The first biharmonic Steklov eigenvalue: positivity preserving and shape optimization

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Abstract. We consider the Steklov problem for the linear biharmonic equation. We survey existing results for the positivity preserving property to hold. These are connected with the first Steklov eigenvalue. We address the problem of minimizing this eigenvalue among suitable classes of domains. We prove the existence of an optimal convex domain of fixed measure.

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1. The Kirchhoff-Love model for a thin plate

Consider a plate, the vertical projection of which is the planar region $\Omega \subset \mathbb{R}^2$. A simple model for its elastic energy is

$$J(u) = \int_\Omega \left(\tfrac{1}{2} (\Delta u)^2 + (1-\sigma) \left(u_{xy}^2 - u_{xx} u_{yy} \right) - f\, u \right) dxdy, \tag{1.1}$$

where f is the external vertical load and u is the deflection of the plate in vertical direction. First order derivatives do not appear in (1.1), which indicates that the plate is free to move horizontally. In (1.1) σ is the Poisson ratio given by $\sigma = \frac{\lambda}{2(\lambda+\mu)}$ with the so-called Lamé constants λ, μ that depend on the material. Usually, $\mu > 0$ and $\lambda \geq 0$ so that $0 \leq \sigma < \frac{1}{2}$. However, some exotic materials have a negative Poisson ratio, see [19]. In any case, it always holds true that

$$-1 < \sigma < 1, \tag{1.2}$$

see [22]. Notice that for $\sigma > -1$ the quadratic part of the functional J is positive.

The somehow modern variational formulation in (1.1) is due to Friedrichs [11] in 1927, although a discussion for a boundary value problem for a thin elastic plate in an old fashioned notation was made much earlier, in 1850, by Kirchhoff

[16]. For hinged plates the natural boundary conditions lead to minimize J in the Hilbert space $H^2\cap H^1_0(\Omega)$. In turn, this leads to the weak Euler-Lagrange equation

$$\int_\Omega (\Delta u\Delta v+(1-\sigma)\,(2u_{xy}v_{xy}-u_{xx}v_{yy}-u_{yy}v_{xx})-f\,v)\;dxdy=0$$

for all $v\in H^2\cap H^1_0(\Omega)$. Formally, an integration by parts gives

$$\begin{aligned}0 \;=\; & \int_\Omega \left(\Delta^2 u-f\right)\; v\;dxdy\;+\int_{\partial\Omega}\left(\frac{\partial}{\partial\nu}\Delta u\right)\;v\;ds\\ & +(1-\sigma)\int_{\partial\Omega}\Big(\left(\nu_1^2-\nu_2^2\right)u_{xy}-\nu_1\nu_2\left(u_{xx}-u_{yy}\right)\Big)\,\frac{\partial}{\partial\tau}v\;ds\\ & +\int_{\partial\Omega}\Big(\Delta u+(1-\sigma)\left(2\nu_1\nu_2u_{xy}-\nu_2^2u_{xx}-\nu_1^2u_{yy}\right)\Big)\,\frac{\partial}{\partial\nu}v\;ds.\end{aligned}$$

On $\partial\Omega$ we have $u=0$ and we may rewrite the second boundary condition that appears from the above integral as

$$\begin{aligned}&\Delta u+(1-\sigma)\left(2u_{xy}\nu_1\nu_2-u_{xx}\nu_2^2-u_{yy}\nu_1^2\right)\\ &=\;\sigma\Delta u+(1-\sigma)\left(2u_{xy}\nu_1\nu_2+u_{xx}\nu_1^2+u_{yy}\nu_2^2\right)\\ &=\;\sigma\Delta u+(1-\sigma)\,u_{\nu\nu}=\sigma\left(u_{\nu\nu}+\kappa u_\nu\right)+(1-\sigma)\,u_{\nu\nu}\\ &=\;u_{\nu\nu}+\sigma\kappa u_\nu=\Delta u-(1-\sigma)\,\kappa u_\nu.\end{aligned}$$

Here κ is the curvature of the boundary, with the sign convention that $\kappa\geq 0$ for convex boundary parts and $\kappa\leq 0$ for concave boundary parts.

Written in a strong form, the above equation and boundary conditions become

$$\begin{cases}\Delta^2u=f & \text{in }\Omega,\\ u=\Delta u-(1-\sigma)\,\kappa u_\nu=0 & \text{on }\partial\Omega.\end{cases}\tag{1.3}$$

The differential equation $\Delta^2u=f$ is called the Kirchhoff-Love model [16, 22] for the vertical deflection of a thin elastic plate, whereas the boundary conditions are named after Steklov due the first appearance in [31]. In this situation, with an integration by parts, the elastic energy J in (1.1) becomes

$$J(u)=\int_\Omega\left(\tfrac{1}{2}\left(\Delta u\right)^2-f\;u\right)\,dx-\frac{1-\sigma}{2}\int_{\partial\Omega}\kappa\,u_\nu^2\,d\omega.\tag{1.4}$$

Minimizers of this functional over $H^2\cap H^1_0(\Omega)$ are weak solutions to (1.3).

Several different aspects of (1.3) are of some interest. Firstly, the so-called positivity preserving property, namely conditions which ensure that the deflection u has the same sign as the the vertical load f. In Section 2 we survey results in this respect, in any bounded domain $\Omega\subset\mathbb{R}^n$ $(n\geq 2)$. If the coefficient $d=(1-\sigma)\kappa$ is assumed to be constant (which occurs if Ω is a ball or in presence of heterogeneous materials having nonconstant Poisson ratio σ), it turns out that a crucial role is played by the first Steklov eigenvalue, namely the smallest value of d for which (1.3) admits nontrivial solutions for $f=0$. In Section 3 we recall a duality principle due to Fichera [10] and its generalization due to Bucur-Ferrero-Gazzola [5] which relates this eigenvalue with a priori L^2-estimates for harmonic functions in Ω.

Then, in Section 3, we turn to the optimization of the first Steklov eigenvalue in suitable classes of domains. We recall that two conjectures were disproved and no Faber-Krahn-type result holds in general domains. Finally, Theorem 4.6 contains the main original contribution of the present paper, namely the existence of an optimal domain which minimizes the Steklov eigenvalue among *convex* domains of given measure. We prove this result by showing that the first Steklov eigenvalue is continuous with respect to Hausdorff convergence of convex domains (see Theorem 5.1), a statement which is not straightforward due to the lack of a trivial extension operator in the space $H^2 \cap H^1_0$. We combine this continuity with some estimates by Payne [25] which enable us to apply Blaschke selection Theorem.

2. Positivity preserving

Throughout this section, Ω is a bounded domain of $\mathbb{R}^n$ ($n \geq 2$) with $\partial\Omega \in C^2$. Let $a \in C^0(\partial\Omega)$, $f \in L^2(\Omega)$, and consider the boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u - a u_\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We say that u is a weak solution to (2.1) if $u \in H^2 \cap H^1_0(\Omega)$ and

$$\int_\Omega \Delta u \Delta v \; dx - \int_{\partial\Omega} a \; u_\nu v_\nu \; d\omega = \int_\Omega f v \; dx \qquad \text{for all } v \in H^2 \cap H^1_0(\Omega).$$

Let us mention that standard elliptic regularity results are available, see [13].

In this section we are interested in finding conditions on Ω and on a such that the assumption $f \geq 0$ implies that the solution u exists and is positive. For any continuous function ϕ, the notation $\phi \gneqq 0$ means $\phi(x) \geq 0$ for all x and $\phi \not\equiv 0$. If ϕ is not continuous the same is intended a.e.

Consider the set $\mathcal{H} := [H^2 \cap H^1_0] \setminus H^2_0(\Omega)$ and define

$$d_1(\Omega) \; := \; \inf_{u \in \mathcal{H}} \; \frac{\displaystyle\int_\Omega |\Delta u|^2 \, dx}{\displaystyle\int_{\partial\Omega} u_\nu^2 \, d\omega} \; . \tag{2.2}$$

The infimum is achieved and $d_1^{-1/2}$ is the norm of the compact linear operator

$$H^2 \cap H^1_0(\Omega) \to L^2(\partial\Omega) \qquad u \mapsto u_\nu|_{\partial\Omega}.$$

Note that with a suitable scaling, for any bounded domain Ω and any $k > 0$, one has

$$d_1(k\Omega) = k^{-1} d_1(\Omega). \tag{2.3}$$

The number d_1 in (2.2) represents the least Steklov eigenvalue, namely the smallest constant value of a for which (2.1) admits a nontrivial solution whenever $f = 0$. In fact, there exists a countable set of eigenvalues. We refer to [9] for a fairly complete study of the spectrum and to [21] for a corresponding Weyl-type asymptotic behaviour.

The following result holds

Theorem 2.1 ([3])**.** *Let $a \in C^0(\partial\Omega)$, $f \in L^2(\Omega)$, and consider the problem*

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u - a u_\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

If $a < d_1$ it admits a unique solution $u \in H^2 \cap H^1_0(\Omega)$. If also $a \geq 0$ and $f \gneqq 0$, then the solution u is strictly superharmonic in $\overline{\Omega}$.

The previous statement proves superharmonicity of the solution, a much stronger condition than positivity. If we relax the lower bound for a, we may still have positivity preserving. Here we denote by $d_{\partial\Omega} = d_{\partial\Omega}(x) > 0$ the distance function from $x \in \Omega$ to $\partial\Omega$.

Theorem 2.2 ([14])**.** *Let $a \in C^0(\partial\Omega)$, $f \in L^2(\Omega)$, and consider the problem* (2.1)*.*
There exists $\delta_c := \delta_c(\Omega) \in [-\infty, 0)$ such that:
1. If $a \geq d_1$ and if $0 \lneqq f \in L^2(\Omega)$, then (2.1) *admits no positive solutions.*
2. If $a = d_1$, then (2.1) *admits a positive eigenfunction $u_1 > 0$ in Ω for $f = 0$. Moreover, u_1 is unique up to multiples.*
3. If $a \lneqq d_1$, then for all $f \in L^2(\Omega)$ there exists a unique solution u to (2.1)*.*
4. If $\delta_c \leq a \lneqq d_1$, then $0 \lneqq f \in L^2(\Omega)$ implies $u \gneqq 0$ in Ω.
5. If $\delta_c < a \lneqq d_1$, then $0 \lneqq f \in L^2(\Omega)$ implies $u \geq c_f\, d_{\partial\Omega}$ in Ω for some $c_f > 0$.
6. If $a < \delta_c$, then there are $0 \lneqq f \in L^2(\Omega)$ with $0 \not\leq u$.
7. If $\Omega = B$, the unit ball, then $\delta_c = -\infty$.

Going back to the hinged plate model discussed in Section 1, the previous results allow to prove the positivity preserving property for the hinged plate in planar convex domains. Recall that the physical bounds for the Poisson ratio are given in (1.2).

Corollary 2.3 ([24])**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with $\partial\Omega \in C^{2,1}$ and assume* (1.2)*. Then for all $f \in L^2(\Omega)$ there exists a unique $u \in H^2 \cap H^1_0(\Omega)$ minimizer of the elastic energy functional J defined in* (1.4)*. The minimizer u is the unique weak solution to* (1.3)*. Moreover, $f \gneqq 0$ implies that there exists $c_f > 0$ such that $u \geq c_f\, d_{\partial\Omega}$ and u is strictly superharmonic in $\overline{\Omega}$.*

3. A priori estimates for harmonic functions

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^2$. Let $g \in L^2(\partial\Omega)$ and consider the problem

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \\ v = g & \text{on } \partial\Omega\ . \end{cases}$$

We are here interested in the optimal constant $\delta_1(\Omega)$ for the a priori estimate

$$\delta_1(\Omega) \cdot \|v\|^2_{L^2(\Omega)} \leq \|g\|^2_{L^2(\partial\Omega)}.$$

In order to characterize variationally δ_1, we introduce the space

$$\mathbf{H} := \text{ closure of } \{v \in C^2(\overline{\Omega});\ \Delta v = 0 \text{ in } \Omega\} \text{ w.r.t. the norm } \|\cdot\|_{L^2(\partial\Omega)}. \tag{3.1}$$

Then we define

$$\delta_1 = \delta_1(\Omega) := \inf_{h \in \mathbf{H}\setminus\{0\}} \frac{\displaystyle\int_{\partial\Omega} h^2\, d\omega}{\displaystyle\int_\Omega h^2\, dx} . \tag{3.2}$$

The infimum is achieved. To see this, combine the continuous map (for weakly harmonic functions) $H^{-1/2}(\partial\Omega) \subset L^2(\Omega)$ (see Theorem 6.6 in Chapter 2 in [20]) with the compact embedding $L^2(\partial\Omega) \subset H^{-1/2}(\partial\Omega)$.

Fichera's principle of duality reads

Theorem 3.1 ([10]). *If $\partial\Omega \in C^2$, then $\delta_1(\Omega) = d_1(\Omega)$.*

It is shown in [10] that $u \gneqq 0$ minimizes d_1 in (2.2) if and only if $h = -\Delta u$ minimizes δ_1 in (3.2). One then wonders whether this principle remains true also if $\partial\Omega \notin C^2$. Following Adolfsson [1] we say that an open domain $\Omega \subset \mathbb{R}^n$ satisfies the outer ball condition if for each $x \in \partial\Omega$ there exists an open ball $B_R \subset \mathbb{R}^n \setminus \Omega$ such that $x \in \partial B_R$. We say that it satisfies a uniform outer ball condition if the radius R of the ball B_R can be taken independently of $x \in \partial\Omega$. Clearly, if $\partial\Omega$ is smooth (C^2) or if Ω is convex, then Ω satisfies the uniform outer ball condition. Theorem 3.1 may be extended to this class of domains. We first state

Theorem 3.2 ([5]). *Assume that $\Omega \subset \mathbb{R}^n$ is open bounded with Lipschitz boundary which satisfies a uniform outer ball condition. Then $d_1(\Omega)$ admits a positive minimizer $u \in [H^2 \cap H^1_0(\Omega)]\setminus H^2_0(\Omega)$ which is unique up to a constant multiplier.*

This result appears somehow sharp since, in view of [23], we believe that a minimizer might not exist in domains with a concave corner. We are now ready to generalize Fichera's principle of duality to nonsmooth domains.

Theorem 3.3 ([5]). *If $\Omega \subset \mathbb{R}^n$ is open bounded with Lipschitz boundary, then $\delta_1(\Omega)$ admits a minimizer $h \in \mathbf{H}\setminus\{0\}$. If we also assume that Ω satisfies a uniform outer ball condition then this minimizer is positive, unique up to a constant multiplier and $\delta_1(\Omega) = d_1(\Omega)$.*

4. Minimization of the first Steklov eigenvalue

From [4] we first recall that for the second order Steklov problem

$$\Delta u = 0 \quad \text{in } \Omega , \qquad u_\nu = \lambda u \quad \text{on } \partial\Omega ,$$

the first (nontrivial) eigenvalue satisfies $\lambda_1(\Omega) \le \lambda_1(\Omega^*)$, where Ω^* denotes a ball having the same measure as Ω. However, as we shall see, the fourth order Steklov problem appears completely different.

Smith [29] conjectures that a Faber-Krahn-type inequality also holds for the first Steklov eigenvalue

Conjecture 4.1 ([29]). *For any bounded domain $\Omega \subset \mathbb{R}^2$, one has*

$$d_1(\Omega) \geq d_1(\Omega^*). \tag{4.1}$$

Smith [29] also gives a proof of Conjecture 4.1. However, in the "Note added in proof" Smith [30] writes that Kuttler and Sigillito pointed out that his proof contains a gap. Smith then concludes by saying that

Although the result is probably true, a correct proof has not yet been found.

From [3] we know that $d_1(B) = n$, where B is the unit ball in $\mathbb{R}^n$. Hence, in particular, for planar domains Ω of measure π (as the unit disk), (4.1) would mean that $d_1(\Omega) \geq 2$. A couple of years later, Kuttler [17] showed that for the square $Q_{\sqrt{\pi}} = (0, \sqrt{\pi})^2$ one has

$$d_1(Q_{\sqrt{\pi}}) < 1.9889...$$

This estimate was subsequently improved in [9] by

$$d_1(Q_{\sqrt{\pi}}) < 1.96256. \tag{4.2}$$

Therefore, (4.1) is false. For this reason, Kuttler [17] suggests a different and weaker conjecture.

Conjecture 4.2 ([17]). *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain such that $|\partial\Omega| = |\partial B|$, where $|\cdot|$ represents the $(n-1)$-Hausdorff measure. Then, $n = d_1(B) \leq d_1(\Omega)$.*

Kuttler gives numerical results on some rectangles which support this conjecture. However, also Conjecture 4.2 is false.

Theorem 4.3 ([5]). *Let $D_\varepsilon = \{x \in \mathbb{R}^2;\ \varepsilon < |x| < 1\}$ and let $\Omega_\varepsilon \subset \mathbb{R}^n$ $(n \geq 2)$ be such that*

$$\Omega_\varepsilon = D_\varepsilon \times (0,1)^{n-2} \ ;$$

in particular, if $n = 2$ we have $\Omega_\varepsilon = D_\varepsilon$. Then,

$$\lim_{\varepsilon \to 0^+} d_1(\Omega_\varepsilon) = 0.$$

Theorem 4.3 disproves Conjecture 4.2: it is not true that the ball has the smallest d_1 among all domains having the same perimeter.

As a straightforward consequence of Theorem 4.3 we have

Corollary 4.4. *Let $B_R = \{x \in \mathbb{R}^n;\ |x| < R\}$. Then*

$$\inf_{\Omega \subseteq B_R} d_1(\Omega) = 0$$

where the infimum is taken over all domains $\Omega \subseteq B_R$ such that $\partial\Omega \in C^\infty$ if $n = 2$ and $\partial\Omega$ is Lipschitzian if $n \geq 3$.

Clearly, the different geometric properties in dimensions $n = 2$ and $n \geq 3$ are due to the shape of Ω_ε in Theorem 4.3. One may then wonder about what happens in annuli in any space dimension $n \geq 2$. It is shown in [5, Theorem 5] that a strange phenomenon appears, the limit when the interior ball shrinks to a point is *discontinuous* with respect to n which is seen as a real parameter since radial functions are involved.

The homogeneity property (2.3) shows that $d_1(k\Omega) \to 0$ as $k \to \infty$. This suggests that $d_1(\Omega)$ becomes "smaller" when the domain Ω becomes "larger". However, in view of Theorem 4.3, we know that the map $\Omega \mapsto d_1(\Omega)$ is not monotone decreasing with respect to domain inclusion. This fact makes the minimization problem very delicate.

When $\partial\Omega$ has positive mean curvature a lower bounds for d_1 is available:

Theorem 4.5 ([9, 26])**.** *Let Ω be a bounded domain of $\mathbb{R}^n$ $(n \geq 2)$ with $\partial\Omega \in C^2$. Let $\kappa(x)$ denote the mean curvature at $x \in \partial\Omega$ and assume that*

$$\underline{\kappa} := \min_{x \in \partial\Omega} \kappa(x) > 0.$$

Then $d_1(\Omega) \geq n\underline{\kappa}$ with equality holding if and only if Ω is a ball of radius $\underline{\kappa}^{-1}$.

Recently, Wang-Xia [33] have extended Theorem 4.5 to compact manifolds with boundary. Theorem 4.5 seems to say that the infimum of d_1 in the class of convex domains might be strictly positive. And this fact, together with some related improved Hardy inequalities which hold when Ω is strictly starshaped (see [2]), suggests to restrict the class of admissible domains for the shape optimization problem. We answer positively to [9, Problem 3] and we prove

Theorem 4.6. *Among all convex domains in $\mathbb{R}^n$ having the same measure as the unit ball B, there exists an optimal one, minimizing d_1.*

Theorem 4.6 should be complemented with the description of the optimal convex shape. This appears quite challenging since, in view of (4.2), we know that the optimal planar domain is not a disk.

5. Proof of Theorem 4.6

The first step of the proof consists in showing the continuity of the map $\Omega \mapsto d_1(\Omega)$, a fact which does not appear trivial since there is no monotonicity with respect to inclusions and no obvious extension operator from $H^2 \cap H^1_0(\Omega)$ to $H^2(\mathbb{R}^n)$. It was proved in [5] that d_1 is continuous with respect to C^2-diffeomorphism of $\mathbb{R}^n$. Here, we prove the same result in a much weaker topology for convergence of domains. We emphasize that it is essential for the domains to be convex, see Theorem 4.3.

Theorem 5.1. *In the class of bounded convex domains, the map $\Omega \mapsto d_1(\Omega)$ is continuous with respect to Hausdorff convergence of domains.*

Proof. We show both upper and lower semicontinuity of the map $\Omega \mapsto d_1(\Omega)$. To this end, consider a sequence of bounded convex domains $\{\Omega_m\}$ such that $\Omega_m \to \Omega$ in the Hausdorff topology, for some bounded convex domain Ω. Up to a finite number of Ω_m, we know that there exist $0 < r < R$ such that $B_r \subset \Omega_m \subset B_R$ for all m.

Upper semicontinuity. By Theorem 3.3, we can show that the map $\Omega \mapsto \delta_1(\Omega)$ is upper semicontinuous. Consider a sequence $\{t_m\} \subset (0,1)$ such that $t_m \to 1$

and $t_m\Omega_m \subset \Omega$ for all m. Since $t_m\Omega_m \to \Omega$ in the Hausdorff topology, for any $h \in C^2(\overline{\Omega})$ such that $h \not\equiv 0$ and $\Delta h = 0$ in Ω we have

$$\int_{t_m\Omega_m} h^2\,dx \to \int_\Omega h^2\,dx\ , \qquad \int_{\partial(t_m\Omega_m)} h^2\,d\omega \to \int_{\partial\Omega} h^2\,d\omega\ .$$

Hence, by the variational characterization of δ_1 in (3.2),

$$\frac{\displaystyle\int_{\partial\Omega} h^2\,d\omega}{\displaystyle\int_\Omega h^2\,dx} \;=\; \lim_{m\to\infty} \frac{\displaystyle\int_{\partial(t_m\Omega_m)} h^2\,d\omega}{\displaystyle\int_{t_m\Omega_m} h^2\,dx} \;\geq\; \limsup_{m\to\infty} \delta_1(t_m\Omega_m)\ . \tag{5.1}$$

By (2.3) and Theorem 3.3 we know that $\delta_1(t_m\Omega_m) = t_m^{-1}\delta_1(\Omega_m)$. Moroever, (5.1) holds for any h as above. Hence, by considering the space $\mathbf{H}$ defined in (3.1) and by taking the infimum of the l.h.s. in (5.1), by (3.2) we infer

$$\delta_1(\Omega) \geq \limsup_{m\to\infty} \delta_1(\Omega_m).$$

This proves upper semicontinuity of $\Omega \mapsto \delta_1(\Omega)$.

Lower semicontinuity. For any m, let $u_m \in H^2 \cap H_0^1(\Omega_m)$ be a normalized first eigenfunction of the Steklov problem in Ω_m, that is

$$\int_{\partial\Omega_m} (u_m)_\nu^2\,d\omega = 1\ , \qquad d_1(\Omega_m) = \int_{\Omega_m} |\Delta u_m|^2\,dx\ .$$

By the just proved upper semicontinuity, we know that there exists $K_1 > 0$ such that $\int_{\Omega_m} |\Delta u_m|^2 \leq K_1$ for all m. Hence, by [12, Corollary 1], there exists $K_2 > 0$ (independent of m since $\Omega_m \subset B_R$) such that

$$\|u_m\|_{H^2(\Omega_m)} \leq K_2\,. \tag{5.2}$$

Since Ω is convex, it satisfies an interior cone property. Since $\Omega_m \to \Omega$, all the Ω_m satisfy a (uniform) interior cone property so that, by [6, Theorem II.1], there exist extensions $\widehat{u_m} \in H^2(B_R)$ such that $\widehat{u_m} = u_m$ in Ω_m and $\|\widehat{u_m}\|_{H^2(B_R)} \leq K_3$ for some $K_3 > 0$, independent of m. Therefore there exists $\widehat{u} \in H^2(B_R)$ such that, up to a subsequence, $\widehat{u_m} \rightharpoonup \widehat{u}$ in $H^2(B_R)$ and

$$\widehat{u_m} \to \widehat{u} \text{ in } H^1(B_R), \quad \nabla\widehat{u_m} \to \nabla\widehat{u} \text{ and } \widehat{u_m} \to \widehat{u} \text{ a.e. in } B_R. \tag{5.3}$$

The pointwise convergence tells us that $u \in H^2 \cap H_0^1(\Omega)$, where u is the restriction of $\widehat{u}$ to Ω. Let χ_m and χ denote the characteristic functions of Ω_m and Ω respectively. Take any $\varphi \in C_c^\infty(B_R)$ and let us estimate

$$\begin{aligned} I_m &:= \left|\int_{B_R} (\chi_m\Delta\widehat{u_m} - \chi\Delta\widehat{u})\varphi\,dx\right| \\ &\leq \left|\int_{B_R} (\Delta\widehat{u_m} - \Delta\widehat{u})\chi_m\varphi\,dx\right| + \left|\int_{B_R} (\chi_m - \chi)\varphi\Delta\widehat{u}\,dx\right|. \end{aligned}$$

Since $\chi_m\varphi \to \chi\varphi$ in $L^2(B_R)$ and $(\Delta\widehat{u_m} - \Delta\widehat{u}) \rightharpoonup 0$ in $L^2(B_R)$, we have

$$\left|\int_{B_R} (\Delta\widehat{u_m} - \Delta\widehat{u})\chi_m\varphi\, dx\right| \to 0.$$

Moreover, since $\chi_m \to \chi$ in $L^2(B_R)$, by Hölder's inequality we have

$$\left|\int_{B_R} (\chi_m - \chi)\varphi\Delta\widehat{u}\, dx\right| \leq \|\chi_m - \chi\|_{L^2(B_R)} \cdot \|\varphi\Delta\widehat{u}\|_{L^2(B_R)} \to 0.$$

This shows that $I_m \to 0$ and, by arbitrariness of φ, that $\chi_m\Delta\widehat{u_m} \rightharpoonup \chi\Delta\widehat{u}$ in $L^2(B_R)$. In turn, by lower semicontinuity of the L^2-norm with respect to weak convergence, we finally obtain

$$\liminf_{m\to\infty} \|\Delta u_m\|^2_{L^2(\Omega_m)} = \liminf_{m\to\infty} \|\chi_m\Delta\widehat{u_m}\|^2_{L^2(B_R)} \geq \|\chi\Delta\widehat{u}\|^2_{L^2(B_R)} = \int_\Omega |\Delta u|^2\, dx.$$

At this point, we need the following convergence result, whose proof is given below:

$$\lim_{m\to\infty} \int_{\partial\Omega_m} (u_m)^2_\nu\, d\omega = \int_{\partial\Omega} u^2_\nu\, d\omega\,. \tag{5.4}$$

By combining these two limits we obtain

$$\liminf_{m\to\infty} d_1(\Omega_m) \geq \frac{\displaystyle\int_\Omega |\Delta u|^2\, dx}{\displaystyle\int_{\partial\Omega} u^2_\nu\, d\omega} \geq d_1(\Omega)$$

where the last inequality follows by the variational characterization of $d_1(\Omega)$, see (2.2). This proves lower semicontinuity of $\Omega \mapsto d_1(\Omega)$.

Proof of (5.4). For simplicity, we put $v_m := |\nabla\widehat{u_m}|$ and $v_0 := |\nabla\widehat{u}|$ so that (5.4) follows if we show that

$$\lim_{m\to\infty} \int_{\partial\Omega_m} v^2_m\, d\omega = \int_{\partial\Omega} v^2_0\, d\omega\,. \tag{5.5}$$

We now need some tools from the theory of convex bodies. For the definitions of a support function and of Gauss map we refer to [28]. To simplify subsequent notations we put $\Omega_0 = \Omega$. Then for any $m = 0, 1, ...$ and any $\theta \in \mathbb{S}^{n-1}$ we define

$$\rho_m(\theta) = \sup\{\rho > 0;\ \rho\theta \in \Omega_m\}\,, \qquad r_m(\theta) = \rho_m(\theta)\theta\,,$$

respectively the radial function and the radial map of Ω: $r_m(\theta)$ is the unique intersection of $\partial\Omega$ with the ray from the origin in the direction of θ. Also denote by $h_m : \mathbb{S}^{n-1} \to \mathbb{R}_+$ and by $\nu_m : \partial\Omega_m \to \mathbb{S}^{n-1}$, respectively the support function and the Gauss map of Ω_m. Finally, let

$$\Phi_m(\theta) := \frac{\rho^n_m(\theta)}{h_m(\nu_m(r_m(\theta)))} \qquad (\theta \in \mathbb{S}^{n-1}).$$

It is clear that the functions Φ_m are uniformly bounded on $\mathbb{S}^{n-1}$. Moreover, by [15, Remark 1.6] (see also (17) in [7]) we know that

$$\Phi_m(\theta) \to \Phi_0(\theta) \quad \text{a.e. in } \mathbb{S}^{n-1} \tag{5.6}$$

where a.e. is intended with respect to the $(n-1)$-Hausdorff measure. Inspired by [7, (16)] we shall make use of the following change of variables

$$\int_{\partial\Omega_m} f(\omega)\,d\omega = \int_{\mathbb{S}^{n-1}} f(r_m(\theta))\,\Phi_m(\theta)\,d\theta \tag{5.7}$$

valid for any $f \in L^1(\partial\Omega_m)$. We apply (5.7) to v_m^2 to get

$$\int_{\partial\Omega_m} v_m^2(\omega)\,d\omega = \int_{\mathbb{S}^{n-1}} v_m^2(r_m(\theta))\,\Phi_m(\theta)\,d\theta. \tag{5.8}$$

Note that (5.2) implies that the sequence $\{|\nabla u_m|\}$ is bounded in $H^1(\Omega_m)$. In turn, since the Ω_m are convex, this implies that $\{|\nabla u_m|\}$ is bounded in $H^{1/2}(\partial\Omega_m)$. By embedding Theorems (recall that $\partial\Omega_m$ is $(n-1)$-dimensional!) we know that $\{|\nabla u_m|\}$ is bounded in $L^q(\partial\Omega_m)$ for $1 \le q \le \frac{2(n-1)}{n-2}$ if $n \ge 3$ and all $1 \le q < \infty$ if $n = 2$. Therefore, the sequence $\{v_m^2(r_m(\theta))\,\Phi_m(\theta)\}$ in (5.8) is bounded in $L^{q/2}(\mathbb{S}^{n-1})$ for some $q/2 > 1$. Since by (5.3) and (5.6) it also converges pointwise a.e. to $v_0^2(r_0(\theta))\,\Phi_0(\theta)$ we may apply Vitali's version of Lebesgue's Theorem [32] to obtain

$$\int_{\mathbb{S}^{n-1}} v_m^2(r_m(\theta))\,\Phi_m(\theta)\,d\theta \ \to \ \int_{\mathbb{S}^{n-1}} v_0^2(r_0(\theta))\,\Phi_0(\theta)\,d\theta\ .$$

By (5.7) this implies (5.5) and, subsequently, (5.4). □

Remark 5.2. As a byproduct of this proof we see that $\|\Delta u_m\|_{L^2(\Omega_m)} \to \|\Delta u\|_{L^2(\Omega)}$, a kind of continuity of normalized eigenfunctions.

A second crucial tool needed for the proof of Theorem 4.6 is a lower bound for d_1. By comparison with the solution to the torsion problem, Payne [25, Formulae (5.11)-(5.12)] was able to prove

Lemma 5.3. *Let $\Omega \subset \mathbb{R}^n$ be a convex bounded domain and let ρ_Ω denote the minimal distance between parallel planes which define a strip containing Ω. Then*

$$d_1(\Omega) \ge \frac{2}{\rho_\Omega}.$$

The result in [25] is obtained with some regularity assumptions on the boundary. This restriction has been removed in [27], see also [18] for planar rectangles.

We are now ready to give the proof of Theorem 4.6. Consider a sequence $\{\Omega_m\} \subset \mathbb{R}^n$ of convex domains having the same measure as the unit ball $B \subset \mathbb{R}^n$ such that $d_1(\Omega_m) \to \inf d_1$, where the infimum of $d_1(\Omega)$ is taken among all convex sets of measure $|B|$. By Lemma 5.3 we know that there exists $R > 0$ such that $\Omega_m \subset B_R$ for all m, since otherwise $d_1(\Omega_m) \to +\infty$. This fact, combined with Blaschke selection Theorem [28, Theorem 1.8.6], guarantees that, up to a subsequence, $\{\Omega_m\}$ converges to a convex domain Ω of measure $|B|$. Hence, by Theorem 5.1, the infimum is achieved by this limit domain.

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