REVERSE FABER-KRAHN AND MAHLER INEQUALITIES FOR THE CHEEGER CONSTANT

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Abstract. We prove a reverse Faber-Krahn inequality for the Cheeger constant, stating that every convex body in $\mathbb{R}^2$ has an affine image such that the product between its Cheeger constant and the squareroot of its area is not larger than the same quantity for the regular triangle. An analogous result holds true for centrally symmetric convex bodies, with the regular triangle replaced by the square. We also prove a Mahler-type inequality for the Cheeger constant, stating that every axisymmetric convex body in $\mathbb{R}^2$ has a linear image such that the product between its Cheeger constant and the Cheeger constant of its polar body is not larger than the same quantity for the square.

1. Introduction and statement of the results

In this paper we prove two new inequalities for the Cheeger constant of planar convex bodies: a reverse Faber-Krahn inequality and a Mahler-type inequality, the latter restricted to the axisymmetric setting.

To introduce the topic of reverse-type inequalities, let us start with a short review about the case of the classical isoperimetric inequality. Formulated within the class $\mathcal{K}^n$ of convex bodies in $\mathbb{R}^n$, it states that balls solve the minimization problem

$$\inf_{K\in\mathcal{K}^n} \frac{|\partial K|}{|K|^{\frac{n-1}{n}}}\,, \tag{1}$$

where $|K|$ and $|\partial K|$ denote respectively the volume and the surface area measure of K.

If one replaces the infimum in (1) with a supremum, it is evident that the problem is not well-posed, since a sequence of thinning domains may have fixed volume and arbitrarily large perimeter. However, there is a natural way to "reverse" the classical isoperimetric inequality, which amounts to consider convex bodies modulo affine transformations: the celebrated reverse isoperimetric inequality proved by K. Ball in [2] establishes that, for every n-dimensional convex body, there is an affine image of it for which the isoperimetric quotient appearing in (1) is not larger than the corresponding expression for the regular n-dimensional simplex. In other words, regular simplices solve the shape optimization problem

$$\sup_{K\in\mathcal{K}^n} \inf_{T\in A_n} \frac{|\partial T(K)|}{|T(K)|^{\frac{n-1}{n}}}\,, \tag{2}$$

where A_n denotes the class of invertible affine transformations from $\mathbb{R}^n$ into itself. A variant of such reverse isoperimetric inequality, as well establised in [2], states that the same

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problem restricted to the class $\mathcal{K}^n_*$ of centrally symmetric n-dimensional bodies modulo invertible linear transformations, *i.e.*

$$\sup_{K\in\mathcal{K}^n_*}\inf_{T\in GL_n}\frac{|\partial T(K)|}{|T(K)|^{\frac{n-1}{n}}}\,, \tag{3}$$

is solved by n-dimensional cubes.

In the plane, the reverse isoperimetric inequality was firstly proved by Gustin [21] and by Behrend [6], who studied systematically extremal problems of the form

$$\sup_{K\in\mathcal{K}_n}\inf_{T\in A_n} f(T(K))\,, \tag{4}$$

for several continuous, translation invariant, and zero homogeneous geometric functionals f over $\mathcal{K}^n$ which attain a known minimum but are unbounded from above (for instance, f can be a quotient involving quantities among diameter, thickness, perimeter, area).

It is clear that a central issue in this matter is to understand features of bodies in special *positions*, a position being the image under a nondegenerate affine transformation. A body is said to be in *John position* when its ellipsoid of maximal volume is a ball (see Section 2 for more details). The key discovery by K. Ball which allowed him to prove the reverse isoperimetric inequality in n-dimensions was that, when put in John position, a convex body gives rise to an isotropic measure to which a general analytic inequality due to Brascamp-Lieb applies. Such powerful inequality has been later generalized, complemented with a dual counterpart, and equipped with equality conditions (see [5, 27]). In particular, the equality conditions for the reverse isoperimetric inequality have been supplied by Barthe in [5].

For wider and enlightening presentations of these topics, we refer to the handbook article [4] by K. Ball, to the survey paper [19] by Gardner, and to Section 10.13 in [32], where it is possible to find also a rich and up-to-date collection of related bibliographical references.

An unexplored challenge at the frontier between Calculus of Variations and Convex Geometry is to obtain reverse isoperimetric inequality for variational energies, namely to study shape optimization problems of the form (4) when $f(K)$ is not directly driven from the geometry of K but involves some minimization problem set on K. In this new context, it seems already quite delicate to obtain reverse inequalities in the planar setting.

In this paper we focus our attention on the case when the involved variational energy is the *Cheeger constant* of K. We recall that, given $K \in \mathcal{K}^2$, its Cheeger constant $h(K)$ is obtained by minimizing over the subsets of K the quotient between perimeter and area; equivalently, $h(K)$ can be defined as the limit of the first Dirichlet eigenvalue $\lambda_{1,p}(K)$ of the p-Laplacian on (the interior of) K as $p \to 1^+$.

The Cheeger constant satisfies a Faber-Krahn inequality in which balls are optimal domains. Indeed, denoting by K^* a ball with the same volume as K, by $C(K)$ a Cheeger set of K, and by $C^*(K) \subseteq K^*$ a ball with the same volume as $C(K)$, it holds

$$h(K) = \frac{\mathrm{Per}(C(K),\mathbb{R}^2)}{|C(K)|} \geq \frac{\mathrm{Per}(C^*(K),\mathbb{R}^2)}{|C^*(K)|} \geq h(K^*)\,.$$

Since $h(\cdot)$ is homogeneous of degree -1 under dilations, denoting by B the unit ball, we can reformulate the above inequality as

$$h(K)|K|^{1/2} \geq h(B)|B|^{1/2}\,. \tag{5}$$

In the same perspective as illustrated above for the classical isoperimetric inequality, we search for a reverse form of the Faber-Krahn inequality (5). Namely, we study the shape optimization problem

$$\sup_{K\in\mathcal{K}^2}\inf_{T\in A_2} h(T(K))|T(K)|^{1/2} \tag{6}$$

and its centrally symmetric variant

$$\sup_{K\in\mathcal{K}^2_*}\inf_{T\in GL_2} h(T(K))|T(K)|^{1/2}\,. \tag{7}$$

Our result reads:

Theorem 1. *The regular triangle Δ and the square Q solve respectively problems* (6) *and* (7)*, namely:*

(i) *for every $K\in\mathcal{K}^2$ there is an affine image $\widetilde{K}$ of K (given by K in John position) which satisfies*
$$h(\widetilde{K})|\widetilde{K}|^{1/2}\le h(\Delta)|\Delta|^{1/2}=\inf_{T\in A_2} h(T(\Delta))|T(\Delta)|^{1/2}=3^{3/4}+\sqrt{\pi}\,;$$

(ii) *for every $K\in\mathcal{K}^2_*$ there is a linear image $\widetilde{K}$ of K (given by K in John position) which satisfies*
$$h(\widetilde{K})|\widetilde{K}|^{1/2}\le h(Q)|Q|^{1/2}=\inf_{T\in GL_2} h(T(Q))|T(Q)|^{1/2}=2+\sqrt{\pi}\,.$$

The proof is obtained by strongly relying on the characterization of bodies in John position. Thanks to such characterization, we are able to obtain Theorem 1 by combining several features of the Cheeger constant of convex planar bodies (including: the uniqueness of the Cheeger set, its characterization via inner parallel sets, a representation formula for the Cheeger constant of polygons whose sides are all touched by the Cheeger set, the Faber-Krahn inequality for the Cheeger constant of N-gons) with some purely geometric ingredients (such as: the reverse isoperimetric inequality by K. Ball and the Brunn-Minkowski theorem).

In order to make the link between reverse isoperimetric and Mahler-type inequalities, one has to restrict attention to the class $\mathcal{K}^2_\sharp$ of axisymmetric convex bodies. Even more specifically, it is enough to consider the class $\mathcal{O}$ of octagons in $\mathcal{K}^2_\sharp$ (meant as the class of polygons with at most 8 sides, so that it includes hexagons and squares). Restricted to such class, the variant of problems (6)-(7) reads

$$\sup_{K\in\mathcal{O}}\inf_{T\in D_2} h(T(K))|T(K)|^{1/2}\,, \tag{8}$$

where D_2 denotes the class of invertible diagonal transformations of $\mathbb{R}^2$ into itself.

Clearly, by Theorem 1 (ii), problem (8) is solved by the square. However, it is useful for its implications on Mahler inequality to show that problem (8) can be solved by putting octagons $K\in\mathcal{O}$ in a different position $\widetilde{K}$ than John's one, which can be characterized by having intersection segments with the coordinate axes of equal length. Notice that, up to a homothety, such a linear image $\widetilde{K}$ turns out to be wedged between the two squares Q_- and Q_+, Q_- being the square with vertices $(\pm1,0)$ and $(0,\pm1)$, and Q_+ being the square with vertices $(\pm1,\pm1)$. For this reason we say that $\widetilde{K}$ is in *$Q_\pm$-position.* We have:

Theorem 2. *For every $K\in\mathcal{O}$, if $\widetilde{K}$ denotes the image of K in $Q_\pm$-position, it holds*
$$h(\widetilde{K})|\widetilde{K}|^{1/2}\le h(Q)|Q|^{1/2}\,.$$

As a by-product of Theorem 2, we are able to show that the Cheeger constant of planar axisymmetric convex bodies satisfies a Mahler-type inequality. To introduce it, let us recall few facts about geometric inequalities involving polar duality. An important affine invariant, which is usually called *volume product*, is given by the product between the volume of K and the volume of its polar body $K^o := \{y \in \mathbb{R}^2 \ : \langle x, y\rangle \leq 1 \ \forall x \in K\}$. A well-known inequality due to Blaschke-Santaló establishes that, in any space dimension, such product is maximal on balls [7, 31]. In [28] Mahler proved that, in dimension 2 and for centrally symmetric bodies, the volume product is minimal on the square (and on its affine images):

$$|K||K^o| \geq |Q||Q^o| \qquad \forall K \in \mathcal{K}^2_* \,;$$

an analogous inequality holds true modulo a translation for non-centered planar bodies, with triangles as optimal domains.

In the same paper, dating back to 1939, Mahler conjectured that an analogous result should be valid in any space dimension, namely that the infimum of the volume product over $\mathcal{K}^n_*$ should be reached on n-cubes and their linear images (and similarly on n-simplexes and their affine images when dropping the condition of central symmetry). This conjecture is still unproved, except for some restricted classes of bodies, and it is one of the central open questions in Convex Geometry.

We give up providing a full list of references about progresses on Mahler conjecture, and we limit ourselves to indicate the monograph [32] and the blog notes [34], where the interested reader can find the main partial achievements about the problem, its history, and a lot of additional bibliography.

As well as reverse isoperimetric-type inequalities, also inequalities involving the notion of polarity for convex bodies are almost unexplored when dealing with variational functionals. In our recent paper [9], we have moved a first step in this direction, by proving a Blaschke-Santaló inequality for the first Dirichlet eigenvalue of the Laplacian in any space dimension, and a Mahler-type inequality for axisymmetric bodies in dimension 2. The latter required a reverse Faber-Krahn inequality for axisymmetric convex octagons analogous to Theorem 2, which was obtained by mixing analytic and rigorous computer assisted arguments. (In fact, a similar combination of theoretical and numerical arguments is successfully applied to prove Theorem 2.)

When considering the Cheeger constant, it is immediate to see that the Blaschke-Santaló inequality continues to hold. Namely, the same proof as in [9, Theorem 1] shows that

$$h(K)h(K^o) \geq h(B)h(B^o) \qquad \forall K \in \mathcal{K}^n_* \,. \tag{9}$$

Concerning the converse inequality, in the simplified planar axisymmetric setting the problem can be formulated as

$$\sup_{K \in \mathcal{K}^2_\sharp} \inf_{T \in D_2} h(T(K))h((T(K))^o) \,. \tag{10}$$

Notice that it is necessary to insert to the infimum over T in (10) because, differently from the volume product, the shape functional $h(K)h(K^o)$ is *not* invariant under invertible linear transformations, and taking directly its supremum over $\mathcal{K}^2_\sharp$ it is easy to see that one would get $+\infty$.

Our result reads:

Theorem 3. *The square Q solves problem* (10)*, namely, for every $K \in \mathcal{K}^2_\sharp$ there is a diagonal image $\widetilde{K}$ of K (given by K in $Q_\pm$-position) which satisfies*

$$h(\widetilde{K})h(\widetilde{K}^o) \leq h(Q)h(Q^o) = \inf_{T \in D_2} h(T(Q))h((T(Q))^o) = \frac{\sqrt{2}}{4}(2+\sqrt{\pi})^2 \,.$$

As mentioned above, the analogous of Theorems 2 and 3 for λ_1, the first Dirichlet eigenvalue of the Laplacian, have been already obtained in [9]. We stress that, in spite, the analogous of Theorem 1 for λ_1 is still unproved. We conjecture that it holds true, and we address it as an interesting and challenging open problem.
The paper is organized as follows:
- the proof of Theorem 1 is given in Section 3;
- the proof of Theorem 2 is given in Section 4;
- the proof of Theorem 3 is given in Section 5.

In order to make the paper more readable and self-contained, we devote Section 2 to recall the basic facts about Cheeger sets and bodies in John position which intervene in the proofs.

2. Preliminaries

2.1. **The Cheeger problem.** Given $K \in \mathcal{K}^2$, its Cheeger constant is defined by

$$h(K) := \inf \Big\{ \frac{\mathrm{Per}(E, \mathbb{R}^2)}{|E|} \ : \ E \text{ measurable}\,,\ E \subseteq K \Big\}\,. \tag{11}$$

where $\mathrm{Per}(E, \mathbb{R}^2)$ denotes the perimeter of E in the sense of De Giorgi.
In recent years, the minimization problem (11), named after Cheeger who introduced it in [14], has attracted an increasing interest: without any attempt of completeness, see [1, 8, 11, 10, 12, 13, 16, 17, 18, 24, 25, 33]; further references can be found in the review papers [26, 29].
Below we recall the main results that we shall need to exploit about the Cheeger problem. We state them as a series of propositions. First we recall an uniqueness property holding for convex bodies which has been proved in [1] (and is actually true in any space dimension).

Proposition 4. (uniqueness of the Cheeger set) *For every $K \in \mathcal{K}^2$, the minimization problem* (11) *admits a unique solution, which is called the Cheeger set of K.*

In the sequel, the Cheeger set of K will be denoted by $C(K)$. Next we recall some useful results whose proof can be found in [24]. Below and throughout the paper, we denote by B the Euclidean unit ball.

Proposition 5. (characterization via inner parallel sets) *For every $K \in \mathcal{K}^2$, there exists a unique value $t = t^*$ such that, setting $K^t = \{x \in K \ : \ \mathrm{dist}(x, \partial K) \geq t\}$, it holds $|K^t| = \pi t^2$. Then the Cheeger constant and the Cheeger set of K can be characterized respectively as*

$$h(K) = \frac{1}{t^*} \qquad \textit{and} \qquad C(K) = K^{t^*} + t^* B\,. \tag{12}$$

If $K \in \mathcal{K}^2$ is a convex polygon, the above equality shows in particular that

$$\partial C(K) \cap \mathrm{int}(K) = \bigcup_j \Gamma_j\,, \tag{13}$$

where Γ_j is an arc of circle of radius $t^* = (h(K))^{-1}$ tangent to ∂K.

For polygons which are Cheeger-regular according to the definition below, a representation formula for the Cheeger constant even more explicit than (12) is available. We recall it in Proposition 7.

Definition 6. If $K \in \mathcal{K}^2$ is a polygon, we say that it is *Cheeger-regular* if its Cheeger set $C(K)$ meets every side of ∂K.

Given a convex polygon $K \in \mathcal{K}^2$, we denote by $\theta_1, \dots, \theta_N$ the inner angles of K (ordered in a counter-clockwise order), and we set

$$\Lambda(K) := \sum_i \cot\left(\frac{\theta_i}{2}\right). \tag{14}$$

Recall that the isoperimetric inequality for convex polygons reads (see for instance [15]),

$$|\partial K|^2 \geq 4\Lambda(K)|K|\,,$$

with equality in case K is circumscribed (namely there is a ball to which every side of ∂K is tangent).

Proposition 7. (the case of Cheeger-regular polygons) *Let $K \in \mathcal{K}^2$ be a polygon with sides ℓ_i and inner angles θ_i, $i = 1, \dots, N$. Then K is Cheeger-regular if and only if $|K| - \overline{r}|\partial K| + \overline{r}^2(\Lambda(K) - \pi) \leq 0$, with*

$$\overline{r} := \min_{i=1,\dots,N-1} \frac{\ell_i}{\cot\left(\frac{\theta_i}{2}\right) + \cot\left(\frac{\theta_{i+1}}{2}\right)}. \tag{15}$$

In this case, $(h(K))^{-1}$ is given by the smallest root of the polyonmial $|K| - r|\partial K| + r^2(\Lambda(K) - \pi)$, namely it holds

$$h(K) = \frac{|\partial K| + \sqrt{|\partial K|^2 - 4|K|(\Lambda(K) - \pi)}}{2|K|}. \tag{16}$$

In particular, if K is circumscribed, we have

$$h(K) = \frac{|\partial K| + \sqrt{4\pi|K|}}{2|K|}. \tag{17}$$

Finally, we need to recall the a discrete version of Faber-Krahn inequality (5) that we recently proved in [10].

Proposition 8. (Faber-Krahn inequality for the Cheeger constant of N-gons) *Among all simple polygons with a given area and at most N sides, the regular N-gon minimizes the Cheeger constant.*

2.2. **Bodies in John position.** Among the ellipsoids contained into a given convex body, there is one of largest volume, which is usually called *John ellipsoid* (since the result was first proved by John in 1948). Hence, each $K \in \mathcal{K}^n$ can be put in *John position*, that is transformed through an affine function which maps its John ellipsoid into the Euclidean unit ball.

We recall below some facts about bodies in John position. Though they are valid in n dimensions, we prefer to present directly the 2-dimensional version needed for our purposes. A first quite useful result is a characterization of bodies in John position (for proofs, extensions, and applications, see [22, 30, 3, 4, 20, 23]).

Proposition 9. (John's Theorem) *Let I denote the 2×2 identity matrix.*
(i) Let $K \in \mathcal{K}^2$ containing B. Then K is in John position if and only if there exist vectors $u_i \in \partial B \cap \partial K$ and positive numbers λ_i, for $i = 1, \dots, N$, $3 \leq i \leq 5$, such that

$$I = \sum_i \lambda_i u_i \otimes u_i \qquad \text{and} \qquad 0 = \sum_i \lambda_i u_i \,.$$

(ii) Let $K \in \mathcal{K}^2_*$ containing B. Then K is in John position if and only if there exist vectors $u_i \in \partial B \cap \partial K$ and positive numbers λ_i, for $i = 1, \dots, N$, $2 \leq i \leq 3$, such that

$$I = \sum_i \lambda_i u_i \otimes u_i \,.$$

Next we recall a volume ratio estimate for bodies in John position. It was established by K. Ball in [2], and in particular it served as a key ingredient for the proof of his reverse isoperimetric inequality. The latter, already recalled in the Introduction, is restated for convenience below.

Proposition 10. (K. Ball's volume ratio estimate)
(i) Let $K \in \mathcal{K}^2$ be in John position, and let Δ_1 denote the regular triangle circumscribed around B. Then it holds

$$|K| \leq |\Delta_1| = 3^{\frac{3}{2}} \,.$$

(ii) Let $K \in \mathcal{K}^2_*$ be in John position, and let Q_1 denote the square circumscribed around B. Then it holds

$$|K| \leq |Q_1| = 4 \,.$$

Proposition 11. (K. Ball's reverse isoperimetric inequality)
(i) For every $K \in \mathcal{K}^2$ there is an affine image $\widetilde{K}$ of K (given by K in John position) which satisfies

$$\frac{|\widetilde{\partial K}|}{|\widetilde{K}|^{\frac{1}{2}}} \leq \frac{|\partial \Delta|}{|\Delta|^{\frac{1}{2}}} \,,$$

where Δ is a regular triangle.
(ii) For every $K \in \mathcal{K}^2_*$ there is a linear image $\widetilde{K}$ of K (given by K in John position) which satisfies

$$\frac{|\partial \widetilde{K}|}{|\widetilde{K}|^{\frac{1}{2}}} \leq \frac{|\partial Q|}{|Q|^{\frac{1}{2}}} \,,$$

where Q is a square.

3. Proof of Theorem 1

We set

$$\mathcal{J} := \Big\{ K \in \mathcal{K}^2 \text{ in John position} \Big\} \,, \qquad \mathcal{J}_* := \mathcal{J} \cap \mathcal{K}^2_* \,.$$

Let us introduce a sequence of approximating problems. For every $N \in \mathbb{N}$, we denote by $\mathcal{P}^N$ the class of polygons in $\mathcal{K}^2$ with at most N sides, and we set

$$\mathcal{J}^N := \mathcal{J} \cap \mathcal{P}^N \,, \qquad \mathcal{J}^N_* := \mathcal{J}^* \cap \mathcal{P}^N \,.$$

Notice that such classes are closed under Hausdorff convergence. Then, since the shape functional $h(K)|K|^{1/2}$ is continuous on $\mathcal{K}^2$ in the Hausdorff topology, for every $N \in \mathbb{N}$ the following problems admit a solution:

$$\sup_{K \in \mathcal{J}^N} h(K)|K|^{\frac{1}{2}} \qquad \text{and} \qquad \sup_{K \in \mathcal{J}^N_*} h(K)|K|^{\frac{1}{2}} . \tag{18}$$

In Proposition 15 below, we are going to establish that the regular triangle and the square are optimal for any of these maximization problems for polygons.
To achieve this goal, we need some preliminary lemmas. The first one states a general property of polygons in John position.

Lemma 12. *Let $K \in \mathcal{J}^2$ be a polygon, and let $C(K)$ denotes its Cheeger set. Then it holds*

$$B \subset C(K) .$$

Proof. We exploit the characterization of the Cheeger constant and of the Cheeger set given by Proposition 5. Let t^* the unique value such that $|K^t| = \pi t^2$. In view of (12), and recalling that B has unit radius, to obtain the inclusion $B \subset C(K)$ it is enough to show the inequality

$$t^* < 1 . \tag{19}$$

Consider the inner parallel sets $K^t = \{x \in K \, : \, \mathrm{dist}(x, \partial K) \geq t\}$. The values of t such that $|K^t| < \pi t^2$ are those larger than t^*. Therefore, if there holds $|K^1| < \pi$, the required inequality (19) is fulfilled. Let us show that the other possible situation, *i.e.* $|K^1| \geq \pi$ cannot occur. Assume by contradiction that $|K^1| \geq \pi$. Let us evaluate the measure of K. Consider the set

$$K \setminus K^1 = \Big\{ x \in K \, : \, \mathrm{dist}(x, \partial K) < 1 \Big\} .$$

By the coarea formula, the isoperimetric inequality, and the assumption $|K^1| \geq \pi$ (which implies $|K^t| \geq \pi$ for all $t \in [0, 1]$), we obtain

$$\begin{aligned} |K \setminus K^1| &= \int_{K \setminus K^1} |\nabla \mathrm{dist}(x, \partial K)| \, dx \\ &= \int_0^1 |\partial K^t| \, dt \geq 2\sqrt{\pi} \int_0^1 |K^t|^{\frac{1}{2}} \, dt \geq 2\pi . \end{aligned}$$

We infer that

$$|K| = |K \setminus K^1| + |K^1| \geq 3\pi .$$

This contradicts K. Ball's volume ratio estimate (*cf.* Proposition 10). □

Next, with the help of Lemma 12, we establish that if K_o is an optimal polygon for any of the problems in (18), then K_o is circumscribed around B. To prove such result, we are going to exploit in different ways a one-parameter family of perturbations of K_o, obtained by a parallel movement of one of its sides which brings it at an oriented distance t from its initial position. More precisely, let us introduce the following

Definition 13. If $K \in \mathcal{K}^2$ be a polygon, let S denote a fixed side in ∂K, and S_1 and S_2 denote the two sides of ∂K adjacent to S. We denote by K_t the polygon obtained by keeping fixed the other sides and replacing the three sides (S, S_1, S_2) by the new sides (S^t, S_1^t, S_2^t) defined as follows (see Figure 1 below):

- S^t lies on the straight-line parallel to S having signed distance t from S (precisely, by signed distance t from S, we mean $+t$ or $-t$ according to whether, respectively, S^t does not intersect or intersects K_{opt});
- S_1^t and S_2^t lie on the same straight-line containing respectively S_1 and S_2;
- the lengths of S^t, S_1^t and S_2^t, are chosen so that the three sides are consecutive (namely (S^t, S_1^t) and (S^t, S_t^2) have one point in common).

If $K \in \mathcal{K}_*^2$, we still denote by K_t the polygon obtained by making the displacement described above not only to the fixed side S but also to its symmetrical with respect to the origin (so that K_t still belongs to $\mathcal{K}_*^2$).

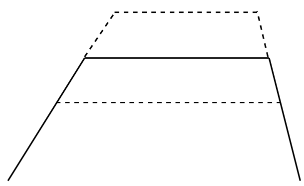

FIGURE 1. Parallel displacement according to Definition 13

Lemma 14. *Let K_o be a solution to any of the maximization problems in* (18)*. Then K_o is circumscribed around B.*

Proof. We divide the proof in three steps.

Step 1: None of the arcs of circles contained in $\partial C(K_o)$ according to (13) *is tangent to three different sides of ∂K.*

Assume by contradiction that there is one of the arcs of circles appearing in $\partial C(K_o)$ according to (13) which is tangent to three different sides of ∂K. Denote these sides by S_1, S, S_2 (ordered in the counterclockwise order), let ℓ be the length of S, and let θ_1 and θ_2 be the angles formed by S respectively with S_1 and S_2. Then we have

$$h(K) = \frac{1}{\overline{r}}, \qquad \overline{r} = \frac{\ell}{\cot\left(\frac{\theta_1}{2}\right) + \cot\left(\frac{\theta_2}{2}\right)}, \tag{20}$$

with $\overline{r}$ defined by (15).

Now, for $t > 0$ small, let K_t be the polygon obtained by applying Definition 13 to the side S of K_o. We claim that $h(K_t) = h(K_o)$, Indeed we observe that, since $\overline{r}$ is given by the second equality in (20), we have

$$(K_o)^{\overline{r}} := \Big\{x \in K_o \ : \ \operatorname{dist}(x, \partial K_o) \geq \overline{r}\Big\} = \Big\{x \in K_t \ : \ \operatorname{dist}(x, \partial K_t) \geq \overline{r}\Big\} =: (K_t)^{\overline{r}}, \tag{21}$$

see Figure 2.

By the first equality in (20), (21), and Proposition 5, we have that $|(K_t)^{\overline{r}}| = \pi\overline{r}^2$. Since there is a unique value $s > 0$ such that $|(K_t)^s| := \big|\big\{x \in K_t \ : \ \operatorname{dist}(x, \partial K_t) \geq s\big\}\big| = \pi s^2$, according to Proposition 5 we conclude that $h(K_t) = \overline{r}^{-1}$. We have thus proved that $h(K_t) = h(K_o)$.

In view of the strict inequality $|K_t| > |K_o|$, we have contradicted the optimality of K_o provided we show that K_t is admissible, namely that it is in John position. By Lemma 12,

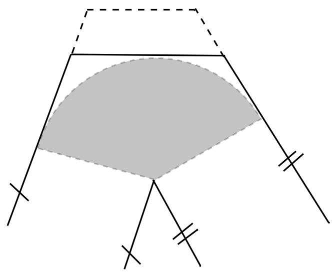

FIGURE 2. The equality $(K_o)^{\overline{r}} = (K_t)^{\overline{r}}$.

we have $B \subset C(K_t)$ and hence B cannot be tangent to S. Therefore, the contact points u_i between ∂K_o and ∂B given by Proposition 9 are not moved when we make the parallel displacement of S. Hence these contact points still lie on the intersection between ∂K_t and ∂B, so that K_t is in John position.

Step 2: K_o is Cheeger-regular
Assume by contradiction that there is at least one side S in ∂K_o which is not touched by the Cheeger set $C(K_o)$ of K_o. For $t > 0$ small, let K_t be the polygon obtained by applying Definition 13 to the side S of K_o, so that

$$K_t \supseteq K_o \qquad \text{and} \qquad \lim_{t \to 0^+} d_H(K_t, K_o) = 0\,. \tag{22}$$

We observe that, similarly as in Step 1, the polygon K_t belongs to $\mathcal{P}^N$ (or to $\mathcal{P}^N_*$). Indeed, since by assumption the Cheeger set of K_o does not touch S, by Lemma 12 also B does not touch S. In particular, this ensures that the contact points u_i given by Proposition 9 are not moved when we make the parallel displacement of S. Hence they still lie on the intersection between ∂K_t and ∂B, so that K_t is in John position.
Consider now the Cheeger set of K_t, that we denote by $C(K_t)$.
We claim that $C(K_t)$ meets S. Indeed, if this is not the case, we have $C(K_t) \subset K_o$, and we can distinguish two cases: either $C(K_t) = C(K_o)$, or $C(K_t) \neq C(K_o)$. In case $C(K_t) = C(K_o)$, since K_t belongs to $\mathcal{P}^N$, and $|K_t| > |K_o|$, the optimality of K_o would be contradicted. In case $C(K_t) \neq C(K_o)$, by the uniqueness of the Cheeger set of K_o, $C(K_t)$ cannot be a Cheeger set for K_o, and hence we would have

$$h(K_o) < \frac{|\partial C(K_t)|}{|C(K_t)|} = h(K_t)\,,$$

against (22) and the decreasing monotonicity of $h(\cdot)$ by inclusions.
Now, by Blaschke selection Theorem, we can find a sequence $t_j \to 0^+$ such that the convex bodies $\{C(K_{t_j})\}$ converge in Hausdorff distance to some convex body C^*. Since $C(K_{t_j})$ meets S for every j, the same occurs for the limit set C^*. On the other hand, by the continuity of the Cheeger constant in the Haudorff topology on $\mathcal{K}^2$, we have

$$h(K_o) = \lim_j h(K_{t_j}) = \lim_j \frac{|\partial C(K_{t_j})|}{|C(K_{t_j})|} = \frac{|\partial C^*|}{|C^*|}\,.$$

We conclude that C^* is a Cheeger set for K_o. Recalling the uniqueness of the Cheeger set in $\mathcal{K}^2$ (*cf.* Proposition 4), we have reached a contradiction, since we assumed $C(K_o) \cap S = \emptyset$.

Step 3: K_o is circumscribed around B.

Assume that there is at least one side S in ∂K_o which is not tangent to B. For $t \in [-\delta, \delta]$, let K_t be the polygon obtained by applying Definition 13 to the side S of K_o. If δ is chosen small enough so that $K_{-\delta} \supseteq B$, by applying Proposition 9 as done in the proof of the previous steps we see that K_t is in John position for every $t \in [-\delta, \delta]$. Therefore the function

$$\Phi(t) := h(K_t)|K_t|^{\frac{1}{2}}$$

attains its maximum over the interval $[-\delta, \delta]$ at $t = 0$.

Now we observe that, setting $K' := K_{-\delta}$ and $K'' := K_\delta$, there exists a continuous map $\alpha : [0, 1] \to [-\delta, \delta]$ (with $\alpha(0) = -\delta$ and $\alpha(1) = \delta$) such that $(1-t)K' + tK'' = K_{\alpha(t)}$; in particular, there exists $\bar{t} \in (0, 1)$ such that $\alpha(\bar{t}) = 0$, namely $(1-\bar{t})K' + \bar{t}K'' = K_o$.

We deduce that the function

$$\Psi(t) := \Phi(\alpha(t)) = h(K_{\alpha(t)})|K_{\alpha(t)}|^{\frac{1}{2}} = h((1-t)K' + tK'')|(1-t)K' + tK''|^{\frac{1}{2}}$$

attains its maximum over the interval $[0, 1]$ at $t = \bar{t} \in (0, 1)$.

Let us show that this is not possible. Since we know from Step 2 that K_o is Cheeger regular, the Cheeger constant of K_o can be expressed as in (16). Moreover, since we know from Step 1 that none of the arcs of circles contained in $\partial C(K_o)$ is tangent to three different sides of ∂K, for δ small enough all the sets K_t are Cheeger-regular too, and their Cheeger constant is given as well by formula (16) (*cf.* the proof of Proposition 7 in [24] and Remark 22 in [10]). Finally notice that, since all the polygons $(1-t)K' + tK''$ have all the same inner angles, if Λ is the quantity defined in (14), there holds

$$\Lambda\big((1-t)K' + tK''\big) = \Lambda(K_o) \qquad \forall t \in [0, 1] .$$

Thus we can write explicitly $\Psi(t)$ just in terms of the area and the perimeter of the Minkowski linear combination $(1-t)K' + tK''$, as

$$\Psi(t) = \frac{|\partial((1-t)K' + tK'')|}{2|(1-t)K' + tK''|^{\frac{1}{2}}} + \frac{1}{2}\sqrt{\frac{|\partial((1-t)K' + tK'')|^2}{|(1-t)K' + tK''|} - 4(\Lambda(K_o) - \pi)} .$$

We see that

$$\Psi(t) = g\big(q(t)\big) , \qquad q(t) := \frac{|(1-t)K' + tK''|^{\frac{1}{2}}}{|\partial((1-t)K' + tK'')|} ,$$

where the function $g(s) = \frac{1}{2s} + \frac{1}{2}\sqrt{\frac{1}{s^2} - 4(\Lambda(K_o) - \pi)}$ is strictly monotone decreasing.

Then the fact that Ψ attains its maximum over the interval $[0, 1]$ at an interior point implies that the isoperimetric quotient $q(t)$ attains its mimumum over $[0, 1]$ at an interior point $\bar{t} \in (0, 1)$. But this is not possible, since by using the Brunn-Minkowski theorem and the linearity of perimeter under Minkowski addition in $\mathcal{K}^2$ (see for instance [32]), we get

$$\frac{|(1-\bar{t})K' + \bar{t}K''|^{\frac{1}{2}}}{|\partial((1-\bar{t})K' + \bar{t}K'')|} > \frac{(1-\bar{t})|K'|^{\frac{1}{2}} + \bar{t}|K''|^{\frac{1}{2}}}{(1-\bar{t})|\partial K'| + \bar{t}|\partial K''|} \geq \min\left\{\frac{|K'|^{\frac{1}{2}}}{|\partial K'|}, \frac{|K''|^{\frac{1}{2}}}{|\partial K''|} .\right\}$$

Notice in particular that the first inequality above is strict because K' and K'' are not homothetic. We have thus reached a contradiction, and we conclude that all the sides of K_o must be tangent to B. $\square$

Proposition 15. *For every $N \in \mathbb{N}$, it holds*

$$\sup_{K \in \mathcal{P}^N} h(K)|K|^{\frac{1}{2}} = h(\Delta)|\Delta|^{\frac{1}{2}} = 3^{3/4} + \sqrt{\pi} , \quad \sup_{K \in \mathcal{P}^N_*} h(K)|K|^{\frac{1}{2}} = h(Q)|Q|^{\frac{1}{2}} = 2 + \sqrt{\pi} . \tag{23}$$

Proof. Let K_o be a solution to one of the maximization problems in (18). By Lemma 14, we know that K_o is circumscribed around B. Then, by the representation formula (17), its Cheeger constant is given by

$$h(K_o) = \frac{|\partial K_o| + \sqrt{4\pi|K_o|}}{2|K_o|},$$

so that

$$h(K_o)|K_o|^{\frac{1}{2}} = \frac{|\partial K_o|}{2|K_o|^{\frac{1}{2}}} + \sqrt{\pi}$$

By K. Ball's reverse isoperimetric inequality (*cf.* Proposition 11), the right hand side of the above equality is bounded from above by

$$\frac{|\partial\Delta|}{2|\Delta|^{\frac{1}{2}}} + \sqrt{\pi} = 3^{3/4} + \sqrt{\pi} \quad \text{or} \quad \frac{|\partial Q|}{2|Q|^{\frac{1}{2}}} + \sqrt{\pi} = 2 + \sqrt{\pi}\,. \tag{24}$$

Then the proof of (23) is concluded because, by using again (17), we see that the expressions in (24) coincide respectively with $h(\Delta)$ and $h(Q)$. □

We can now easily obtain the

Conclusion of the proof of Theorem 1.
Notice first that Theorem 1 follows if we show that

$$\sup_{K\in\mathcal{J}} h(K)|K|^{\frac{1}{2}} = h(\Delta)|\Delta|^{\frac{1}{2}} (= 3^{3/4} + \sqrt{\pi})\,, \ \sup_{K\in\mathcal{J}_*} h(K)|K|^{\frac{1}{2}} = h(Q)|Q|^{\frac{1}{2}} (= 2 + \sqrt{\pi})\,. \tag{25}$$

Namely, for every $K \in \mathcal{K}^2$ there exists $T \in A_2$ such that $\widetilde{K} := T(K) \in \mathcal{J}$. Therefore, using the first equality in (25), we have

$$\sup_{K\in\mathcal{K}^2} \inf_{T\in A_2} h(K)|K|^{\frac{1}{2}} \geq \sup_{K\in\mathcal{J}} h(K)|K|^{\frac{1}{2}} = h(\Delta)|\Delta|^{\frac{1}{2}}\,.$$

Hence Theorem 1 (in the non centrally symmetric case) follows provided

$$h(\Delta)|\Delta|^{\frac{1}{2}} = \inf_{T\in A_2} h(T(\Delta))|T(\Delta)|^{\frac{1}{2}}\,.$$

Such inequality is satisfied since we know from [9] that the regular triangle minimizes the functional $h(K)|K|^{1/2}$ among triangles.
In the centrally symmetric case, the above argument can be repeated just replacing $\mathcal{K}^2$ by $\mathcal{K}^2_*$, A_2 by GL_2, $\mathcal{J}$ by $\mathcal{J}_*$, and Δ by Q.
In order to prove (25), we observe that, since the shape functional $h(K)|K|^{1/2}$ is continuous on $\mathcal{K}^2$ in the Hausdorff topology and the classes $\mathcal{J}$, $\mathcal{J}^*$ are closed in the same topology, any of the maximization problems in (25) admits a solution K_{opt}. We claim that it is possible to find a sequence of polygons $\{P_j\} \subset \mathcal{J}$ which approximate K_{opt} in the Hausdorff distance d_H. Indeed, let $\{u_i\}$ be the contact points between ∂K_{opt} and B given by Proposition 9. According to such proposition, any polygon containing the vectors u_i's on its boundary is in John position. Thus, it is enough to take a sequence of polygons $\{P_j\}$ satisfying $\lim_j d_H(P_j, K_{opt}) = 0$ and such that, for every j, all the vectors u_i's belong to ∂P_j.
By Proposition 15, for every $j \in \mathbb{N}$ the product $h(P_j)|P_j|^{\frac{1}{2}}$ is bounded from above by $h(\Delta)|\Delta|^{\frac{1}{2}}$ (or by $h(Q)|Q|^{\frac{1}{2}}$ in the centrally symmetric case). We conclude that the same will be true in the limit as $j \to +\infty$ for $h(K_{opt})|K_{opt}|^{\frac{1}{2}}$. This implies (25), which in turn yields Theorem 1. □

4. Proof of Theorem 2

Consider the maximization problem

$$\sup\Big\{h(\Omega)|\Omega|^{1/2}\ :\ \Omega\in\mathcal{O}\ :\ Q_-\subseteq\Omega\subseteq Q_+\Big\}\,. \tag{26}$$

Clearly it admits a solution, since the class of admissible domains is closed in the Hausdorff topology.
Such family can be parametrized by the two coordinates of one of the vertices not lying on the coordinate axes. For any $(a,b)\in[0,1]^2$ with $b\geq 1-a$, we denote by $\Omega_{(a,b)}$ the octagon with vertices $(\pm1,0)$, $(0,\pm1)$, $(\pm a,\pm b)$ (which may degenerate into a hexagon or a square). Notice that, up to a rotation, we can reduce ourselves to maximize the functional $h(\Omega)|\Omega|^{1/2}$ over the class

$$\mathcal{U}=\Big\{\Omega_{(a,b)}\ :\ (a,b)\in[0,1]^2\,,\quad b\geq\max\{a,1-a\}\Big\}\,. \tag{27}$$

The maximization of the functional $h(\Omega)|\Omega|^{1/2}$ over $\mathcal{U}$ will come to maximize an explicit function of two variables. Indeed, we shall find the explicit expression of $h(\Omega_{(a,b)})|\Omega_{(a,b)}|^{1/2}$, which is a rational function depending on a,b and $\sqrt{P(a,b)}$, for some polynomial P. Then we have to show that

$$h(\Omega_{(a,b)})|\Omega_{(a,b)}|^{1/2}-h(Q)|Q|^{1/2}\leq 0\qquad\forall(a,b)\in[0,1]^2\ :\ b\geq\max\{a,1-a\}\,. \tag{28}$$

To that aim, we shall exploit a theoretical-numerical argument which is similar to the one used in [9] and runs roughly speaking as follows. The inequality (28) will be justified via a theoretical analysis in an explicit region (called "confidence zone") around the set of points where the maximum is achieved (in our case the point $(1,1)$ and the line segment $\{(a,b)\in[0,1]^2 : a+b=1\}$). In a second step, we use a computational argument (which leads to a rigorous mathematical proof of the inequality) to prove the inequality outside the confidence zone. Here are the main steps:

- We cover the complement of the confidence zone with a grid, which is chosen to be fine enough.
- For every square of the grid we find an analytical bound of the quantity
$$h(\Omega_{(a,b)})|\Omega_{(a,b)}|^{1/2}-h(Q)|Q|^{1/2},$$
valid for every (a,b) ranging in this square. The analytical bound has an explicit expression, involving elementary functions (polynomials of degree 4 and square roots).
- For every such square, we compute this bound using MATLAB. The computations are carried with machine precision, of order 10^{-16}. We decide that the value of the bound is negative, if it is a negative number of order 10^{-4}. These computations are carried successfully since the confidence zone covers the critical values together with a large enough region around them, and the grid step is chosen small enough.

We give all the computational details at the end of the section.

Let us now start with the proof of (28). We begin by showing the following claim:

Claim: If $\Omega_{(\overline{a},\overline{b})}$ is an optimal octagon for problem (26), either $\overline{b}=1$ (so that $\Omega_{(\overline{a},1)}$ is a hexagon), or $\Omega_{(\overline{a},\overline{b})}$ is Cheeger-regular.

Namely, assume by contradiction that $\Omega_{(\overline{a},\overline{b})}$ is an optimal octagon for problem (26) with $\overline{b}<1$ and *not* Cheeger-regular. Since we work on the class (27), we infer that the side

S having endpoints $(0,1)$ and $(\overline{a},\overline{b})$ is not touched by the Cheeger set of $\Omega_{(\overline{a},\overline{b})}$. Then, consider for small $\varepsilon > 0$, the perturbed octagon $\Omega_{(a_\varepsilon,b_\varepsilon)}$, obtained by moving the vertex $(\overline{a},\overline{b})$ along the straight line through $(\overline{a},\overline{b})$ and $(1,0)$ into

$$a_\varepsilon := \overline{a} - \varepsilon\,, \qquad b_\varepsilon := \frac{\overline{b}}{\overline{a}-1}(a_\varepsilon - 1)\,.$$

Since S is not touched by the Cheeger set of $\Omega_{(\overline{a},\overline{b})}$, by arguing as in Step 1 in the proof of Lemma 14, we infer that $\Omega_{(a_\varepsilon,b_\varepsilon)}$ and $\Omega_{(\overline{a},\overline{b})}$ have the same Cheeger constant. Since $|\Omega_{(a_\varepsilon,b_\varepsilon)}| > |\Omega_{(\overline{a},\overline{b})}|$, we have contradicted the optimality of $\Omega_{(\overline{a},\overline{b})}$.
Thanks to the claim just proved, we can divide the remaining of the proof in two parts, where we show respectively that the inequality $h(\Omega_{(a,b)})|\Omega_{(a,b)}|^{1/2} \leq h(Q)|Q|^{1/2}$ holds true for admissible hexagons (*i.e.* for $b = 1$) and for admissible Cheeger regular octagons.

Case 1: Hexagons. Set for brevity $\Omega := \Omega_{(a,1)}$. We are going to compute explicitly $h(\Omega)$ as a function of the parameter $a \in [0,1]$ by exploiting Propositions 5 and 7.

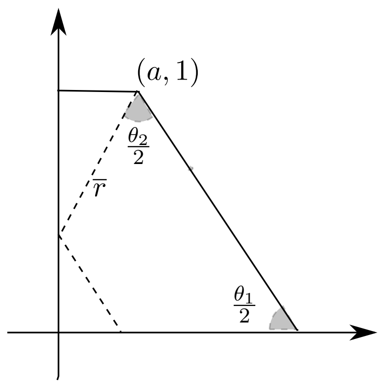

FIGURE 3.

We observe that the inner angles of Ω are

$$\theta_1 = 2\,\mathrm{arccot}(1-a)\,, \qquad \theta_2 = \pi - \mathrm{arccot}(1-a)\,.$$

and the radius $\overline{r}$ defined by (15) is given by

$$\overline{r} = a \tan\Big(\frac{\theta_2}{2}\Big) = a\cot\Big(\frac{1}{2}\,\mathrm{arccot}(1-a)\Big)\,.$$

Also the quantites $|\Omega|$, $|\partial\Omega|$, and $\Lambda(\Omega)$ are easily computed in terms of a as

$$\begin{aligned} |\Omega| &= 2(a+1) \\ |\partial\Omega| &= 4\Big(a + \sqrt{1+(1-a)^2}\Big) \\ \Lambda(\Omega) &= 2\cot\big(\tfrac{\theta_1}{2}\big) + 4\cot\big(\tfrac{\theta_2}{2}\big)\,. \end{aligned} \tag{29}$$

By Proposition 7, we know that Ω is Cheeger regular if and only if

$$g(a) := |\Omega| - |\partial\Omega|\overline{r} + (\Lambda(\Omega) - \pi)\overline{r}^2 \leq 0$$

By using (29), it is easy to see that g is a rational function of a and $\sqrt{a+(1-a)^2}$, which is strictly positive on $[0, 0.19]$ and strictly negative on $[0.21, 1]$. Moreover, the derivative

$g'(a)$ is strictly negative on the interval $[0.19, 0.21]$. Hence, there exists a unique zero $a^* \in (0.19, 0.21)$, and it holds $g(a) \geq 0$ for $a \in [0, a^*]$ and $g(a) \leq 0$ for $a \in [a^*, 1]$, see the plot represented in Figure 4, obtained with Mathematica.

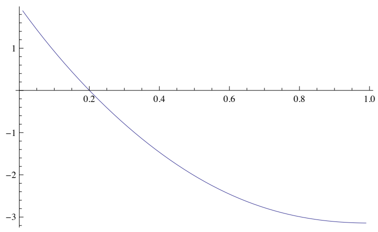

FIGURE 4. Plot of the map $a \mapsto g(a)$ on the interval $[0, 1]$

Then we distinguish the two possibilities $a \geq a^*$ and $a < a^*$.

(i) *Cheeger-regular hexagons:* $a \geq a^*$. In this case we know from Proposition 7 that $h(\Omega)^{-1}$ agrees with the smallest root of the polynomial $p(r) = |\Omega| - |\partial\Omega| r + (\Lambda(\Omega) - \pi) r^2$, or equivalently that $h(\Omega)$ is given by formula (16). Then, using this formula and the expressions in (29), we can write explicitly the function $F(a) := h(\Omega)|\Omega|^{1/2} - (2 + \sqrt{\pi})$ for $a \in [a^*, 1]$. It is easy to check that the first derivative of the function $F(a)$ at $a = 1^-$ is strictly positive, so that it is possible to determine $\overline{\varepsilon} > 0$ so that the function $F(a)$ is strictly negative in a confidence interval $(1 - \overline{\varepsilon}, 1)$. Then, the inequality $F(a) < 0$ can be proved analytically also outside such confidence interval. Indeed, through elementary estimates, one can show that there holds $F(a) < 0$ on $[0.19, \overline{\varepsilon}]$. In particular, we have that F remains strictly negative on the interval $(a^*, 1)$ (see the plot represented in Figure 5 right, obtained with Mathematica).

(i) *Not Cheeger-regular hexagons:* $a < a^*$. In this case the value t^* such that $|\Omega^{t^*}| = \pi(t^*)^2$ according to Proposition 5 is larger than $\overline{r}$. (Recall that $\Omega^t := \{x \in \Omega(a) : \mathrm{dist}(x, \partial\Omega(a)) \geq t\}$). In order to compute $h(\Omega) = (t^*)^{-1}$, we observe that, for $t > \overline{r}$, we have

$$t = \overline{r} + \varepsilon \quad \Rightarrow \Omega^t = (\Omega^{\overline{r}})^{\varepsilon}\,,$$

Hence, using Steiner's formula, we have

$$|\Omega^t| = |\Omega^{\overline{r}}| - |\partial\Omega^{\overline{r}}|\varepsilon + \Lambda(\Omega^{\overline{r}})\varepsilon^2\,,$$

where the quantities $|\Omega^{\overline{r}}|$, $|\partial\Omega^{\overline{r}}|$, and $\Lambda(\Omega^{\overline{r}})$ are easily computed in terms of a as

$$\begin{aligned} |\Omega^{\overline{r}}| &= 2(1-\overline{r})^2 \cot\left(\tfrac{\theta_1}{2}\right) \\ |\partial\Omega^{\overline{r}}| &= 4(1-\overline{r})\sqrt{1+\cot^2\left(\tfrac{\theta_1}{2}\right)} \\ \Lambda(\Omega^{\overline{r}}) &= 2\cot\left(\tfrac{\theta_1}{2}\right) + 2\cot\left(\tfrac{\pi-\theta_1}{2}\right). \end{aligned} \tag{30}$$

Then, imposing $|\Omega^t| = \pi t^2$, we can determine $h(\Omega)^{-1}$ as $t^* = \overline{r} + \varepsilon^*$, with ε^* equal to the smallest root of the polynomial $p(\varepsilon) = |\Omega^{\overline{r}}| - |\partial\Omega^{\overline{r}}|\varepsilon + \Lambda(\Omega^{\overline{r}})\varepsilon^2 - \pi(\overline{r} + \varepsilon)^2$, namely

$$\varepsilon^* = \frac{2(|\Omega^{\overline{r}}| - \pi\overline{r}^2)}{|\partial\Omega^{\overline{r}}| + 2\pi\overline{r} + \sqrt{(|\partial\Omega^{\overline{r}}| + 2\pi\overline{r})^2 - 4(\Lambda(\Omega^{\overline{r}}) - \pi)(|\Omega^{\overline{r}}| - \pi\overline{r}^2)}}\,.$$

Consequently, we can write explicitly the function $F(a) := h(\Omega)|\Omega|^{1/2} - (2 + \sqrt{\pi})$ for $a \in [0, a^*)$. It is easy to check that, while the the first derivative of the function $F(a)$ at $a = 0^+$ vanishes, the second order derivative is strictly negative. Then, similarly as above, it is possible to determine $\overline{\varepsilon} > 0$ so that the function $F(a)$ is strictly negative in a confidence interval $(0, \overline{\varepsilon})$. Then, through elementary computations, one can show analytically that $F(a) < 0$ on the interval $[\overline{\varepsilon}, 0.21]$. In particular, we have that F remains strictly negative on the interval $(0, a^*)$ (see the plot represented in Figure 5 left, obtained with Mathematica).

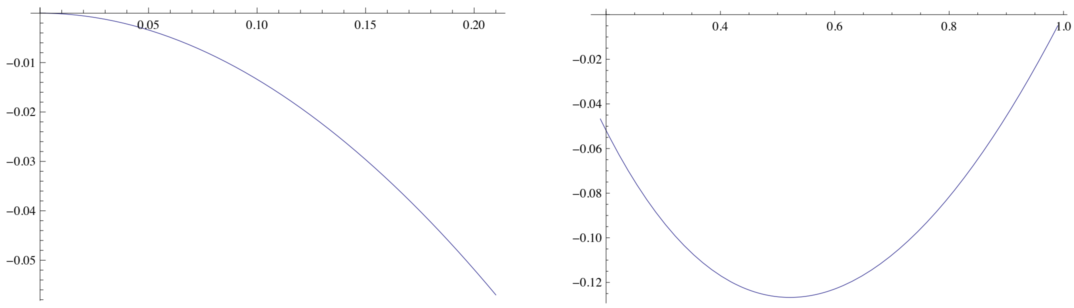

FIGURE 5. Plots of the map $a \mapsto F(a)$ on the interval $[0, 0.21]$ (left) and on the interval $[0.19, 1]$ (right)

Case 2: Cheeger-regular octagons. Set for brevity $\Omega := \Omega_{(a,b)}$. Under the assumptions that Ω is Cheeger-regular, we are going to compute explicitly $h(\Omega)$ as a function of the parameters (a, b) by means of formula (16).

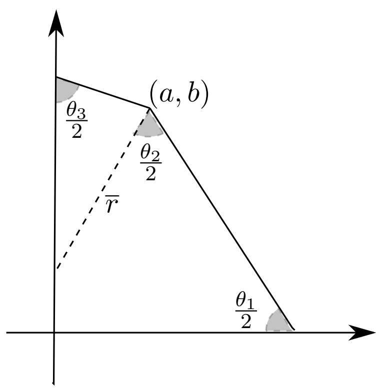

FIGURE 6.

For $(a, b) \in (0, 1)^2$ with $b > \max\{a, 1 - a\}$, the inner angles of Ω are

$$\theta_1 = 2 \arctan\left(\tfrac{b}{1-a}\right)$$
$$\theta_2 = \tfrac{3\pi}{2} - \arctan\left(\tfrac{b}{1-a}\right) - \arctan\left(\tfrac{a}{1-b}\right)$$
$$\theta_3 = 2 \arctan\left(\tfrac{a}{1-b}\right),$$

and the radius $\overline{r}$ defined by (15) is given by

$$\overline{r} = \frac{\sqrt{a^2+(1-b)^2}}{\cot\left(\frac{\theta_2}{2}\right)+\cot\left(\frac{\theta_3}{2}\right)}\,.$$

The expressions of $|\Omega|$, $|\partial\Omega|$, and $\Lambda(\Omega)$ in terms of (a,b) read

$$\begin{aligned}
&|\Omega| = 2(a+b)\\
&|\partial\Omega| = 4\Big(\sqrt{a^2+(1-b)^2}+\sqrt{b^2+(1-a)^2}\Big)\\
&\Lambda(\Omega) = 2\Big[\cot\left(\tfrac{\theta_1}{2}\right)+2\cot\left(\tfrac{\theta_2}{2}\right)+\cot\left(\tfrac{\theta_3}{2}\right)\Big]\\
&\quad = 2\,\frac{1-a}{b}+2\,\frac{1-b}{a}+4\frac{a+b-1}{\sqrt{a^2+(1-b)^2}+\sqrt{b^2+(1-a)^2}+a+b-a^2-b^2}\,.
\end{aligned} \tag{31}$$

We know that Ω is Cheeger regular if and only if $g(a,b) := |\Omega|-|\partial\Omega|\overline{r}+(\Lambda(\Omega)-\pi)\overline{r}^2 \le 0$. Figure 7, obtained with Mathematica, represents the admissible region of points where $g(a,b)\le 0$. For such values of (a,b), we can write explicitly the Cheeger constant in terms of (a,b) by using formula (16) and the expressions in (31).

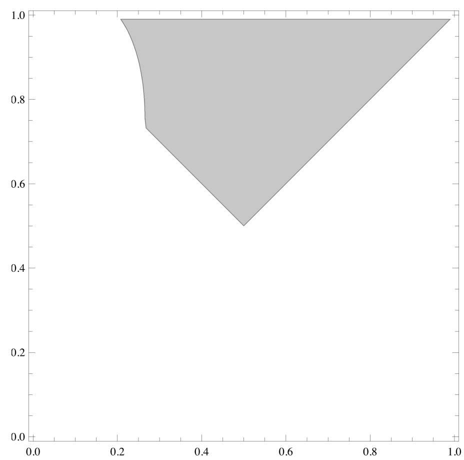

FIGURE 7. The region of points $(a,b)\in(0,1)^2$ with $b>\max\{a,1-a\}$ such that $g(a,b)\le 0$.

Then we proceed as follows. As a first step we determine a confidence zone in an explicit way, near the points where we expect to find the maximum of the map $(a,b)\mapsto h\big(\Omega_{(a,b)}\big)\big|\Omega_{(a,b)}\big|^{1/2}$, namely the vertex $(a,b)=(1,1)$ and the line segment $S:=\{(a,b) : a\in[0,1/2]\,, b=1-a\}$.

– *Confidence region near* $(a,b)=(1,1)$.

We take a point (a, b) of the form $(1-\varepsilon, 1-\delta)$. Starting from the expressions in (31), and making use only of elementary arithmetic estimates, it is easy to get the inequalities

$$\begin{aligned}
&|\partial\Omega| \leq 4\Big(2-(\varepsilon+\delta)+(\varepsilon^2+\delta^2)\Big)\,,\\
&|\Omega|^{-1/2} \leq \frac{1}{2}\Big(1+\frac{(\varepsilon+\delta)}{4}+\frac{(\varepsilon+\delta)^2}{4}\Big)\,,\\
&\Lambda(\Omega) \geq 4-2(\varepsilon+\delta)+4\varepsilon\delta\,.
\end{aligned}$$

As a consequence, using formula (16) and further straightforward computations, we obtain the following upper bound:

$$h\big(\Omega_{(a,b)}\big)\big|\Omega_{(a,b)}\big|^{1/2} \leq 2+\sqrt{\pi}-\frac{\varepsilon+\delta}{2}+2(\varepsilon+\delta)^2\,.$$

We infer that, provided $(\varepsilon+\delta) \leq \frac{1}{4}$, the function $h\big(\Omega_{(a,b)}\big)\big|\Omega_{(a,b)}\big|^{1/2}$ does not exceed the value $2+\sqrt{\pi}$. In other words, the neighbourhood of $(1,1)$ given by points $(a,b) \in (0,1)^2$ with $b \geq -a+1.75$ is a confidence region where we have just proved the required inequality analytically.

– *Confidence region near the segment S.*
We take now a point (a, b) of the form $(x+\varepsilon, 1-x+\varepsilon)$. Since it is enough to establish our inequality in the region where $g(a, b) \leq 0$, we can assume that $x \in [0.25, 0.5]$. Then we use again the expressions in (31) and elementary arithmetic estimates to obtain

$$\begin{aligned}
&|\partial\Omega| \leq 4\sqrt{2}+\frac{2\sqrt{2}\varepsilon^2}{xy}\,,\\
&|\Omega| = 2(1+2\varepsilon)\,,\\
&\Lambda(\Omega) \geq 4-2\tfrac{\varepsilon(1+2\varepsilon)}{xy}\Big(1-\tfrac{\varepsilon}{x}+\tfrac{\varepsilon^2}{x^2}\Big)\Big(1-\tfrac{\varepsilon}{y}+\tfrac{\varepsilon^2}{y^2}\Big)+\frac{4\varepsilon}{xy\Big(1+\frac{\varepsilon^2}{2x^2}\Big)\Big(1+\frac{\varepsilon^2}{2y^2}\Big)+xy}\,.
\end{aligned}$$

Then, we exploit as usual formula (16): by applying the above inequalities and further tedious but elementary computations, we arrive at the upper bound

$$h\big(\Omega_{(a,b)}\big)\big|\Omega_{(a,b)}\big|^{1/2} \leq 2+\sqrt{\pi}+\frac{1}{2\sqrt{2}}\Big(\frac{\sqrt{8}(\pi-4)}{\sqrt{\pi}}\varepsilon+\frac{49\sqrt{2}}{8\sqrt{\pi}}\frac{\varepsilon^2}{xy}+\frac{34\sqrt{2}}{8\sqrt{\pi}}\frac{\varepsilon^3}{xy}\Big)\,.$$

Now we observe that, on the line segment we are working, the product xy is bounded below by 0.1875. Then the above inequality readily implies that the difference $h\big(\Omega_{(a,b)}\big)\big|\Omega_{(a,b)}\big|^{1/2}-2+\sqrt{\pi}$ remains negative provided $\varepsilon \leq 0.01$. In other words, the neighbourhood of S given by points $(a,b) \in (0,1)^2$ with $a \in [0.25, 0.5]$ and $1-a \leq b \leq 1.01-a$ is a confidence region where we have just proved the required inequality analytically.

We summarize what we have obtained until now in Figure 8 below, where the two confidence regions determined above are displayed.
Eventually, as a second and final step, we have to prove the inequality outside the confidence regions determined so far.
Figure 9, obtained with Mathematica, shows that the region of points where $F(a, b) \leq 0$ (in light grey) contains the admissible region of points where $g(a, b) \leq 0$ (in dark grey). A rigorous proof can be obtained by using some Matlab computations. A similar strategy was used to prove a Mahler type inequality for the first Dirichlet eigenvalue in our recent paper [9], but in the present case the proof is much simpler.

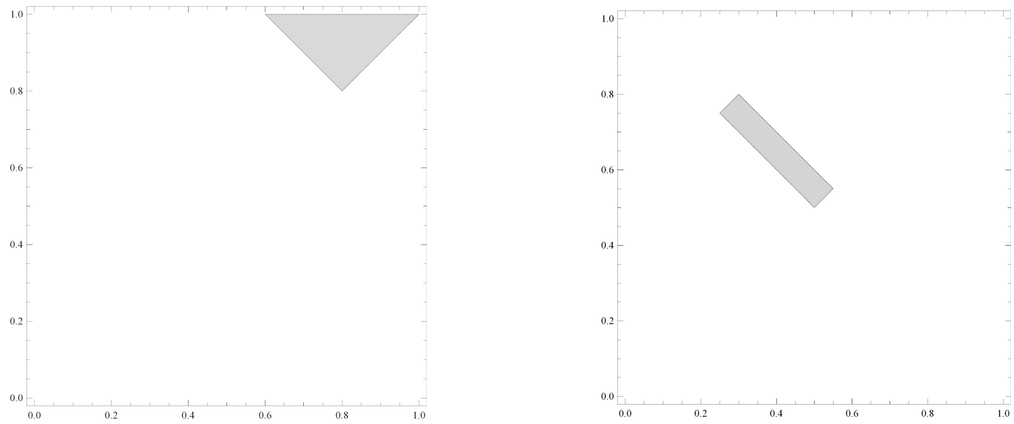

FIGURE 8. The confidence regions near the point $(1, 1)$ and near the segment S.

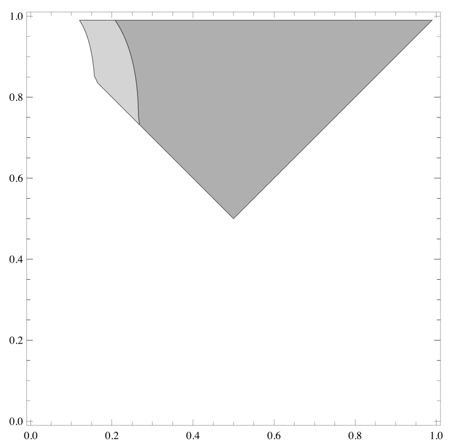

FIGURE 9. The regions of points $(a, b) \in (0, 1)^2$ with $b > \max\{a, 1 - a\}$ such that $g(a, b) \leq 0$ (in dark grey) and $F(a, b) \leq 0$ (in light grey).

The method consists in dividing the computation region (which is the complement in $\{g(a, b) \leq 0\}$ of the confidence regions) in a square grid $\{(\frac{i}{n}, \frac{j}{n}) : 0 \leq i \leq j \leq n, i + j \geq n - 1\}$ for some large enough $n \in \mathbb{N}$. Assume that for some (i, j) we prove

$$h(\Omega_{(\frac{i}{n}, \frac{j}{n})})|\Omega_{(\frac{i}{n}+\frac{1}{n}, \frac{j}{n}+\frac{1}{n})}|^{1/2} - h(Q)|Q|^{1/2} < 0. \tag{32}$$

Then we claim that the inequality (28) holds true for every (a, b) in the square of size $\frac{1}{n}$ having a low left vertex at $(\frac{i}{n}, \frac{j}{n})$.
This is indeed true, thanks to a simple monotonicity argument, which runs as follows. For every point (x, y) belonging to the this square, both the Cheeger constant and and the Lebesgue measure are monotone with respect to domain inclusion (respectively, monotone decreasing and monotone increasing). Consequently, we have

$$h(\Omega_{(x,y)})|\Omega_{(x,y)}|^{1/2} \leq h(\Omega_{(\frac{i}{n}, \frac{j}{n})})|\Omega_{(\frac{i}{n}+\frac{1}{n}, \frac{j}{n}+\frac{1}{n})}|^{1/2} < h(Q)|Q|^{1/2}. \tag{33}$$

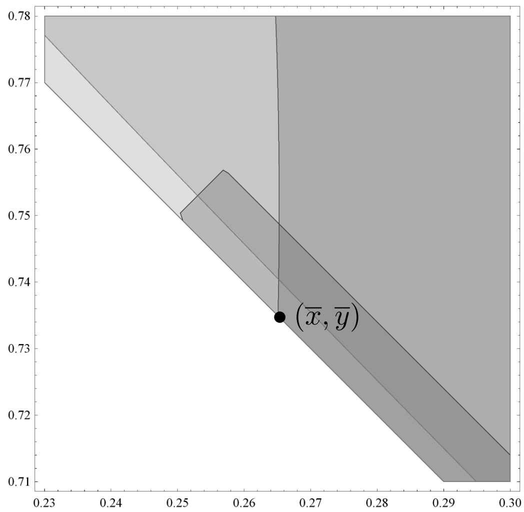

FIGURE 10. A zoom of the intersection between confidence and non-confidence region.

Hence, it remains to prove inequality (32) for a family of squares covering the complement of the confidence zone. The value $h(\Omega_{(\frac{i}{n},\frac{j}{n})})|\Omega_{(\frac{i}{n}+\frac{1}{n},\frac{j}{n}+\frac{1}{n})}|^{1/2}$ has an elementary expression, being a fraction involving polynomials of degree 4 and square roots of polynomials of degree 2. The computation of this value is carried with a precision which can be controlled a priori. Provided the computed number is negative and the computational error is smaller than its absolute value, we can conclude that inequality (32) holds true.

We used a grid of size 10^{-3} and performed the computations with machine precision, of order 10^{-16}. We decided that the value in (32) is negative, if it is a negative number of order not smaller than 10^{-4}. Our computations were carried using MATLAB on a 1.8 GHz computer with 4 Go Ram. The computation time is less than one minute. The results are available at `http://www.lama.univ-savoie.fr/~bucur/Matlab-Mahler-web/`.

The confidence zone around the point $(1,1)$ is very large and does create any difficulty.

Around the segment S, the positivity of the quantity in (32) occurs only in a narrow strip of horizontal width equal to $8 \cdot 10^{-3}$ around the segment S. It is absolutely normal to get positive values in a small neighborhood of S, since we introduce an error by choosing to compute the area on a larger octagon. It turns out that all the positive values are convered by the confidence zone, as Figure 10 shows.
A special attention must be paid when applying the monotonicity argument close to the intersection point $(\overline{x}, \overline{y})$ between the confidence region near S and the boundary of the admissible region $\{g(a,b) \leq 0\}$. In this respect, we point out that our inequality remains true at $(\overline{x}, \overline{y})$, which can be directly checked by using the estimates $0.264 \leq \overline{x} \leq 0.266$, $0.734 \leq \overline{y} \leq 0.736$.

□

5. Proof of Theorem 3

The equality $h(Q)h(Q^o) = \frac{\sqrt{2}}{4}(2+\sqrt{\pi})^2$ is immediately checked by using formula (17). The equality $h(Q)h(Q^o) = \inf_{T\in D_2} h(T(Q))h((T(Q))^o)$ follows by by using the Faber-Krahn inequality for the Cheeger constant for quadrilaterals we proved in [10], and the invariance of the volume product under invertible linear transformations. Indeed, for any $T \in D_2$, we have

$$\begin{aligned} h(T(Q))h(T(Q)^o) &= \frac{h(T(Q))|T(Q)|h((T(Q))^o)|(T(Q))^o|}{|T(Q)||(T(Q))^o|} \\ &\geq \frac{h(Q)|Q|h(Q^o)|Q^o|}{|Q||Q^o|} = h(Q)h(Q^o) \end{aligned}$$

To conclude the proof of Theorem 3 it remains show that, for every $K \in \mathcal{K}^2_\sharp$, there is a diagonal image $\widetilde{K}$ of K which satisfies

$$h(\widetilde{K})h(\widetilde{K}^o) \leq \frac{\sqrt{2}}{4}(2+\sqrt{\pi})^2\,. \tag{34}$$

Let $e_1 = (1,0)$ and $e_2 = (0,1)$, and let $\langle e_1\rangle, \langle e_2\rangle$ be the coordinate axes.
If $K \cap \langle e_i\rangle = [-\ell_i, \ell_i]$ for $i = 1, 2$, we are going to show that (34) holds true by choosing $\widetilde{K} := T(K)$, being T the map

$$T(x_1,x_2) := \big(\frac{\ell_2}{\ell_1}x_1, x_2\big) \qquad \forall (x_1,x_2)\in\mathbb{R}^2\,. \tag{35}$$

Notice that the transformed body $T(K)$ satisfies the equality condition

$$|T(K)\cap\langle e_1\rangle| = |T(K)\cap\langle e_2\rangle| (= \ell_2)\,.$$

Moreover, since $K^o \cap \langle e_i\rangle = -[\frac{1}{\ell_i}, \frac{1}{\ell_i}]$ for $i = 1, 2$, the dual transformed body $(T(K))^o = (T^t)^{-1}(K^o)$ (where $(T^t)^{-1}$ is the transformation $(T^t)^{-1}(y_1,y_2) = \big(\frac{\ell_1}{\ell_2}y_1, y_2\big)$) satisfies the same kind of equality condition, namely

$$|(T(K))^o\cap\langle e_1\rangle| = |(T(K))^o\cap\langle e_2\rangle| (= \frac{1}{\ell_2})\,.$$

Set for brevity $\widetilde{K} := T(K)$, and $\ell := \frac{1}{2}|\widetilde{K}\cap\langle e_1\rangle| = \frac{1}{2}|\widetilde{K}\cap\langle e_2\rangle|$. Let (x_1,x_2) be a fixed point in $\widetilde{K}$, with $x_1 > 0$, $x_2 > 0$. Denote by $\Omega_{(x_1,x_2)}$ the convex axisymmetric octagon with vertices at $(\pm x_1, \pm x_2)$, $(\pm\ell, 0)$, $(0, \pm\ell)$. Notice that $|\Omega_{(x_1,x_2)}| = 2\ell(x_1 + x_2)$. Since by convexity $\widetilde{K} \supseteq \Omega_{(x_1,x_2)}$, by using the monotonicity of the Cheeger constant under inclusions and Theorem 2, we obtain

$$h^2(\widetilde{K})2\ell(x_1+x_2) \leq h^2(\Omega_{(x_1,x_2)})2\ell(x_1+x_2) = h^2(\Omega_{(x_1,x_2)})|\Omega_{(x_1,x_2)}| \leq (2+\sqrt{\pi})^2\,.$$

Therefore,

$$h^2(\widetilde{K})(x_1+x_2) \leq \frac{(2+\sqrt{\pi})^2}{2\ell} \qquad \forall (x_1,x_2)\in\widetilde{K}\,. \tag{36}$$

The same argument applied to the polar body $\widetilde{K}^o$ yields

$$h^2(\widetilde{K}^o)(x_1+x_2) \leq \frac{(2+\sqrt{\pi})^2\ell}{2} \qquad \forall (x_1,x_2)\in\widetilde{K}^o\,. \tag{37}$$

The inequalities (36) and (37) imply respectively that

$$v := \Big(\frac{h^2(\widetilde{K})}{\frac{(2+\sqrt{\pi})^2}{2\ell}}, \frac{h^2(\widetilde{K})}{\frac{(2+\sqrt{\pi})^2}{2\ell}} \Big) \in \widetilde{K}^o \qquad \text{and} \qquad w := \Big(\frac{h^2(\widetilde{K}^o)}{\frac{(2+\sqrt{\pi})^2\ell}{2}}, \frac{h^2(\widetilde{K}^o)}{\frac{(2+\sqrt{\pi})^2\ell}{2}} \Big) \in \widetilde{K}\,.$$

Therefore, the scalar product of v and w is less than or equal to 1, which gives the required inequality (34). $\square$

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