 we get $u \in H^1_{cond,K}(\Omega)$.

Proposition 4.8 *Suppose there exists $\varepsilon > 0$ such that K_n are ε-detectable. The conclusion of Theorem 4.3 holds provided that K and $\tilde{K}$ are ε-stable.*

5 Application: approximation by finite elements

We prove in this section that the unknown defects can be formally approached using finite elements, regardless their regularity. Basicly, this is one of the main applications of the stability result established in the previous section. All previous stability results, which give finer estimates for the stability, assume *a priori* the smoothness of the defects and suppose known their (uniform) Lipschitz character. With this regard, Theorem 4.3 does not give a quantitative estimate for the stability, but provides a rigorous justification of the approach by finite elements. For a similar argument related to shape optimization problems with homogeneous Dirichlet conditions on the free boundaries, we refer to [8].

We discuss both problems (3) and (4). It will be quite surprising to notice that, formally, problem (4) is easier to treat from a numerical point of view, since a unique mesh can be used at each step for both capturing the defect Γ and to compute the finite element approximation of the solution. This is also the case for homogeneous Dirichlet problems [8]. For problem (3) the defect Γ is captured on a mesh while the finite element approximation of the solution needs a refinement of the mesh. This is precisely what is done in practice.

Let F be "the design region", i.e. a subdomain of Ω containing all defects. Let $(\mathcal{T}_h)_h$ denote a family of triangulations of Ω made of elements which are triangles (the extension to quadrilaterals is standard). The maximal size of elements is the discretization parameter, denoted by h. In addition, we assume that each triangulation satisfies the usual admissibility assumptions, i.e., the intersection of two different elements is either empty, a vertex, or a whole edge, and $\mathcal{T}_h$ is assumed to be "regular", i.e., the ratio between the diameter of any element $K \in \mathcal{T}_h$ and the diameter of its largest inscribed ball is bounded by a constant σ independent of K and h.

Let $K^* \subseteq F$ be a defect such that $\sharp K^* \leq M$ which gives the measures w_1, w_2 corresponding to the input fluxes, ψ_1, ψ_2, respectively. We solve the finite dimensional problem

$$\min_{\substack{K \subset \mathcal{T}_h \cap F \\ \#K \leq M}} \int_{\partial\Omega} |w_{K,\psi_1} - w_1|^2 \, d\sigma + \int_{\partial\Omega} |w_{K,\psi_2} - w_2|^2 \, d\sigma, \tag{24}$$

which admits at least one solution, denoted K_h. The following convergence result holds.

Theorem 5.1 *For $h \to 0$, there exits a subsequence such that*

$$K_h \stackrel{H}{\to} \tilde{K} \quad \text{and } G_{K^*} = G_{\tilde{K}}.$$

Proof By compactness we can extract a subsequence $K_h \stackrel{H}{\to} \tilde{K}$. First, we notice that

$$\int_{\partial\Omega} |w_{K_h,\psi_1} - w_1|^2 + \int_{\partial\Omega} |w_{K_h,\psi_2} - w_2|^2 \longrightarrow 0, \text{ as } h \longrightarrow 0. \tag{25}$$

Indeed, we define

$$K_h^* = \bigcup_{\substack{T \in \mathcal{T}_h \cap F \\ \overline{T} \cap K^* \neq \emptyset}} \overline{T}. \tag{26}$$

Then, $d(K_h^*, K^*) \leq h$, $\#K_h^* \leq M$ and $K_h^* \subset F$. Moreover, $K_h^* \stackrel{H}{\to} K^*$ and following the stability result for the direct problem [6], we have

$$\int_{\partial\Omega} |w_{K_h,\psi_1} - w_1|^2 + \int_{\partial\Omega} |w_{K_h,\psi_2} - w_2|^2 \longrightarrow 0, \text{ as } h \longrightarrow 0.$$

By the choice of K_h in (24) we get (25). Second, since (25) holds, we use Theorem 4.3 and get $G_{K^*} = G_{\tilde{K}}$ which means that K_h is an approximation of K^*. □

Remark 5.2 Notice that in the least square approximation (problem (24)), the continuous solutions $w_{K,\psi_1}, w_{K,\psi_2}$ are chosen to be compared to the measures w_1, w_2. In practice, instead of w_{K_h,ψ_i}, we use a finite element approximation, say $w^j_{K_h,\psi_i}$, obtained on finer mesh. This approximation can be chosen such that $\|w^j_{K_h,\psi_i} - w_{K_h,\psi_i}\|_{L^2(\partial\Omega)} \leq j,\ j < h$. Consequently, as h goes to zero, the result of Theorem 5.1 still holds.

In the sequel we consider the approximation problem for the perfectly conducting case. Let $K^* \subseteq F$ be a defect such that $\sharp K^* \leq M$ which gives the measures u_1, u_2 corresponding to the input fluxes, ψ_1, ψ_2, respectively. We solve the following finite dimensional problem:

$$\min_{\substack{K \subset \mathcal{T}_h \cap F \\ \#K \leq M}} \int_{\partial\Omega} |u^h_{K,\psi_1} - u_1|^2 \, d\sigma \, + \int_{\partial\Omega} |u^h_{K,\psi_2} - u_2|^2 \, d\sigma. \tag{27}$$

Theorem 5.3 *For $h \longrightarrow 0$, there exits a subsequence such that*

$$K_h \overset{H}{\to} \tilde{K} \quad \textit{and } G_{K^*} = G_{\tilde{K}}.$$

Proof By compactness, we can extract a subsequence $K_h \overset{H}{\to} \tilde{K}$. We prove that for the continuous solutions we have

$$\int_{\partial\Omega} |u_{K_h,\psi_1} - u_1|^2 \, d\sigma \, + \int_{\partial\Omega} |u_{K_h,\psi_2} - u_2|^2 \, d\sigma \longrightarrow 0, \text{ as } h \longrightarrow 0. \tag{28}$$

For $i = 1, 2$ we have that

$$\int_{\partial\Omega} |u_{K_h,\psi_i} - u_i|^2 \, d\sigma \leq 2\big(\int_{\partial\Omega} |u^h_{K_h,\psi_i} - u_i|^2 \, d\sigma + \int_{\partial\Omega} |u_{K_h,\psi_i} - u^h_{K_h,\psi_i}|^2 \, d\sigma\big).$$

We construct K_h^* as in (26). Then, for $i = 1, 2$

$$\begin{aligned}\int_{\partial\Omega} |u^h_{K_h,\psi_i} - u_i|^2 \, d\sigma &\leq \int_{\partial\Omega} |u^h_{K_h^*,\psi_i} - u_i|^2 \, d\sigma \\ &\leq 2\big(\int_{\partial\Omega} |u_{K_h^*,\psi_i} - u_i|^2 \, d\sigma + \int_{\partial\Omega} |u^h_{K_h^*,\psi_i} - u_{K_h^*,\psi_i}|^2 \, d\sigma.\end{aligned}$$

From the stability of the direct problem (see [6]) we get

$$\int_{\partial\Omega} |u_{K_h^*,\psi_i} - u_i|^2 \, d\sigma \longrightarrow 0, \text{ as } h \longrightarrow 0.$$

In order to get (28) we have to prove that

$$\int_{\partial\Omega} |u_{K_h,\psi_i} - u^h_{K_h,\psi_i}|^2 \, d\sigma + \int_{\partial\Omega} |u_{K_h^*,\psi_i} - u^h_{K_h^*,\psi_i}|^2 \, d\sigma \longrightarrow 0, \text{ as } h \longrightarrow 0.$$

In fact, it is enough to prove that if

$$K_h \overset{H}{\to} \tilde{K},\ \#K_h \le M,\ K_h \subset F,$$

then, $u^h_{K_h,\psi_i} \overset{L^2(\partial\Omega)}{\to} u_{\tilde{K},\psi_i}$. This is a consequence of the Mosco convergence of the spaces

$$V_h = \{u \in C(\overline{\Omega}),\ u \in P_1(T), \forall T \in \mathcal{T}_h,\\ u = \text{ constant on each connected component of } K_h\}$$

to $H^1_{cond,\tilde{K}}(\Omega)$.

Indeed, let $v_h \in V_h$, $v_h \overset{H^1(\Omega)}{\rightharpoonup} u$. Following [6], $u \in H^1(\Omega)$, u is constant on each connected component of $\tilde{K}$, hence $u \in H^1_{cond,\tilde{K}}(\Omega)$. Let now $u \in H^1_{cond,\tilde{K}}(\Omega)$. Applying Hedberg's result [13] locally in a neighbourhood of each connected component of $\tilde{K}$, for every $\epsilon > 0$, there exists $\delta > 0$ and $u_\delta \in H^1_{cond,\overline{\tilde{K}^\delta}}(\Omega) \cap C^\infty(\overline{\Omega})$, such that $|u_\delta - u|_{H^1(\Omega)} < \epsilon$. Then, $u_\delta \in H^1_{cond,K_h}(\Omega)$ for h small enough. Thus, for the finite element approximation $u^h_\delta \in V_h$ we have the error estimate $|u_\delta - u^h_\delta| \le h\|u_\delta\|_{H^2(\Omega)}$. By a diagonal procedure, we construct $u^h_{\delta_h} \in V_h$, and $u^h_{\delta_h} \overset{H^1(\Omega)}{\to} u$ □

Remark 5.4 In Theorem 5.3, we have the approximation of $u_{K^*,\psi}$ obtained on $\mathcal{T}_h$, which means that no refinement is necessary. This is mainly due to the Dirichlet type boundary conditions which are easier to handle than the Neumann ones.

6 Appendix

Proof of Example 3.7. We start with the following simple result.

Lemma 6.1 *Let $(c_n)_n$ and $(r_n)_n$ be two sequences of real numbers such that for all n, $0 < r_n \le c_n$, $(r_n)_n$ is decreasing and $(c_n)_n$ converges to zero. Then, there exists a subsequence $(c_{n_k})_{n_k}$, such that: $|c_{n_k} - c_{n_k+1}| \ge \frac{r_{n_k} - r_{n_k+1}}{2}$.*

Proof Assume for contradiction that there exists n_0 such that

$$\forall n \ge n_0, \quad |c_n - c_{n+1}| < \frac{r_n - r_{n+1}}{2}.$$

Then, for all $k > n_0$, we have

$$|c_{n_0} - c_k| < \frac{r_{n_0} - r_k}{2},$$

which yields when k goes to $+\infty$: $c_{n_0} \le \frac{r_{n_0}}{2}$, in contradiction with the hypothesis of the lemma. □

Let now choose r_n as in Example 3.7. Then $\overline{\Omega \setminus K} = \overline{\Omega}$ and take $\phi \in C(\overline{\Omega}) \cap H^1_{cond,K\cap B_{0,r}}(\Omega)$ such that $\phi(0) = 0$. Suppose for contradiction that 0 is not conductive. Then, there exists $C, \delta > 0$ such that $\phi(x) \ge C|x|$ for $x \in K \cap B_{0,\delta}$. Since $\phi \in H^1_{cond,K\cap B_{0,r}}(\Omega)$ we have that ϕ is constant on every vertical line. Let us denote c_n this constant. Writing $\phi(b_n, r_n) \geq C\sqrt{b_n^2 + r_n^2}$ we get that $c_n \geq \sqrt{b_n^2 + r_n^2} \geq r_n$.

We prove in the sequel that the gradient of ϕ has infinite L^2-norm, and this will contradict the hypothesis $\phi \in H^1_{cond,K\cap B_{0,r}}(\Omega)$. We have the following

$$\begin{aligned}\int_\Omega |\nabla\phi|^2 dx &\geq \sum_{n_0}^{\infty} \int_{[b_{n+1},b_n]\times r_{n+1}} |\nabla\phi|^2 dx \\ &\geq \sum_{n_0}^{\infty} \int_{[b_{n+1},b_n]\times r_{n+1}} \Big(\frac{c_{n+1}-c_n}{b_n - b_{n+1}}\Big)^2 dx \\ &\geq \sum_{n_0}^{\infty} \frac{r_{n+1}}{b_n - b_{n+1}} (c_{n+1} - c_n)^2.\end{aligned}$$

Using Lemma 6.1, there exists a subsequence such that

$$\frac{r_{n_k+1}}{b_{n_k} - b_{n_k+1}} (c_{n_k+1} - c_{n_k})^2 \geq \frac{r_{n_k+1}}{4(b_{n_k} - b_{n_k+1})} (r_{n_k+1} - r_{n_k})^2.$$

We observe from the definition of b_n and r_n, that $b_n - b_{n+1} = r_{n+1}(r_{n+1} - r_n)^2$, therefore the series above can not converge since its general term does not converge to zero.

Proof of Proposition 3.8.

Let K be a compact subset of Ω such that $\Omega \setminus K$ is connected, and let $x \in \partial(\Omega \setminus K)$ such that $x \in U_x \subseteq K$, where U_x is a continuum of positive diameter. Following Proposition 3.5, in order to prove that x is conductive for $\Omega \setminus K$ it is enough to prove the existence of a continuum of positive diameter F such that $x \in F \subseteq \partial(\Omega \setminus K)$.

One can mimic the proof of the second case in the proof of Theorem 3.9. The only difference is that a curve γ joining x to $\partial\Omega$ and lying in $\Omega \setminus K$ may not exists. Nevertheless, there exists a sequence of points $y_n \in \Omega \setminus K$, $y_n \to x$ and smooth curves γ_n joining y_n to a point of the $\partial\Omega$ and lying in $\Omega \setminus K$. Choosing ε_n as in Theorem 3.9 and choosing y_n such that $|x - y_n| < \varepsilon_n/2$ we define

$$t_n = \min\{t \in [0,1], \gamma_n(t) \in P_n\}. \tag{29}$$

We observe that z_n is well defined, but we do not have necessarily $z_n \to x$. Nevertheless, $\gamma_n([0, t_n])$ is a continuum containing y_n and z_n, and lying in $(\Omega \setminus K) \cap \overline{K}^{\varepsilon_n}$. Two possibilities occur: either, for a subsequence $z_n \to x$ and we apply the same argument as in Theorem 3.9, or $|z_n - x| \geq \alpha > 0$ and in this case any Hausdorff limit of $\gamma_n([0, t_n])$ is a continuum of diameter greater than or equal to α contained in $\partial(\Omega \setminus K)$ and passing through x.

Example 6.2 Example of a Cantor set which is uniquely identifiable by two boundary measurements. Let $\Omega = B(0, 2)$ and define

$$F_1 = [0, 1] \times \{0\},$$

$$F_2 = \Big\{[0, \frac{1}{2} - \varepsilon_1] \cup [\frac{1}{2} + \varepsilon_1, 1]\Big\} \times \{0\},$$

$$F_3 =$$

$\{[0,\frac{1}{2}(\frac{1}{2}-\varepsilon_1)-\varepsilon_2]\cup[\frac{1}{2}(\frac{1}{2}-\varepsilon_1)+\varepsilon_2,\frac{1}{2}-\varepsilon_1]\cup[\frac{1}{2}+\varepsilon_1,\frac{1}{2}(\frac{1}{2}+\varepsilon_1+1)-\varepsilon_2]\cup[\frac{1}{2}(\frac{1}{2}+\varepsilon_1+1)+\varepsilon_2,1]\}\times\{0\}$, etc. Let $(c_n)_n$ be an increasing sequence of positive numbers converging to 1. The value of ε_1 is chosen such that $\forall\varphi\in H^1_{cond,F_1}(\Omega\setminus F_1)\cap C(\overline{\Omega})$, $\varphi(0,0)=0, \int_\Omega|\nabla\varphi|^2dx\le 1$ we have for every $t\in[0,1]$ $\int_0^t\varphi^2(s,0)ds\le c_1t^4$. Such a constant ε_1 exists, if not for a sequence φ_k corresponding to $\varepsilon_1^k\to 0$ we would have

$$\int_0^{t_k}\varphi_k^2(s,0)ds>c_1t_k^4.$$

Assuming $t_k\to t$, we clearly have $t\ge 1/2$ and for the limit function we get (by the continuity of the trace operator) $\int_0^t\varphi^2(s,0)ds\ge c_1t^4$. But from Lemma 2.1 we have $\varphi=0$ on $[0,1]\times\{0\}$, hence we get contradiction.

Note that ε_1 can be chosen such that

$$\forall\varphi\in H^1_{cond,F_1}(\Omega\setminus F_1)\cap C(\overline{\Omega}),\ \varphi(1,0)=0,\int_\Omega|\nabla\varphi|^2dx\le 1 \tag{30}$$

we have

$$\forall t\in[0,1],\int_t^1\varphi^2(s,0)ds\le c_1t^4. \tag{31}$$

As well, we define $\varepsilon_2>0$ such that $\forall\varphi\in H^1_{cond,F_1}(\Omega\setminus F_1)\cap C(\Omega)$, $\varphi(0,0)=0, \int_\Omega|\nabla\varphi|^2dx\le 1$ we have for every $t\in[0,1]$ $\int_0^t\varphi^2(s,0)ds\le c_2t^4$. Note that ε_2 can be chosen such that similar relations as in (30)-(31) holds for the points $(1/2-\varepsilon_1,0)$, $(1,0)$ in the "left" direction and for $(1/2+\varepsilon_1,0)$ in the "right" direction.

By induction, we define F_n and set $F=\cap_{n\in\mathbb{N}}F_n$, which is a totally disconnected Cantor set. Since $\cup_{n\in\mathbb{N}}\partial F_n$ is dense in F, it is enough to prove that F is conductive at every point of ∂F_n. So fix n and chose $x_0\in\partial F_n$. There are two possibilities: either x_0 is a left end point of an interval of F_n or a right end point. Thanks to the construction of ε_k, both situations are treated in the same way. If it is a left end point, the proof is similar to the conductivity of 0, that we give in the sequel.

Let $u\in H^1_{cond,F}(\Omega\setminus F)\cap C(\overline{\Omega})$. There exists $\varphi_\varepsilon\in H^1(\Omega)$ such that $\nabla\varphi_\varepsilon=0$ on F^ε, $\varphi_\varepsilon\to u$ in $H^1(\Omega)$. Moreover, the functions φ_ε can be chosen continuous and maybe translated by a constant such that $\varphi_\varepsilon(0)=0$. It is clear that, even after translations, $\nabla\varphi_\varepsilon\to\nabla u$ strongly in L^2. Since for every $\varepsilon>0$ we have for n large enough $F_n\subseteq F^\varepsilon$ we have from the previous construction that for every $t\in[0,1]$ $\int_0^t\varphi_\varepsilon^2(s,0)ds\le(M+1)t^4$, where $M=\limsup_{n\to\infty}\int_\Omega|\nabla\varphi_n|^2dx$. This implies (for a subsequence) that φ_ε converges weakly in $H^1(\Omega)$ to $\tilde{u}=u+c$, where c is a constant. The continuity of the trace operator gives

$$\forall t\in[0,1],\ \int_0^t\tilde{u}^2(s,0)ds\le(M+1)t^4, \tag{32}$$

and the continuity of $\tilde{u}$ implies $\tilde{u}(0,0)=0$, hence $c=0$, thus u satisfies (32). Consequently $\liminf_{t\to 0}\frac{|u(t,0)|}{t}=0$ hence $(0,0)$ is conductive, otherwise $|u(t,0)|\ge c|t|$ in a neighbourhood of 0, which is in contradiction with (32).

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