A new isoperimetric inequality for the elasticae

Dorin Bucur Antoine Henrot*

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Abstract

For a smooth curve γ, we define its elastic energy as $E(\gamma) = \frac{1}{2}\int_\gamma k^2(s)ds$ where $k(s)$ is the curvature. The main purpose of the paper is to prove that among all smooth, simply connected, bounded open sets of prescribed area in $\mathbb{R}^2$, the disc has the boundary with the least elastic energy. In other words, for any bounded simply connected domain Ω, the following isoperimetric inequality holds: $E^2(\partial\Omega)A(\Omega) \geq \pi^3$. The analysis relies on the minimization of the elastic energy of drops enclosing a prescribed area, for which we give as well an analytic answer.

1 Introduction

Let Ω be a smooth, bounded simply connected open set in the plane (the exact smoothness which is required will be made precise in Section 2) and let us denote by $\partial\Omega$ its boundary. Following L. Euler, we define its elastic energy as

$$E(\partial\Omega) = \frac{1}{2}\int_{\partial\Omega} k^2(s)ds \tag{1}$$

where s is the arc length parameter and k is the curvature. We will denote by $A(\Omega)$ the area of Ω and $L(\Omega)$ its perimeter. The aim of this paper is to prove the following isoperimetric inequality.

Theorem 1.1 *For any bounded, smooth, simply connected, nonempty open set $\Omega \subseteq \mathbb{R}^2$*

$$E^2(\partial\Omega)A(\Omega) \geq \pi^3 \tag{2}$$

where equality holds only for the disc.

Since for any disc B_R we have $E^2(\partial B_R)A(B_R) = \pi^3$, we deduce that for every $A_0 > 0$, the disc is the unique solution for the minimization problem

$$\min\{E(\partial\Omega) : A(\Omega) = A_0, \Omega \text{ bounded, smooth, simply connected open set of } \mathbb{R}^2\}.$$

More precisely, if we perform any scaling of ratio t, we have $E(t\partial\Omega) = t^{-1}E(\partial\Omega)$ and $A(t\partial\Omega) = t^2A(\partial\Omega)$. Therefore, it is classical to prove that the following three minimization problems are equivalent (in the sense that any solution of one gives a solution of the others after a suitable scaling):

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(i) $\min E^2(\partial\Omega)A(\Omega)$

(ii) $\min\{E(\partial\Omega) : A(\Omega) \le A_0\}$

(iii) $\min E(\partial\Omega) + A(\Omega)$

Let us make some comments. For a detailed bibliography on closed elasticae, we refer to the classical [8] or the more recent [9]. Inequality (2) was already known for convex domains. Indeed, by a famous inequality due to M. Gage [6], for any bounded convex domain

$$\frac{E(\partial\Omega)A(\Omega)}{L(\Omega)} \ge \frac{\pi}{2}$$

with equality for the disc. Therefore,

$$E^2(\partial\Omega)A(\Omega) \ge E^2(\partial\Omega)A(\Omega)\frac{4\pi A(\Omega)}{L^2(\Omega)} \ge \frac{\pi^2}{4} \times 4\pi = \pi^3,$$

the first inequality being the classical isoperimetric inequality, and the second the Gage inequality. If the convexity hypothesis is dropped, then the Gage inequality is false (as shown by the counterexample of Figure 1).

The simple connectedness assumption is necessary. Indeed, if we take as a domain Ω the ring

$$\Omega_R = \{(x,y) : R < \sqrt{x^2+y^2} < R + \frac{1}{R}\},$$

we get $E(\partial\Omega_R) = \frac{\pi}{R} + \frac{\pi R}{R^2+1}$, while

$$A(\Omega_R) = \pi(R + \frac{1}{R})^2 - \pi R^2 = 2\pi + \frac{\pi}{R^2}$$

showing that $E^2(\partial\Omega_R)A(\Omega_R) \to 0$ when $R \to +\infty$.

In the same way, the boundedness assumption is also necessary. Let us consider the following unbounded domain, subgraph of a Gaussian function, but with finite area and elastic energy:

$$\Omega_\alpha = \{(x,y) \in \mathbb{R}^2 : -\infty < x < +\infty, 0 < y < e^{-\alpha x^2/2}\}.$$

We have

$$A(\Omega_\alpha) = \int_{-\infty}^{+\infty} e^{-\alpha x^2/2} dx = \sqrt{\frac{2\pi}{\alpha}},$$

while

$$E(\partial\Omega_\alpha) = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{(\alpha^2x^2-\alpha)^2 e^{-\alpha x^2}}{(1+\alpha^2x^2e^{-\alpha x^2})^{\frac{5}{2}}} dx = \frac{\alpha^{\frac{3}{2}}}{2}\int_{-\infty}^{+\infty} \frac{(u^2-1)^2 e^{-u^2}}{(1+\alpha u^2 e^{-u^2})^{\frac{5}{2}}} du,$$

and we see that $E^2(\partial\Omega_\alpha)A(\Omega_\alpha) \to 0$ as $\alpha \to 0$.

This shows that the assumptions in Theorem 1.1 can not be weakened.

Our strategy is to solve the following equivalent version of problem (2)

$$\min\{E(\partial\Omega) + A(\Omega) : \Omega \subseteq \mathbb{R}^2 \text{ open, smooth, bounded, simply connected}\}, \qquad (3)$$

and to prove that the solution is a disc.

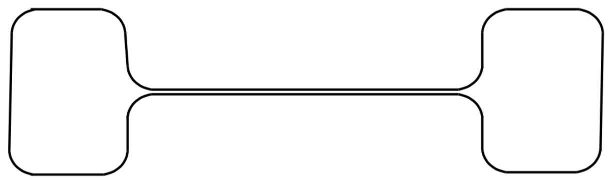

Figure 1: A dumbbell with bounded area and elastic energy with a large perimeter

The proof follows the direct method of the calculus of variations (existence, regularity, analysis of the optimality conditions), but the existence part is by no means easy. In fact we need a control on the perimeter of a minimizing sequence (which is not a priori bounded) and have to handle the fact that the geometric limit of a minimizing sequence may not be anymore smooth, in the sense that tangential self intersections could occur.

Indeed, the boundedness constraints on $E(\partial\Omega)$ and $A(\Omega)$ do not ensure that the perimeter is uniformly bounded, as shown by a counter-example like a dumbell, see Figure 1. In order to deal with minimizing sequences having a diameter going to to infinity, our strategy follows the idea introduced by De Giorgi in [4] for the analysis of the isoperimetric inequality. First, we introduce an artificial boundedness constraint: we shall assume that all our competing sets lie in a ball of radius R centered at the origin. In a second step, we prove that if R is large enough, the optimal set does not touch the boundary of the ball (up to a suitable translation), and so we will be able to write optimality conditions on the full boundary, and consequently deduce that the set is the disc.

In order to handle the self intersection points, we analyse the minimization of the elastic energy of *drops* enclosing a fixed area, i.e. closed loops without self-intersection points, which are smooth except one point, where the tangents are opposite. For this class of sets, we can easily eliminate the situations in which the limit of a minimizing sequence has self intersections. Consequently, we give a complete characterization of the optimal drop, which turns out to be unique. We refer to Section 3 for a precise definition of drops.

Here is our plan.

- Let $R > 0$. We analyze the problem

$$\min\{E(\partial\Omega) + A(\Omega) : \Omega \subseteq B_R \text{ open, smooth, simply connected } \textit{drop}\}. \tag{4}$$

  There exists $R_0 > 0$, such that for $R \geq R_0$ the optimal drop does not touch the boundary of B_R. As a consequence of the optimality conditions, we give an analytic description of the optimal drop and deduce it is unique, independent on R.

- For every $R \geq R_0$ we consider,

$$\min\{E(\partial\Omega) + A(\Omega) : \Omega \subseteq B_R \text{ open, smooth, simply connected}\}. \tag{5}$$

We prove the existence of a solution which does not touch the boundary of B_R. The possible self intersection points of a geometric limit of a minimizing sequence are eliminated by direct comparison with the disc, since their energy would be at least the double of the energy of the optimal drop. Consequently, we can write the optimality conditions on the full boundary and deduce that there are only four sets which satisfy the optimality conditions. By direct observation, the disc is the solution.

- Conclude that the solution of (3) is the disc, and so that inequality (2) holds, with equality if and only if Ω is a disc.

2 Preliminaries

All curves $\gamma : [0, L] \to \mathbb{R}^2$ are parametrized by the arc-length. We denote θ the angle of the tangent to γ with respect to the axis Ox. The curvature of γ at the point $\gamma(s)$ will be denoted $k(s)$ and it is equal to $\theta'(s)$. Since we shall work with curves with finite elastic energy, the function θ belongs to the Sobolev space $H^1(0, L)$. Using the embedding $H^1(0, L) \subseteq C^{0,\alpha}[0, L]$, for any $\alpha < 1/2$, the function θ is, in particular, continuous.

All curves we work in this paper have finite elastic energy

$$E(\gamma) = \frac{1}{2} \int_{[0,L]} |\theta'(s)|^2 ds < +\infty.$$

We start with a series of three technical lemmas.

Lemma 2.1 *Let $\gamma : [0, L] \to \mathbb{R}^2$ be a curve parametrized by the arc length such that $E(\gamma) < +\infty$. Then, for $\delta = \frac{\pi^2}{32E(\gamma)}$ the curve is locally a graph of a 1-Lipschitz function on each interval of size $\delta/\sqrt{2}$.*

Proof Let us fix s_0 and assume that $\theta(s_0) = 0$. By Cauchy-Schwartz inequality we get for every $s \in (s_0, s_0 + \delta)$

$$|\theta(s)| \le \delta^{\frac{1}{2}} (2E(\gamma))^{\frac{1}{2}} \le \frac{\pi}{4},$$

which gives the conclusion.

□

Lemma 2.2 *Let $\gamma : [0, L] \to \mathbb{R}^2$ be a curve parametrized by the arc length such that $E(\gamma) < +\infty$. If $\varepsilon > 0$ is given and $0 \le s < t \le L$ are such that*

$$|\theta(s) - \theta(t)| = \varepsilon,$$

then

$$\int_{[s,t]} |\theta'|^2 ds \ge \frac{\varepsilon^2}{L}.$$

Proof As $\int_{[0,L]} |\theta'|^2 ds < +\infty$, we write

$$|\theta(s) - \theta(t)| = \left| \int_s^t \theta'(u) du \right| \le |t - s|^{\frac{1}{2}} (\int_{[s,t]} |\theta'|^2)^{\frac{1}{2}}$$

which gives the result. □

Remark 2.3 The idea coming out of the lemma is that if there is an ε-variation of the angle, the elastic energy on that section of the curve is at least a constant times ε^2, the constant depending on the global length of the curve.

Let B_R be a ball of radius R.

Lemma 2.4 *Let $\gamma : [0, L] \to \mathbb{R}^2$ be a smooth loop parametrized by the arc length such that $E(\gamma) < +\infty$ and $\gamma([0, L]) \subseteq B_R$. Then*

$$L \leq 2R^2 E(\gamma).$$

Proof Denoting $\gamma(s) = (x(s), y(s))$, we have

$$L = \int_0^L x'^2(s) + y'^2(s) = -\int_0^L x(s)x''(s) + y(s)y''(s)ds.$$

But $|x(s)x''(s)+y(s)y''(s)| \leq (x^2(s)+y^2(s))^{\frac{1}{2}}(x''^2(s)+y''^2(s))^{\frac{1}{2}} \leq R|k(s)|$. Therefore, the conclusion of the lemma follows from the Cauchy-Schwarz inequality

$$L^2 \leq R^2 L \int_0^L k^2(s)ds.$$

□

Assume that a simply connected open set Ω is bounded by a loop γ such that $E(\gamma) < +\infty$, and such that on an interval $(s_0, s_0 + L)$ the curve γ does not have self intersections. Assume moreover that for all perturbations of the form $Id + tV$, $V \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$, such that $(suppV) \cap \partial\Omega = (suppV) \cap \gamma|_{(s_0,s_0+L)}$ the shape derivative of $E(\gamma) + A(\Omega)$ is vanishing at $t = 0$ (see [7, chapter 5] for details on the shape derivative). We shall call such a piece of curve $\gamma|_{(s_0,s_0+L)}$, a free branch and denote it $\tilde{\gamma}$.

Theorem 2.5 (Optimality conditions) *Let $\tilde{\gamma}$ be any free branch of a minimizer Ω of the energy $E(\partial\Omega) + A(\Omega)$. Then $s \mapsto k(s)$ is C^∞ on $\tilde{\gamma}$ and satisfies:*

(B1) $k'' = -\frac{1}{2}k^3 + 1$

(B2) $k'^2 = -\frac{1}{4}k^4 + 2k + 2C$, *for some constant* C

(B3) $\exists Q \in \mathbb{R}^2$, *such that* $\forall M \in \tilde{\gamma}$: $|QM|^2 = 2k + 2C$, *for some constant* C

(B4) $\exists Q \in \mathbb{R}^2$, *such that* $\forall M \in \tilde{\gamma}$: $\overrightarrow{QM}.\nu = \frac{1}{2}k^2$ *where* ν *is the exterior normal vector to* $\partial\Omega$.

Remark 2.6 *The point Q in (B3), (B4) is the same (see the proof below). The constant C in (B2), (B3) is also the same. To see that, take a point M_M on $\tilde{\gamma}$ where the curvature k is maximum. If this point does not exist, just extend the curve with the same ODE. Then, according to (B3), $|QM_M|$ is also maximum and the normal derivative of the boundary at this point is $\overrightarrow{QM}_M/|QM_M|$. Therefore (B4) yields $|QM_M| = \frac{1}{2}k^2$ and plugging into (B3) gives (B2), because $k' = 0$ at this point, with the same constant.*

Proof Let us first prove condition (B3). For that purpose we use the expression of the elastic energy and the area with the parametrization with the angle θ, we have (see [2] for more details):

$$E(\tilde{\gamma}) = \frac{1}{2}\int_{\tilde{\gamma}} \theta'^2 ds := e(\theta), \quad A(\Omega) = \int\int_T \cos\theta(u)\sin\theta(s)\, du\, ds := a(\theta)$$

where T is the triangle $T = \{(u,s) \in \mathbb{R}^2 \; ; \; 0 \le u \le s \le L(\Omega)\}$. We note L for $L(\Omega)$. Thus we are led to minimize the sum $e(\theta) + a(\theta)$ with the following constraints (the starting and the ending point of the branch $\tilde{\gamma}$ are fixed).

$$\int_0^L \cos(\theta(s))\, ds = x(L) - x(0), \qquad \int_0^L \sin(\theta(s))\, ds = y(L) - y(0). \tag{6}$$

The derivative of $e(\theta)$ is (for a perturbation v compactly supported)

$$\langle de(\theta), v\rangle = \int_0^L \theta' v' ds = -\int_0^L \theta'' v ds$$

while the derivative of $a(\theta)$ is given by

$$\langle da(\theta), v\rangle = \int\int_T \cos\theta(u)\cos\theta(s)v(s) - \sin\theta(s)\sin\theta(u)v(u) du ds.$$

Using (6) and Fubini, we can write

$$\int\int_T \sin\theta(s)\sin\theta(u)v(u)duds = (y(L) - y(0))\int_0^L \sin\theta(s)v(s)ds - \int\int_T \sin\theta(u)\sin\theta(s)v(s)duds.$$

Therefore, the optimality condition for the constrained problem reads: there exists Lagrange multipliers λ_1, λ_2 such that, for any v:

$$-\int_0^L \theta'' v ds + \int_0^L \left(\cos\theta(s)\int_0^s \cos(\theta(u)du + \sin\theta(s)\int_0^s \sin(\theta(u)du\right) v(s)ds = \tag{7}$$
$$= (y(L) - y(0))\int_0^L \sin\theta(s)v(s)ds - \lambda_1 \int_0^L \sin\theta(s)v(s)ds + \lambda_2 \int_0^L \cos\theta(s)v(s)ds.$$

which implies (thanks to $x'(s) = \cos\theta(s), y'(s) = \sin\theta(s)$)

$$-\theta'' + x'(x - x(0)) + y'(y - y(0)) = (y(L) - y(0) - \lambda_1)y' + \lambda_2 x' \tag{8}$$

By integration, we get (B3) setting $Q = (x(0) + \lambda_2, y(L) - \lambda_1)$.

Now, the C^∞ regularity of $k(s)$ (and $\theta(s)$) comes from a bootstrap argument and equation (8). The first condition (B1) comes from the classical *shape derivative* of the elastic energy (under the small perturbation defined above). Following e.g. the Appendix in [2], we see that it is given by

$$dE(\partial\Omega, V) = -\int_{\tilde{\gamma}} \left(\frac{1}{2}k(s)^3 + k''(s)\right)\langle V, \nu\rangle \; ds$$

while the derivative of the area is classically

$$dA(\Omega, V) = \int_{\tilde{\gamma}} \langle V, \nu\rangle \; ds$$

Condition (B1) follows since the derivative of $E + A$ must vanish for any admissible V. We obtain condition (B2) multiplying (B1) by k' and integrating.

At last, differentiating twice (B3) we get $k' = \overrightarrow{QM}.\tau$ (where τ is the tangent vector) and $k'' = 1 - k\overrightarrow{QM}.\nu$. Using (B1) we see that $\frac{1}{2}k^3 = k\overrightarrow{QM}.\nu$, so (B4) holds where $k \neq 0$. Since k is a solution of the ODE (B1), and therefore can be written with elliptic functions, it can only vanish on isolated points and thus (B4) holds everywhere by continuity of both members. □

In the following lemma, we assume that the simply connected open set Ω is a minimizer of the energy $E(\partial\Omega) + A(\Omega)$.

Lemma 2.7 *Any free branch of a minimizer Ω has a length L uniformly bounded by*

$$L \leq 146.$$

Proof We work with a free branch of $\tilde{\gamma}$ on $s \in (s_0, s_0 + L)$ and use the optimality conditions above. We also know that the elastic energy of this branch is less than the total energy of the best disc B, so that

$$E(\tilde{\gamma}) \leq E(\partial B) + A(B) = 3\pi 2^{-\frac{2}{3}}. \tag{9}$$

We consider two cases. Assume first that $C \leq 1$ on this branch (C is defined above in (B2), (B3)). Then we know from (ODE3) in the Appendix that

$$k(s) \leq k_M(C) \leq k_M(1) \leq \frac{7}{3} \tag{10}$$

Then, from (B3)

$$|QM|^2 \leq \frac{14}{3} + 2 = \frac{20}{3},$$

hence the arc is contained in the disc centered at Q with radius $R_0 = \sqrt{\frac{20}{3}}$.

On the other hand, if we put the origin at Q

$$L(\tilde{\gamma}) = L = \int_0^L x'^2 + y'^2 dx = (xx' + yy')|_0^L - \int_0^L xx'' + yy'' ds.$$

But $|x(L)x'(L) + y(L)y'(L)| \leq R_0$ and $|x(0)x'(0) + y(0)y'(0)| \leq R_0$ while by Cauchy-Schwarz and (9)

$$|\int_0^L xx'' + yy'' ds| \leq R_0 \int_0^L |k| ds \leq R_0 \sqrt{L 2 E(\tilde{\gamma})} \leq R_0 \sqrt{L 3\pi 2^{\frac{1}{3}}}.$$

Therefore, L satisfies

$$L \leq 2\sqrt{\frac{20}{3}} + \sqrt{\frac{20}{3} \times 3\pi \times 2^{\frac{1}{3}}} \sqrt{L}, \tag{11}$$

which implies (as soon as $C \leq 1$)

$$L \leq 90. \tag{12}$$

Second case: $C \geq 1$ for this branch. In this case we have from (ODE3) in the Appendix

$$k_M(C) \geq k_M(1) \geq \frac{9}{4},$$

$$k_m(C) \leq k_m(1) \leq -\frac{9}{10}.$$

We decompose the interval $I = (s_0, s_0 + L)$ in three parts (some could be empty), $I = I_- \cup I_0 \cup I_+$ where

$$I_- = \{s \in I : k(s) \leq 0\}$$

$$I_0 = \{s \in I : 0 < k(s) < 2^{\frac{1}{3}}\}$$

$$I_+ = \{s \in I : 2^{\frac{1}{3}} \leq k(s)\}$$

and we are going to prove that the length of each part is uniformly bounded, by a controlled constant. First of all, we have seen that the integral of k^2 on a period satisfies (see (ODE4) in the Appendix)

$$\frac{1}{2}\int_0^T k^2 ds \geq \frac{\pi}{4}\sqrt{\frac{22}{3}}.$$

Following (9), this implies that we can not have more than 3 periods on each free branch. We begin with I_+. Obviously

$$E(\tilde{\gamma}) \geq \frac{1}{2}\int_{I_+} k^2 ds \geq \frac{1}{2} 2^{\frac{2}{3}} |I_+|,$$

therefore

$$|I_+| \leq 3\pi \times 2^{-\frac{2}{3}} \times 2^{\frac{1}{3}} \leq 8. \tag{13}$$

For I_0, we consider one of its connected components, say (α, β). Since $k_M(C) \geq \frac{9}{4} > 2^{\frac{1}{3}}$ and $k_m(C) \leq -\frac{9}{10} < 0$, we cannot have any local minimum or local maximum of k in I_0 according to (ODE2) from the Appendix. Therefore, k is either increasing from $k(\alpha)$ to $k(\beta)$, or decreasing from $k(\alpha)$ to $k(\beta)$. Moreover, there are at most 6 such connected components because there are at most 3 periods of k. Let us consider the case of k increasing from $k(\alpha)$ to $k(\beta)$, the other one being similar. We have $0 \leq k(\alpha) \leq k(\beta) \leq 2^{\frac{1}{3}}$. By (B1), $k'' \geq 0$ on (α, β), so that k is convex. Therefore

$$k(\alpha) + k'(\alpha)(s - \alpha) \leq k(s) \tag{14}$$

which implies

$$k(\alpha) + k'(\alpha)(\beta - \alpha) \leq k(\beta) \leq 2^{\frac{1}{3}}.$$

Now $k(\alpha) \geq 0$ and $k'(\alpha) = \sqrt{2C + 2k(\alpha) - \frac{1}{4}k^4(\alpha)} \geq \sqrt{2C}$ thus $\sqrt{2}(\beta - \alpha) \leq \sqrt{2C}(\beta - \alpha) \leq 2^{\frac{1}{3}}$ or $\beta - \alpha \leq 2^{-\frac{1}{6}}$.

Since, there are at most 6 such intervals, we have

$$|I_0| \leq 6 \times 2^{-\frac{1}{6}} \leq 6. \tag{15}$$

At last we consider the case of I_-. The set I_- is not empty only when $C > 0$ and $k_m < 0$. The set I_- is composed of connected components $[\alpha, \beta]$ such that $k(\alpha) = k(\beta) = 0$ or is included in such connected components. Since we want to estimate from above the length of I_-, it suffices to look for the length of these connected components. There are at most 3 of these (identical) components and $k(\frac{\alpha+\beta}{2}) = k_m$ by symmetry.

Now, the elastic energy of such a component satisfies

$$E(\tilde{\gamma}_{\alpha_1,\beta_1}) = \frac{1}{2}\int_{\alpha_1}^{\beta_1} k^2 ds = \int_{\frac{\alpha_1+\beta_1}{2}}^{\beta_1} k^2 ds = \int_{\alpha_1}^{\frac{\alpha_1+\beta_1}{2}} k^2 ds. \tag{16}$$

We denote $L_- = \beta_1 - \alpha_1$ the length of this component. By convexity, on $(\alpha_1, \frac{\alpha_1+\beta_1}{2})$ we have

$$k(s) \leq \frac{2k_m}{L_-}(s - \alpha_1) \leq 0,$$

thus

$$E(\tilde{\gamma}_{\alpha_1,\beta_1}) \geq \int_{\alpha_1}^{\alpha_1 + \frac{L_-}{2}} \frac{4k_m^2}{L_-^2}(s - \alpha_1)^2 ds = \frac{k_m^2}{6} L_-.$$

Now, for $C \geq 1$ we have (see (ODE3) in the Appendix) $k_m^2 \geq k_m^2(1) \geq \frac{81}{100}$ and $E(\tilde{\gamma}_{\alpha_1,\beta_1}) \leq 3\pi 2^{-\frac{2}{3}}$. Therefore

$$L_- \leq \frac{600}{81} \times 3\pi 2^{-\frac{2}{3}} \leq 44,$$

and the total length of

$$|I_-| \leq 3L_- \leq 132. \tag{17}$$

In conclusion, for $C \geq 1$ the total length satisfies (by gathering (13), (15), (17))

$$L \leq 132 + 8 + 6 = 146.$$

□

3 The optimal drop

In this section we prove the existence of a best *drop* minimizing the sum of the elastic energy and the area enclosed. We introduce the class of admissible *Jordan drops* consisting of simply connected open sets Ω bounded by a Jordan curve γ of finite length, which satisfies

$$\theta(0) = \theta(L_\gamma) - \pi, \quad E(\gamma) < +\infty,$$

where L_γ is the length of γ. A drop will be denoted (Ω, γ), Ω being the open set enclosed by the

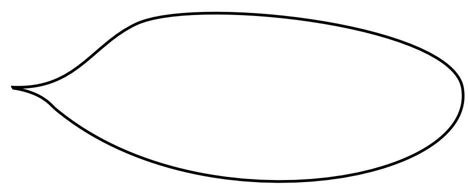

Figure 2: A drop

Jordan curve γ (all Jordan curves are oriented in the positive sense).

The class of Jordan drops is not closed under Haudsdorff convergence, since tangential contacts may occur into the limit of a sequence of Jordan drops. If this situation occurs for a minimizing sequence, we shall focus only the loop which is boundary of *a suitably chosen* connected component of the limit set, which turns out to be a Jordan drop. This selection is possible thanks to a priori geometric information on the minimizing sequence.

For some $R > 0$, we consider the problem

$$\inf\{E(\gamma) + A(\Omega) : (\Omega, \gamma) \text{ is a Jordan drop}, \Omega \subseteq B_R\}. \tag{18}$$

Note that by a similar argument as in Lemma 2.4, the length of Jordan drop γ can not exceed $8R^2 E(\gamma)$. Indeed, the same argument works for the drop, if the singularity lies at the origin, we have $x^2 + y^2 \leq 4R^2$ since the diameter of the drop is less than $2R$.

Here is the main result.

Theorem 3.1 *Problem* (18) *has at least one solution.*

Remark 3.2 With no assumptions on the radius R, it could be possible that the optimal drop (Ω, γ) touches the boundary of the ball but, as we shall prove, it may not have self intersections.

For simplicity of the notation, the ball B_R will be denoted B. We start with the following.

Lemma 3.3 *Let (Ω,γ) be a drop contained in B. If for some $\varepsilon>0$ there exists $0\le s<t\le L_\gamma$ with*

$$\theta(t)=\theta(s)-\pi-\varepsilon$$

then there exists a new drop $(\tilde\Omega,\tilde\gamma)$ in B such that

$$\int_{\tilde\gamma}|\tilde\theta'|^2\le\int_\gamma|\theta'|^2-\frac{\varepsilon^2}{2L_\gamma}\ \ \textit{and}\ \ A(\tilde\Omega)\le A(\Omega).$$

Proof Assume s and t satisfy the hypotheses. Then, from continuity of θ, there exists $s<\overline{s}<\overline{t}<t$ such that

$$\theta(\overline{s})=\theta(s)-\frac{\varepsilon}{2}\ \ \textit{and}\ \ \theta(\overline{t})=\theta(t)+\frac{\varepsilon}{2}.$$

Moreover, there exists $\overline{s}\le s'<t'\le\overline{t}$ such that

$$\theta(t')=\theta(\overline{t}),\theta(s')=\theta(\overline{s})$$

and for every $u\in(s',t')$

$$\theta(u)\in(\theta(t'),\theta(s')).$$

Indeed, we define

$$t'=\inf\{t:t>\overline{s},\theta(t)=\theta(\overline{t})\},$$

and

$$s'=\sup\{s:s<t',\theta(s)=\theta(\overline{s})\}.$$

Then we notice that the curve $\gamma_{[s',t']}$ is a graph in the direction $\theta(s')$, otherwise it would contradict the choice of s' and t'. Setting the orientation of the curve in the trigonometric sense, we are in

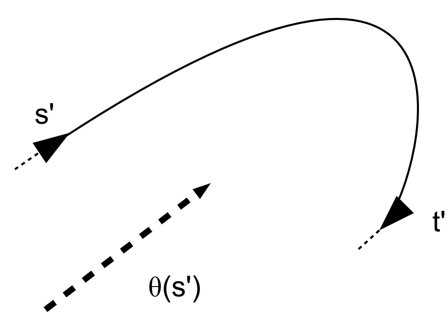

Figure 3: The curve is a graph in the direction $\theta(s')$

configuration similar to Figure 4. Using the graph property, we can translate continuously the piece of the curve $\gamma|_{[s',t']}$ in a parallel way in the direction $\theta(s')$ until this piece touches again γ and add the two parallel segments described by the points $\gamma(s'),\gamma(t')$ in the newly created curve (Figure 4, right).

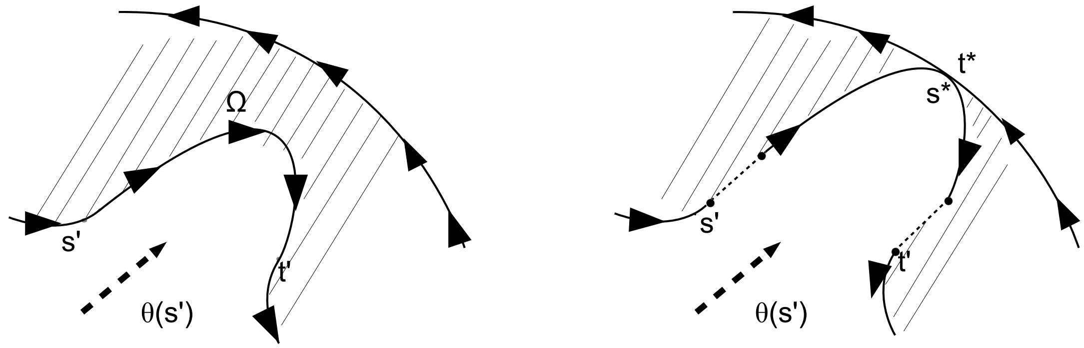

Figure 4: Translation of $\gamma|_{[s',t']}$ in the direction $\theta(s')$

We denote $s_\alpha \in [s',t']$ and $t_\alpha \in [0,L] \setminus [s',t']$ the couples of touching points. We denote s_1, respectively s_2, the minimal and maximal values of s_α. Then, one of the curves starting with s_2 and ending in t_2, or starting in t_1 and ending in s_1 is a drop. Precisely, it is the one which does not contain the point $\gamma(0)$. Without losing generality we can assume it is the curve $s_2 \to t_2$ and rename the point $(s_2,t_2) = (s^*,t^*)$ and denote this curve $\tilde{\gamma}$. We notice that $\tilde{\gamma}$ can not touch any more the piece of curve $\gamma|_{[\overline{s},s']}$. If there would be a contact point, this contact is generated by the translation of $\gamma|_{[s',t']}$ and has to be precisely (s^*,t^*). But in this case, t^* lies in the interval $[\overline{t},s']$, so the curve starting at t^* and ending at s^* is a drop, which does not touch the piece of curve $\gamma|_{[t',\overline{t}]}$.

In this way, we built a new drop $(\tilde{\Omega},\tilde{\gamma})$, which encloses a domain contained in Ω and, in view of Lemma 2.2 has an elastic energy smaller by at least an increment of $\frac{\varepsilon^2}{4L_\gamma}$. □

Now we give the proof of Theorem 3.1.

Proof [of the Theorem 3.1] Let us consider (Ω_n,γ_n) be a minimizing sequence of drops. We may assume that $E(\gamma_n)$, $A(\Omega_n)$ and L_{γ_n} are convergent. Assume that for every n we have $L_{\gamma_n} \le L^*$. In order to work on a fixed Sobolev space $H^1(0,L^*)$, we assume that θ_n is formally extended by the constant $\theta_n(L_{\gamma_n})$ on $(L_{\gamma_n},L^*]$. Up to a subsequence, we can assume that θ_n converges uniformly on $[0,L^*]$ to some function θ. We define the limit curve γ in the following way: $L_\gamma = \lim_{n\to\infty} L_{\gamma_n}$ and $\gamma : [0,L_\gamma] \to \mathbb{R}^2$, $\gamma(s) = \int_0^s e^{i\theta(s)}ds + a$, where $a = \lim_{n\to\infty}\gamma_n(0)$.

Let us fix $\varepsilon > 0$. Then, from the previous lemma, for every $s < t$ and n large enough we have

$$\theta_n(t) \ge \theta_n(s) - \pi - \varepsilon.$$

Indeed, otherwise we would replace (Ω_n,γ_n) by $(\tilde{\Omega}_n,\tilde{g}_n)$ decreasing the energy by a fixed increment $\frac{\varepsilon^2}{4L^*}$, where L^* is a bound of the lengths. This is in contradiction with the minimality of the sequence.

In particular, passing to the limit we get that for every $\varepsilon > 0$ and for every $s < t$

$$\theta(t) \ge \theta(s) - \pi - \varepsilon.$$

Since ε is arbitrary, we get

$$\theta(t) \ge \theta(s) - \pi. \tag{19}$$

From the compactness of the class of closed subsets of $\overline{B}_R$ endowed with the Hausdorff metric, and the embedding of $H^1(0, L^*)$ into $C^{0,\alpha}[0, L^*]$, we may assume that for some open set $\Omega \subseteq B_R$

$$\Omega_n^c \xrightarrow{H} \Omega^c,$$

and the convergence of θ_n leads to

$$\gamma_n([0, L_{\gamma_n}]) \xrightarrow{H} \gamma([0, L_\gamma]).$$

We refer to [3] or [7] for precise properties of the Hausdorff convergence. We know that in general $1_\Omega \leq \liminf_{n\to\infty} 1_{\Omega_n}$, so that $A(\Omega) \leq \lim_{n\to\infty} A(\Omega_n)$. Nevertheless, in our situation the perimeters being uniformly bounded, we get $1_{\Omega_n} \to 1_\Omega$ in $L^1(B_R)$. Moreover, $\partial\Omega \subseteq \gamma([0, L_\gamma])$ and Ω is simply connected (i.e. any loop contained in Ω is homotopic to a point in Ω), but not necessarily connected. The curve γ is possibly self-intersecting, but not crossing, i.e. at every self-intersecting point, the tangent line is the same, while looking locally around the point, the pieces of curve passing through it are (in view of Lemma 2.1) graphs of functions. From the simple connectedness hypothesis, these functions are necessarily ordered. From Lemma 2.2 and the fact that the elastic energy is finite, the number of pieces of curve passing through the touching point is uniformly finite. The situation displayed in Figure 5 may occur.

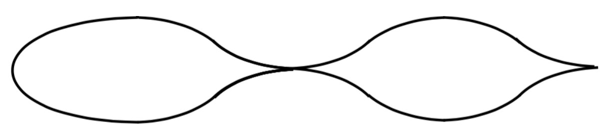

Figure 5: Self touching curve, disconnecting the limit

We shall prove that γ can not have self intersection points, other than the type above, in which case we cut at the self-intersection point, keeping only the drop given by the right loop and decreasing in this way both the elastic energy and the area. The key ingredients are the local representation of the curve as a graph and inequality (19). We shall analyze the different contact types between two pieces of γ. Since the curves are graphs on an interval $[-\frac{l}{\sqrt{2}}, \frac{l}{\sqrt{2}}]$, and the representing functions are ordered, we shall look to the orientation of each piece of curve.

Case 1. Opposite orientation, not disconnecting. Two branches of γ touching at some point $\gamma(s) = \gamma(t)$, are represented as graphs of the functions g_s, g_t, on $[-\frac{l}{\sqrt{2}}, \frac{l}{\sqrt{2}}]$. We assume that $g_s(0) = \gamma(s) = \gamma(t) = g_t(0)$ and choose the couple (s, t) such that for some $\varepsilon > 0$ we have

$$\forall u \in (0, \varepsilon) \;\; g_s(u) > g_t(u),$$

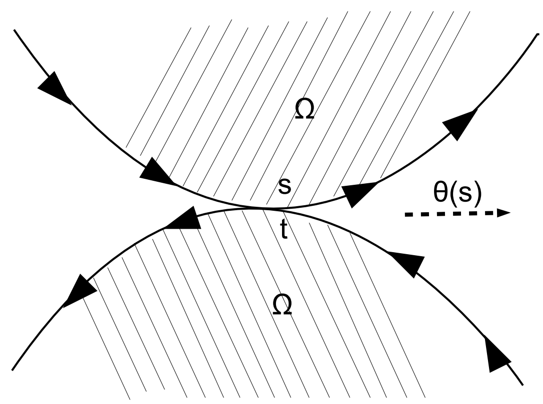

Figure 6: Case 1: opposite orientation, not disconnecting.

otherwise we change the contact point. This inequality would imply the existence of points $s' > s$ and $t' < t$ such that $\theta(t') < \theta(s') - \pi$, which is in contradiction with (19), so that this situation can not occur.

Case 2. Contact of two branches of the same orientation. From the simple connectedness,

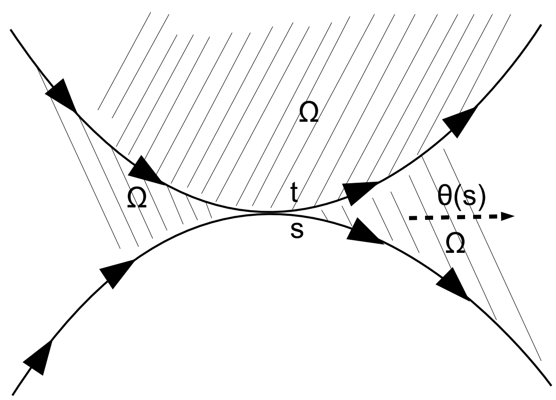

Figure 7: Case 2: same orientation

this situation implies that the touching point $\gamma(s)$ belongs to at least three branches, in particular between the graphs of g_s and g_t, there is a graph corresponding to piece of curve with opposite orientation. There are two possibilities: either this new contact corresponds to a point $t' \in (t, L)$ or to $s' \in (0, s)$. The first situation is in fact the case 1 between the contact points s and t'. The second situation leads also to Case 1, but for the contact points s' and t', so we conclude that the second case can not hold.

Case 3. Opposite orientation, disconnecting. This is the only remaining possibility for self-intersections. There may be several contact points, but every contact point is simple, otherwise we would fall in Case 2. So let us denote $\{(s_\alpha, t_\alpha)\}_\alpha$ the couple of parameters corresponding to the contact points. Because of the simple connectedness and of the absence of contact points as

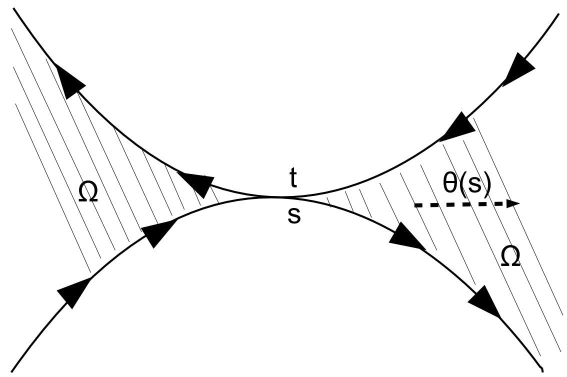

Figure 8: Case 3: simple touch, disconnecting

in cases 1 and 2, we have that if $s_\alpha < s_\beta$ then $t_\beta < t_\alpha$. Consequently, we can identify the contact point (s^*, t^*) such that between s^* and t^* there is no other contact, by setting $s^* = \sup_\alpha s_\alpha$ and $t^* = \inf_\alpha t_\alpha$. Of course, s^* and t^* can not collapse. Indeed, in view of Lemma 2.2 applied to $\gamma|_{[s_\alpha, t_\alpha]}$ if collapse occurs then the elastic energy would blow up. So $\gamma|_{[s^*,t^*]}$ is a Jordan curve for which all the area enclosed is part of Ω, otherwise, because of the simple connectedness, a branch of the curve must pass through the contact point, bringing it to the case 2.

So $\gamma|_{[s^*,t^*]}$ is a drop, with lower elastic energy than γ and enclosing a surface less than or equal to $A(\Omega)$. This means that $\gamma|_{[s^*,t^*]}$ is a solution for problem (18).

□

Lemma 3.4 *There exists R_0, such that if the radius R of the ball B_R in Theorem 3.1 satisfies $R \geq R_0$, then there exists a translation of the optimal drop which does not touch the boundary of B_R.*

Proof The proof relies on Lemma 2.7. Assume that (Ω^*, γ^*) is an optimal drop for problem (18) which touches the boundary, such that there is no translation moving the drop at positive distance from the boundary. This means that the touching points between γ^* and B_R are distributed in such a way that they do not fit in an arc of length less than πR.

Let us fix $R_0 = 300$. This means that the optimal drop which touches the boundary of the ball B_{300} centered at the origin of radius 300, can not touch the boundary of the ball B_{150} of radius 150, otherwise at least one free branch would have the length larger than 146, in contradiction with Lemma 2.7. Two situations may occur, as shown in Figure 9. Either the origin lies inside Ω^*, and so B_{150} has also to lie inside Ω^*, or the origin is not inside in Ω^* and so $\Omega^* \cap B_{150} = \emptyset$. The first situation si excluded since the energy of (Ω^*, γ^*) would be larger than the area of the disc of radius 150: $\pi \cdot 150^2$, in contradiction with its optimality. The second situation is excluded since there would be a free branch having a length larger than 146.

□

Theorem 3.5 *There exists a unique optimal drop (Ω^*, γ^*) which minimizes the energy $E(\gamma)+A(\Omega)$ among all Jordan drops in $\mathbb{R}^2$. This one is fully characterized by the optimality conditions (B1)-(B4) with a unique constant C which can be determined.*

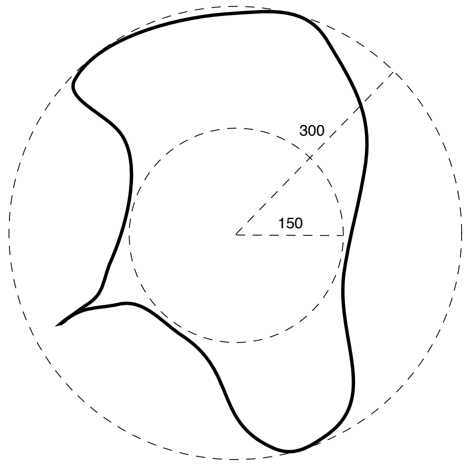

Figure 9: An optimal drop touching the boundary

Moreover

$$E(\gamma^*) + A(\Omega^*) > \pi > 3\pi 2^{-\frac{5}{2}} = \frac{1}{2}[E(\partial B_{2^{-1/3}}) + A(B_{2^{-1/3}})].$$

Remark 3.6 *Figure 10 gives the representation of the optimal drop.*

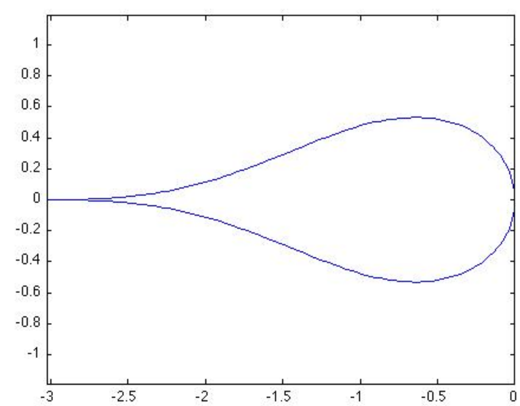

Figure 10: The optimal drop

Proof The proof of existence follows from Theorem 3.1 and Lemma 3.4. The optimality conditions (B1)-(B4) can be written on the whole γ (except at the singularity) according to Theorem 2.5. We start for $s = 0$ at the origin which is the singular point with an horizontal tangent ($\theta(0) = 0$). By (B4) and starshaped property, the point Q is necessarily on the x-axis, the curvature $k(s)$ is negative for $s > 0$ small and $k(s) \to 0$ when $s \to 0$. The function $k(s)$ is periodic but we will prove below (see the end of the proof) that we have only one period for the optimal drop and the curve is symmetric around the x-axis. Therefore to characterize the optimal drop, we can proceed in the following way: for any constant $C > 0$, we solve the ODE

$$\begin{cases} k'' = -\frac{1}{2}k^3 + 1 \\ k(0) = 0 \\ k'(0) = -\sqrt{2C} \end{cases} \tag{20}$$

which has a unique solution. Let us denote by s_M the value where k is maximum with $k(s_M) = k_M$ (respectively s_m and $k_m = k(s_m)$ for the minimum). The point M_M of abscissa s_M is necessarily on the x-axis and its tangent is vertical. Thus, we look for the value of C for which $\theta(s_M) = \int_0^{s_M} k(s)ds = \pi/2$.

We claim that conversely, if we find a value of C for which $\int_0^{s_M} k(s)ds = \pi/2$, then we have found the optimal drop. Indeed, since it satisfies the optimality conditions, it suffices to check that the curve we obtain by $x(s) = \int_0^s \cos\theta(t)dt$ and $y(s) = \int_0^s \cos\theta(t)dt$ with $\theta(s) = \int_0^s k(t)dt$ is an admissible drop. Since M_M is the point where the curvature is maximum, according to (B3), it is the point on γ which is the farthest to Q. But since the tangent is vertical at this point it is necessarily on the x-axis: $y(s_M) = 0$ and the total length of the curve is $2s_M$. Now, since k is symmetric with respect to s_M (see (ODE1) in the Appendix), $k(s_M+t) = k(s_M-t)$ which provides after integration: $\theta(s_M + t) = \pi - \theta(s_M - t)$. This identity gives $\theta(2s_M) = \pi$ and

$$x(2s_M) = \int_0^{s_M} \cos\theta(t)dt + \int_{s_M}^{2s_M} \cos\theta(t)dt = \int_0^{s_M} \cos\theta(t) + \cos(\pi - \theta(t))dt = 0$$
$$y(2s_M) = \int_0^{s_M} \sin\theta(t)dt + \int_{s_M}^{2s_M} \sin\theta(t)dt = \int_0^{s_M} \sin\theta(t) + \sin(\pi - \theta(t))dt = 2y(s_M) = 0$$

which shows that the curve γ is a drop.

Thus to prove uniqueness of the optimal drop, we need to prove that we can find only one $C > 0$ for which $I(C) := \int_0^{s_M} k(s)ds = \pi/2$. Let us write

$$\int_0^{s_M} k(s)ds = \int_0^{2s_m} k(s)ds + \int_{2s_m}^{s_M} k(s)ds = 2\int_0^{s_m} k(s)ds + \int_{2s_m}^{s_M} k(s)ds$$

where we used the symmetry of k with respect to s_m, see (ODE1). This symmetry also shows that $k(2s_m) = 0$. We are going to prove uniqueness of C (and therefore of the optimal drop) by proving that the function $C \mapsto \int_0^{s_M} k(s)ds$ is strictly decreasing. Let us perform the change of variable $u = k(s)$ in each above integral. It comes, using (B2) to express k':

$$\int_{2s_m}^{s_M} k(s)ds = \int_0^{k_M} \frac{u}{\sqrt{2C + 2u - u^4/4}} du \tag{21}$$
$$\int_0^{s_m} k(s)ds = -\int_0^{k_m} \frac{u}{\sqrt{2C + 2u - u^4/4}} du$$

Now to compute the derivative of the first integral $I_1(C)$ with respect to C, we make the change of variable $u = k_M x$, it comes

$$I_1(C) = \int_0^1 \frac{k_M^2 x}{\sqrt{2C + 2k_M x - k_M^4 x^4/4}} dx$$

We compute the derivative of I_1 using $\frac{dk_M}{dC} = 2/(k_M^3 - 2)$ (see (ODE3) in the appendix) and an easy computation gives

$$\frac{dI_1}{dC} = \int_0^1 \frac{6k_M^2 x(x-1)}{(k_M^3 - 2)\left(2C + 2k_M x - k_M^4 x^4/4\right)^{3/2}}\, dx$$

which is clearly negative. In the same way, we get for the second integral $I_2(C) = \int_0^{s_m} k(s)ds$:

$$\frac{dI_2}{dC} = -\int_0^1 \frac{6k_m^2 x(x-1)}{(k_m^3 - 2)\left(2C + 2k_m x - k_m^4 x^4/4\right)^{3/2}}\, dx$$

which is also negative, proving the uniqueness of a solution C for the equation $I_1(C)+2I_2(C) = \pi/2$. Let us remark that a simple computation yields $I(0) = \frac{2\pi}{3}$ while the limit of $I(C)$ when C goes to $+\infty$ is $-\frac{\pi}{2}$ confirming that there exists a solution to our problem.

Let us estimate from below the energy of the optimal drop. Denote by $s_1 = 2s_m$ the first positive zero of k, we recall that s_m is the first minimum of k and $k_m = k(s_m)$, s_M the first maximum of k and $k_M = k(s_M)$. From (B2) k_m and k_M are the real roots of the polynomial (which is concave)

$$P_C(X) = -\frac{1}{4}X^4 + 2X + 2C. \tag{22}$$

The maximum of P_C is at $X = 2^{\frac{1}{3}}$ and $P_C(0) = 2C$. We have

$$k_m < 0 \leq 2^{\frac{1}{3}} \leq k_M \tag{23}$$

(k_m must be negative, otherwise the set Ω^* would be convex).

Moreover, when C increases, $k_M(C)$ is increasing while $k_m(C)$ is decreasing (with increasing absolute value $|k_m(C)|$), because we translate the curve $y = -\frac{1}{4}x^4 + 2x$ up).

If we denote $S = k_M + k_m$ and $P = k_m k_M$ the sum and the product of those two roots, classical elimination and relation between roots provide

$$S^2 = P - \frac{8C}{P} \qquad -\frac{8}{S} = P + \frac{8C}{P} \tag{24}$$

while the two complex roots $z_0, \overline{z}_0$ satisfy $z_0 + \overline{z}_0 = -S$, $z_0\overline{z}_0 = -\frac{8C}{P}$.

Since $P \leq 0$ and $C > 0$, the last equation gives $S > 0$. Let us come back to the computation of the elastic energy of the optimal drop (Ω^*, γ^*)

$$E(\gamma^*) = \int_0^{s_M} k^2 ds.$$

Now $\int_0^{s_M} k^2 ds \geq \int_{s_m}^{s_M} k^2 ds$ and k is increasing from s_m to s_M (since k' can only vanish at zeroes of $P_C(X)$, which only correspond to maxima k_M and minima k_m). We perform the change of variable $x = k(s)$ on this interval $dx = k'(s)ds = \sqrt{2C + 2k - \frac{1}{4}k^4} ds$. Therefore

$$E(\gamma^*) \geq \int_{s_m}^{s_M} k^2 ds = \int_{k_m}^{k_M} \frac{x^2}{\sqrt{2C + 2x - \frac{1}{4}x^4}} dx.$$

We want to find a lower bound of this integral. For this purpose, we write (following (22))

$$P_C(x) = \frac{1}{4}(k_M - x)(x - k_m)(x^2 + Sx - \frac{8C}{P}).$$

Now, the parabola $y = \frac{1}{4}(x^2 + Sx - \frac{8C}{P})$ is symmetric with respect to $-\frac{S}{2}$, and since $\frac{k_M+k_m}{2} = \frac{S}{2} \geq -\frac{S}{2}$, the maximum of y on the interval $[k_m, k_M]$ is equal to

$$F^2 = \frac{1}{4}(k_M^2 + Sk_M - \frac{8C}{P}) = \frac{1}{4}(2k_M^2 + k_m k_M - \frac{8C}{P}) = \frac{1}{4}(3k_M^2 + 2k_m k_M + k_m^2), \tag{25}$$

where we have used (24) for the last equality. Thus

$$\int_{k_m}^{k_M} \frac{x^2}{\sqrt{2C + 2x - \frac{1}{4}x^4}} dx \geq \frac{1}{F} \int_{k_m}^{k_M} \frac{x^2}{\sqrt{(k_M - x)(x - k_m)}} dx.$$

This last integral can be computed explicitly and gives

$$E(\gamma^*) \geq \frac{1}{F} \frac{3k_M^2 + 2k_m k_M + 3k_m^2}{4} \frac{\pi}{2}. \tag{26}$$

We have $F \leq \frac{1}{2}\sqrt{3k_M^2 + 2k_m k_M + 3k_m^2}$ and (26) gives

$$E(\gamma^*) \geq \frac{\pi}{4}\sqrt{3k_M^2 + 2k_m k_M + 3k_m^2}. \tag{27}$$

It remains to get a bound for the quantity $H = 3k_M^2 + 2k_m k_M + 3k_m^2$ which depends only on C. We discuss two cases.

Case A. If $C \geq 1$, $H = k_M^2 + 2k_M(k_M + k_m) + 3k_m^2 \geq k_M^2 + 3k_m^2$. Both mappings $C \mapsto k_m^2$, $C \mapsto k_M^2$ are increasing, thus $C \geq k_M^2(1) + 3k_m^2(1)$. We study $P_1(X) = -\frac{1}{4}X^4 + 2X + 2$

$$P_1(\frac{7}{3}) = -\frac{241}{324} \text{ and } P_1(\frac{9}{4}) = \frac{95}{1024},$$

we get

$$\frac{9}{4} \leq k_M(1) \leq \frac{7}{3}. \tag{28}$$

While from $P_1(-1) = -\frac{1}{4}$ and $P_1(-\frac{9}{10}) = \frac{1439}{40000}$, we get

$$-1 \leq k_m(1) \leq -\frac{9}{10}. \tag{29}$$

It follows that $H \geq \left(\frac{9}{4}\right)^2 + 3(\frac{9}{10}\Big)^2 = \frac{2997}{400} \approx 7.4925$.

Case B. In the case $0 \leq C \leq 1$, we use $k_M^2(C) \geq k_M^2(0) = 4$, $k_m^2(C) \geq 0$ and $|k_M(C)k_m(C)| \leq |k_M(1)k_m(1)| \leq \frac{7}{3}$ to get

$$H = 3k_M^2 + 2k_m k_M + 3k_m^2 \geq 12 - \frac{14}{3} = \frac{22}{3} = 7.333...$$

So in any case, $H \geq \frac{22}{3}$. It follows from (26) that

$$E(\gamma^*) \geq \frac{\pi}{4}\sqrt{\frac{22}{3}}. \tag{30}$$

Now, integrating (B4) on the curve, we get $2A(\Omega^*) = \int_{\gamma^*} \overrightarrow{QM} \cdot \vec{\nu} ds = \frac{1}{2}\int_{\gamma^*} k^2 ds = E(\gamma^*)$.

Therefore

$$E(\gamma^*) + A(\Omega^*) = \frac{3}{2}E(\gamma^*) \geq \frac{3\pi}{8}\sqrt{\frac{22}{3}} > \pi > 3\pi 2^{-\frac{5}{3}}. \tag{31}$$

Let us now conclude by proving that the optimal drop has only one period of the function $k(s)$. The estimate (30) we get is actually true on any possible period. Therefore, if we have a solution (γ_2^*, Ω_2^*) with at least two periods, we would have $E(\gamma_2^*) \geq \frac{\pi}{2}\sqrt{\frac{22}{3}}$, therefore like in (31) its total energy would satisfy $E(\gamma_2^*) + A(\Omega_2^*) > 2\pi$. Now, proceeding in a similar way as we did for the estimate from below, we can get (details omitted) an estimate from above for an optimal drop with only one period which is

$$E(\gamma^*) + A(\Omega^*) \leq 2\pi$$

(the exact value is $E(\gamma^*) + A(\Omega^*) \simeq 4.6823$) therefore, any critical point with more than one period cannot be optimal.

□

4 Proof of Theorem 1.1

With the notation settled in Sections 2 and 3 we return to problem (3), and write

$$\inf\{E(\gamma) + A(\Omega) : \Omega \text{ smooth, bounded, simply connected set }, \partial\Omega = \gamma\}. \tag{32}$$

First of all we recall that among all circles, the optimal one has the radius $r = 2^{-\frac{1}{3}}$. Let us consider $R \geq 300$ and solve the problem

$$\inf\{E(\gamma) + A(\Omega) : \Omega \text{ smooth, bounded, simply connected set}, \Omega \subseteq B_R, \partial\Omega = \gamma\}. \tag{33}$$

Using the same arguments as in Section 3, a minimizing sequence will converge to a couple (Ω, γ). Two possibilities occur. Assume first that there are self intersections. In this case the limiting couple (Ω, γ) contains at least two drops, as in Case 3 of Theorem 3.1. Following Theorem 3.5, this configuration can not be optimal since the energy of Ω is larger than the double of the optimal energy of a drop, so it is excluded.

The second situation is that (Ω, γ) does not have self-intersections. Since the radius is large enough, for a suitable translation the loop does not touch the boundary of the ball, as in Lemma 3.4. Moreover, in this case the optimality conditions $OM.\nu = \frac{1}{2}k^2$ can be written on the full boundary.

Remark 4.1 *This condition recalls the result of Ben Andrews (see Theorem 1.5 in [1]) which, under the hypothesis of positive curvature would allow directly to conclude that the curve is a circle with a radius equal to $2^{-\frac{1}{3}}$ (by direct computation). As the curvature is not known to be positive, we use again the optimality conditions. Actually, Andrews's result does not hold true for non convex curves. Indeed, Figure 11 (which have been obtained using the optimality conditions) shows a curve which satisfies $OM.\nu = \frac{1}{2}k^2$ on the whole boundary.*

If the curvature is not constant, we can assert that Ω is star shaped and the structure of γ is a union of periods consisting of two branches γ_1 and γ_2, where $\gamma_1 : [0, l] \to \mathbb{R}^2$ is a branch of the curve with increasing curvature such that $\gamma(0) = k_m, \gamma(l) = k_M$ and $\gamma_2 : [l, 2l] \to \mathbb{R}^2$ is a congruent branch with decreasing curvature from k_M to k_m. Following (9) and (ODE4), γ consists of one, two or three periods (γ_1, γ_2) (as explained in the proof of Theorem 3.5). From the optimality conditions $(B_1) - (B_4)$ one can eliminate any of those three configurations, since their energy is much larger than the one of the ball. Indeed, in the case of two or three periods, a couple (γ_1, γ_2) has a cap $\gamma_{(l-a,l+a)}$, where a is chosen such that $\nu_{\gamma(a)}$ is orthogonal to the segment $O\gamma(l)$. As on Figure 11 (left), we can cut and reflect along the line $\gamma_(l-a), \gamma(l+a)$ to get a new domain with smaller area and smaller elastic energy. If there is only one period, the argument is similar since we can

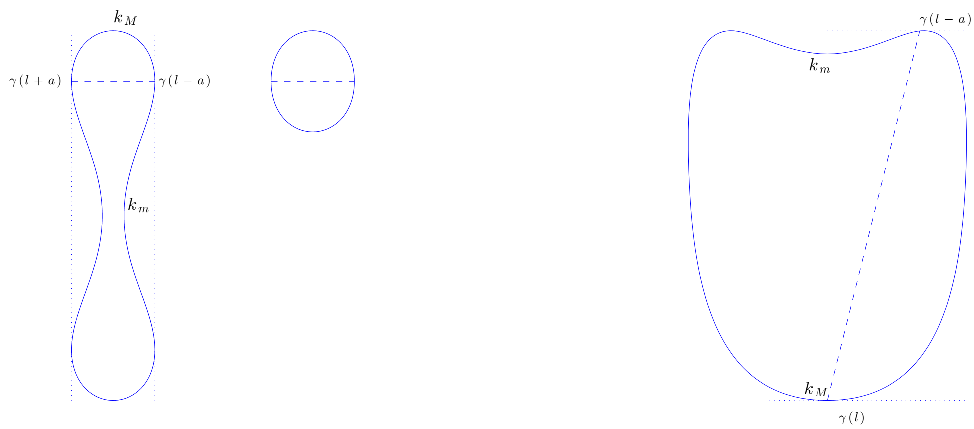

Figure 11: The case of more than one period (left). The case of one period (right).

center-symmetrize the branch from $\gamma(l-a)$ to $\gamma(l)$, where a is chosen such that the normal at $\gamma(l-a)$ is parallel to the segment $O\gamma(l)$ (see Figure 11, righ). The symmetrization center is the middle of the segment joining $\gamma(l-a)$ to $\gamma(l)$.

Both previous constructions are admissible as a consequence of the optimality conditions $(B_1)-(B_4)$.

5 Appendix: analysis of the ODE issued from optimality conditions

In this section, we give several properties of the following ODE in nonstandard form

$$k'^2 = -\frac{1}{4}k^4 + 2k + 2C,$$

where $C \in \mathbb{R}$ is a constant. This ODE is issued from the optimality conditions on a free branch of a minimizer for our problem, see Theorem 2.5. We also refer the reader to reference [2] for related analysis.

Clearly, $C \geq -\frac{3}{4}2^{\frac{1}{3}} \approx -0.944$, otherwise the right hand side is negative. We denote $k_m(C) \leq k_M(C)$ the two real roots of the polynomial $P_C(X) = -\frac{1}{4}X^4 + 2X + 2C$, or if there is no ambiguity simply k_m, k_M.

Here we gather some immediate facts concerning this ODE.

(ODE1) The solution of the ODE is periodic (the period is denoted by T), symmetric with respect to its minimum or maximum.

(ODE2) The only local minima (maxima) are actually global minima (maxima, respectively) and correspond to $k = k_m$ ($k = k_M$, respectively), and k is monotone between these two values.

(ODE3) The mapping $C \mapsto k_M(C)$ is increasing and its range is from $2^{\frac{1}{3}}$ to $+\infty$, while the mapping $C \mapsto k_m(C)$ is decreasing and its range is from $-\infty$ to $2^{\frac{1}{3}}$. Moreover, $k_m(C) < 0$ when $C > 0$, $\frac{9}{4} \leq k_M(1) \leq \frac{7}{3}$, $-1 \leq k_m(1) \leq -\frac{9}{10}$, $-C \leq k_m(C)$. As well, $k_M(C) \geq 2 + C$ for $-\frac{3}{2} \times 2^{\frac{1}{3}} \leq C \leq 0$.

(ODE4) The integral $\frac{1}{2}\int_0^T k^2 ds$ on one period is estimated from below

$$\frac{1}{2}\int_0^T k^2 ds \geq \frac{\pi}{4}\sqrt{\frac{22}{3}}.$$

The proof of (ODE1) is classical, either working with the closed orbit, or using an explicit form of the solution thanks to elliptic functions.

The proof of (ODE2) is easy since k' can vanish only at the zeroes of P_C.

For the proof of (ODE3) we notice that $\frac{dk_M}{dC} = \frac{2}{k_M^3 - 2} > 0$ and $\frac{dk_m}{dC} = \frac{2}{k_m^3 - 2} < 0$, $k_m(0) = 0, k_M(0) = 2$, $P_C(-C) < 0 \Longrightarrow k_m(C) \geq -C$, $P_C(2 + C) = -C[\frac{1}{4}C^3 + 2C^2 + 6C + 4] \geq 0$ and the bounds for $k_m(1), k_M(1)$ have been obtained in (28), (29).

The proof of (ODE4): we have already proved this inequality in Section 3, when $C \geq 0$. It remains the case $-\frac{3}{4}2^{\frac{1}{3}} \leq C < 0$. In this case, we have $k_m \geq -C > 0$ and $k_M \geq 2 + C > 0$, so $3k_M^2 + 2k_m k_M + 3k_m^2 \geq 4C^2 + 8C + 12 \geq 8 \geq \frac{22}{3}$, and the result follows in the same way.

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Dorin Bucur, Laboratoire de Mathématiques, CNRS UMR 5127, Université de Savoie, Campus Scientifique, 73376 Le-Bourget-Du-Lac, France, dorin.bucur@univ-savoie.fr

Antoine Henrot, Institut Elie Cartan de Lorraine, CNRS UMR 7502 and Université de Lorraine, BP 70239 54506 Vandoeuvre-lès-Nancy, France, antoine.henrot@univ-lorraine.fr