Selected Topics in BSDEs Theory

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Ph. Briand, 2019-08-09

Introduction and Motivation

- A Backward Stochastic Differential Equation (BSDE) is an equation

$$dY_t = -f(t, Y_t, Z_t)\,dt + Z_t\,dB_t, \quad 0 \le t \le T, \qquad Y_T = \xi.$$

  - ⋆ Backward : $Y_T = \xi$
  - ⋆ ξ terminal condition
  - ⋆ f generator or driver

- Why two components in the solution?

  - ⋆ (Y, Z) has to be adapted to $\mathscr{F}^B$; Z makes Y adapted to $\mathscr{F}^B$

- Example: $f \equiv 0$

  - ⋆ $-dY_t = 0$, $Y_T = \xi$
  - ⋆ $Y_t = \xi$ not adapted
  - ⋆ The best adapted approximation : $Y_t = \mathbb{E}\left(\xi \mid \mathscr{F}_t^B\right)$
  - ⋆ Y is a Brownian martingale and

$$Y_t = Y_0 + \int_0^t Z_s dB_s, \quad Z \in \mathrm{L}^2, \qquad -dY_t = 0dt - Z_t dB_t$$

- Heat Equation

$$\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x), \quad t > 0,\ x \in \mathbf{R}^n, \quad u(0,x) = u_0(x), \qquad u(t,x) = \mathbb{E}\left[u_0(x + B_t)\right].$$

- Nonlinear (semilinear) Heat Equation

$$\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + f\left(u(t,x), \nabla_x u(t,x)\right), \quad t > 0,\ x \in \mathbf{R}^n, \quad u(0,x) = u_0(x).$$

  - ⋆ $T > 0$ is fixed. Set $Y_t^x = u(T-t, x+B_t)$, $Z_t^x = \nabla_x u(T-t, x+B_t)$
  - ⋆ We have if the PDE has a smooth solution

$$\begin{aligned} dY_t^x &= \left(-\partial_t u(T-t, x+B_t) + \frac{1}{2}\Delta u(T-t, x+B_t)\right) dt + \nabla_x u(T-t, x+B_t) dB_t \\ &= -f\left(u(T-t, x+B_t), \nabla_x u(T-t, x+B_t)\right) dt + \nabla_x u(T-t, x+B_t) dB_t \\ &= -f(Y_t^x, Z_t^x) dt + Z_t^x dB_t. \end{aligned}$$

  - ⋆ Since $Y_T^x = u_0(x + B_T)$, (Y^x, Z^x) solves the BSDE

$$Y_t^x = u_0(x + B_T) + \int_t^T f(Y_s^x, Z_s^x)\,ds - \int_t^T Z_s^x\,dB_s, \qquad u(T,x) = Y_0^x.$$

Contents

Lecture I. Stochastic Calculus: Prerequisite

1. Brownian Motion, Martingales, etc.

- $(\Omega, \mathscr{F}, \mathbb{P})$ a complete probability space

1.1. Stochastic Processes

Definition 1. A stochastic process, X, in $\mathbf{R}^d$ is a family $(X_t)_{t\geq 0}$ of random variables i.e. measurable applications from $(\Omega, \mathscr{F})$ to $\big(\mathbf{R}^d, \mathscr{B}(\mathbf{R}^d)\big)$.

- A stochastic process can be viewed as a random map: $\omega \longmapsto (t \longmapsto X_t(\omega))$
- A stochastic process X is measurable whenever the map $(t,\omega) \longmapsto X_t(\omega)$ from $\mathbf{R}_+ \times \Omega$ to $\mathbf{R}^d$ is measurable w.r.t. the σ-algebras $\mathscr{B}(\mathbf{R}_+) \otimes \mathscr{F}$ and $\mathscr{B}(\mathbf{R}^d)$.
  - ⋆ We will always deal with measurable processes.
- X and Y two stochastic processes
  - ⋆ X is a modification of Y if $\forall\, t \geq 0$, $\mathbb{P}(X_t = Y_t) = 1$
  - ⋆ X and Y are indistinguishable if $\mathbb{P}\,(X_t = Y_t, \forall\, t \geq 0) = 1$
- A stochastic process X is continuous if, $\mathbb{P}$-a.s., the map $t \longmapsto X_t$ is continuous

Exercise.
1. What is the stronger notion between "modification" and "indistinguishability"?
2. Show that, if X and Y are continuous stochastic processes, they are indistinguishable as soon as they are modifications

- Let $\{\mathscr{F}_t\}_{t\geq 0}$ be a filtration of $(\Omega, \mathscr{F})$: $\{\mathscr{F}_t\}_{t\geq 0}$ is an increasing family of σ-algebras

- X is adapted w.r.t. $\{\mathscr{F}_t\}_{t\geq 0}$ if X_t is $\mathscr{F}_t$–measurable for each t
  - ⋆ The smallest filtration for which X is adapted is $\mathscr{F}_t = \sigma(X_s : s \leq t)$
  - ⋆ We will always add the $\mathbb{P}$-null sets of $\mathscr{F}$, $\mathscr{N}$: $\mathscr{F}_t^X = \sigma(\mathscr{N}, X_s : s \leq t)$
- X is said to be progressively measurable if, for each t, the map $(s,\omega) \longmapsto X_s(\omega)$ from $[0,t]\times\Omega$ to $\mathbf{R}^d$ is measurable w.r.t. $\mathscr{B}([0,t])\otimes\mathscr{F}_t$ and $\mathscr{B}(\mathbf{R}^d)$
  - ⋆ A progressively measurable process is measurable and adapted
  - ⋆ If X is continuous and adapted then X is progressively measurable

1.2. Stopping times

Definition 2. A r.v. τ with values in $\overline{\mathbf{R}}_+$ is a stopping time of $\{\mathscr{F}_t\}_{t\geq 0}$ if

$$\forall t \geq 0, \qquad \{\tau \leq t\} \in \mathscr{F}_t.$$

- If τ is a stopping time,
$$\mathscr{F}_\tau = \{A \in \mathscr{F}_\infty,\ A \cap \{\tau \leq t\} \in \mathscr{F}_t, \forall t\}$$
is a σ-algebra
  - ⋆ $\mathscr{F}_\infty = \sigma(\mathscr{F}_t : t \geq 0)$
  - ⋆ The events in $\mathscr{F}_\tau$ can be thought as events that may occur before τ
- If X is progressively measurable and τ is a stopping time then the stopped process X^τ is also progressively measurable w.r.t. $\mathscr{F}_{t\wedge\tau}$
  - ⋆ $X_t^\tau = X_{\tau\wedge t} : X_t^\tau(\omega) = X_{\tau(\omega)\wedge t}(\omega)$

1.3. Brownian Motion

Definition 3. A real valued stochastic process B is a *Brownian motion* if :

1. $B_0 = 0$ $\mathbb{P}$–a.s.
2. For $0 \leq s < t$, $B_t - B_s$ is independent of $\sigma\{B_u, u \leq s\}$ and is a gaussian r.v. with mean 0 and variance $t-s$;
3. continuous paths: $\mathbb{P}$–a.s. $t \longmapsto B_t(\omega)$ is continuous;

- For $t > 0$, the density of B_t is given by $(2\pi t)^{-1/2}\exp\{-x^2/(2t)\}$
- If the filtration $\{\mathscr{F}_t\}_{t\geq 0}$ is given, B is said to be a $\{\mathscr{F}_t\}_{t\geq 0}$-BM if B is adapted with continuous paths and
$$\forall u \in \mathbf{R}, \quad \forall 0 \leq s \leq t, \qquad \mathbb{E}\left(e^{iu(B_t-B_s)} \,|\, \mathscr{F}_s\right) = \exp\left\{-u^2(t-s)/2\right\}.$$
- If B is a BM, the filtration $\mathscr{F}_t^B = \sigma(\mathscr{N}, B_s : s \leq t)$ is right continuous and complete and B is a BM w.r.t. this filtration

- ⋆ We will always work in this setting

Exercise. 1. Let $X_t = \sup_{s\le t} B_s$. Is X and adapted process? A progressively measurable process?

2. Let $Y_t = B_t + B_{2t}$. Is Y and adapted process?

3. Let $c>0$. Show that $\{cB_{t/c^2}\}_{t\ge 0}$ is a BM.

Theorem 1 (Paths regularity)**.** *Let B a BM. Then $\mathbb{P}$-a.s.*

1. *$t \longmapsto B_t(\omega)$ is not of finite variation on any interval*
2. *$t \longmapsto B_t(\omega)$ is locally Hölder continuous of order α for $\alpha < 1/2$.*
3. *$t \longmapsto B_t(\omega)$ is not differentiable at any point*

Definition 4. A BM with values in $\mathbf{R}^d$ is a vector $B = \left(B^1, \dots, B^d\right)$ where B^i are independent real BM.

1.4. Martingales

Definition 5. A real stochastic process X is a supermartingale w.r.t. $\{\mathscr{F}_t\}_{t\ge 0}$ if:

1. for $t \ge 0$, X_t is $\mathscr{F}_t$–mesurable (X is adapted)
2. for $t\ge 0$, X_t is integrable: $\mathbb{E}[|X_t|] < +\infty$
3. for $0 \le s \le t$, $\mathbb{E}(X_t \mid \mathscr{F}_s) \le X_s$

X is a submartingale if $-X$ is a supermartingale: $\mathbb{E}(X_t \mid \mathscr{F}_s) \ge X_s$.
X is a martingale if X is a supermartingale and a submartingale: $\mathbb{E}(X_t \mid \mathscr{F}_s) = X_s$.

- If X is a martingale, S and T two bounded stopping times with $S \le T$ then

$$\mathbb{E}(X_T|\mathscr{F}_S) = X_S, \quad \mathbb{P} - a.s.$$

Example. Let B be a BM. Then B, $\left\{B_t^2 - t\right\}_{t\ge 0}$ and $\left\{\exp\left(\sigma B_t - \sigma^2 t/2\right)\right\}_{t\ge 0}$ are martingales.

Theorem 2 (Doob Maximal Inequalities)**.** *Let X be a martingale (or a nonnegative submartingale) with right-continuous paths. Then,*

1. *$\forall p \ge 1$, $\forall a > 0$, $\quad a^p\, \mathbb{P}\left(\sup_t |X_t| \ge a\right) \le \sup_t \mathbb{E}[|X_t|^p]$;*
2. *$\forall p > 1$, $\quad \mathbb{E}\left[\sup_t |X_t|^p\right] \le q^p \sup_t \mathbb{E}[|X_t|^p]$ where $q = p(p-1)^{-1}$.*

- We will always work with continuous stochastic processes

Definition 6. Let $\{\mathscr{F}_t\}_{t\ge 0}$ be a filtration.

An adapted continuous stochastic process X is a local martingale if there exists a nondecreasing sequence of stopping times $\{\tau_n\}_{n\ge 1}$ s.t. $\lim_{n\to\infty} \tau_n = +\infty$ $\mathbb{P}$–a.s and, for all $n \ge 1$, X^{τ_n} is a martingale.

Theorem 3. *Let X be a continuous local martingale. There exists a unique nondecreasing and continuous process, $\langle X,X\rangle$, s.t. $\langle X,X\rangle_0 = 0$ and $X^2 - \langle X,X\rangle$ is a local martingale.*

Example. If B is a BM, $\langle B,B\rangle_t = t$.

Theorem 4 (BDG inequalities)**.** *Let $p > 0$. There exist two constant c_p et C_p s.t., if X is a continuous local martingale with $X_0 = 0$,*

$$c_p \mathbb{E}\left[\langle X,X\rangle_\infty^{p/2}\right] \le \mathbb{E}\left[\sup_{t\ge 0}|X_t|^p\right] \le C_p \mathbb{E}\left[\langle X,X\rangle_\infty^{p/2}\right].$$

- BDG = Burkholder–Davis–Gundy
- In particular, for any real $T > 0$,

$$c_p \mathbb{E}\left[\langle X,X\rangle_T^{p/2}\right] \le \mathbb{E}\left[\sup_{0\le t\le T}|X_t|^p\right] \le C_p \mathbb{E}\left[\langle X,X\rangle_T^{p/2}\right].$$

2. Itô Calculus

2.1. Stochastic Integration

- Define the integral $\int_0^t H_s\,dB_s$ where B is a BM
  - ⋆ This is not so easy since the paths of B are not of finite variation
- Let $T > 0$ and $H = (H_t)_{0\le t\le T}$ a simple process i.e. a stochastic process of the form

$$H_t = \phi_0 \mathbf{1}_0(t) + \sum_{i=1}^{p} \phi_i \mathbf{1}_{]t_{i-1},t_i]}(t),$$

  where $0 = t_0 < t_1 < \ldots < t_p = T$, ϕ_0 is a r.v. $\mathscr{F}_0$–measurable and bounded, and, for $i = 1,\ldots,p$, ϕ_i is a r.v. $\mathscr{F}_{t_{i-1}}$–measurable and bounded.

- We set, for $0 \le t \le T$,

$$\int_0^t H_s\,dB_s = \sum_{i=1}^{p} \phi_i (B_{t_i\wedge t} - B_{t_{i-1}\wedge t})$$

  - ⋆ If $t \in]t_k, t_{k+1}]$,

$$\int_0^t H_s\,dB_s = \sum_{i=1}^{k} \phi_i (B_{t_i} - B_{t_{i-1}}) + \phi_{k+1}(B_t - B_{t_k}).$$

Proposition 5. *If H is a simple process, then $\left(\int_0^t H_s\,dB_s\right)_{0\le t\le T}$ is a continuous martingale s.t.*

$$\forall t \in [0,T], \qquad \mathbb{E}\left[\left|\int_0^t H_s\,dB_s\right|^2\right] = \mathbb{E}\left[\int_0^t |H_s|^2\,ds\right].$$

- Since simple processes are dense in the space

$$\mathcal{M}^2 = \left\{(H_t)_{0\le t\le T},\ \text{progressively measurable, } \mathbb{E}\left[\int_0^T |H_s|^2\, ds\right] < \infty\right\}$$

  one can define the stochastic integral for $H \in \mathcal{M}^2$ and the results of the previous proposition are still true

Proposition 6. *Let $H \in \mathcal{M}^2$. Then, we have*

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t H_s\, dB_s\right|^2\right] \le 4\mathbb{E}\left[\int_0^T H_s^2\, ds\right],$$

and, if τ is a stopping time,

$$\int_0^\tau H_s\, dB_s = \int_0^T \mathbf{1}_{s\le\tau} H_s\, dB_s, \quad \mathbb{P}\text{–a.s.}$$

- Finally, we can relax the integrability assumption on H
- We can define the stochastic integral for H in the space

$$\mathcal{M}^2_{\text{loc}} = \left\{(H_t)_{0\le t\le T},\ \text{progressively measurable, } \int_0^T |H_s|^2\, ds < \infty\ \mathbb{P}\text{–a.s.}\right\}$$

- In this case, the stochastic integral is a local martingale s.t.

$$\langle \int_0^\cdot H_s\, dB_s \rangle_t = \int_0^t |H_s|^2\, ds.$$

2.2. Itô Processes

- An Itô process is a process X of the form

$$\forall 0 \le t \le T, \qquad X_t = X_0 + \int_0^t K_s\, ds + \int_0^t H_s\, dB_s,$$

  where X_0 is $\mathcal{F}_0$–measurable, K and H two progressively measurable processes s.t. $\mathbb{P}$–a.s.:

$$\int_0^T |K_s|\, ds + \int_0^T |H_s|^2\, ds < +\infty.$$

- In differential form, we have

$$dX_t = K_t\, dt + H_t\, dB_t, \quad t \ge 0.$$

- If X and Y are two such processes, we set

$$\langle X, Y\rangle_t = \int_0^t H_s H'_s\, ds$$

  ⋆ This is the quadratic variation of the martingale parts of X and Y

Proposition 7 (Integration by part formula)**.** *If X and Y are two Itô processes*

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s\, dY_s + \int_0^t Y_s\, dX_s + \langle X, Y\rangle_t.$$

- The usual formula can not be true since B_t^2 is not a martingale!
- The extra term comes from the fact that $\langle B\rangle_t = t$:

$$\langle B\rangle_t = \lim_{|P|\to 0} \sum \left(B_{t_i} - B_{t_{i-1}}\right)^2$$

  ⋆ $P = (t_i)$ subdivision of $[0, T]$, $|P| = \max(t_i - t_{i-1})$

- If X has finite variation paths then $\langle X\rangle_t = 0$.

Theorem 8 (Itô's formula)**.** *Let* $(t, x) \longmapsto f(t, x)$ *be a* $\mathscr{C}^{1,2}$ *function and X an Itô process. Then*

$$f(t, X_t) = f(0, X_0) + \int_0^t f_s'(s, X_s)\, ds + \int_0^t f_x'(s, X_s)\, dX_s + \frac{1}{2}\int_0^t f_{xx}''(s, X_s)\, d\langle X, X\rangle_s.$$

- The result is still true if X is a continuous local martingale
- In the case of an Itô process X, the formula rewrites

$$f(t, X_t) = f(0, X_0) + \int_0^t \left(\partial_s f(s, X_s) + \partial_x f(s, X_s)\, K_s\right) ds$$
$$+ \frac{1}{2}\int_0^t \partial_{xx}^2 f(s, X_s) H_s^2\, ds + \int_0^t \partial_x f(s, X_s)\, H_s\, dB_s.$$

Example. 1. Let $X_t = \exp\left(\sigma B_t - \sigma^2 t/2\right)$. Show that

$$X_t = 1 + \sigma \int_0^t X_s\, dB_s, \quad t \geq 0.$$

2. Show that the stochastic differential equation

$$dX_t = \alpha X_t\, dt + \sigma\, dB_t, \quad t \geq 0, \qquad X_0 = x \in \mathbf{R},$$
$$X_t = x + \alpha \int_0^t X_s\, ds + \sigma B_t, \quad t \geq 0,$$

has a unique solution. Hint: $Y_t = e^{-\alpha t} X_t$.

- Let X be an Itô process in $\mathbf{R}^n$ meaning that, for $i = 1, \dots, n$,

$$X_t^i = X_0^i + \int_0^t K_s^i\, ds + \sum_{k=1}^d \int_0^t H_s^{i,k}\, dB_s^k, \quad t \geq 0.$$

- If f is a smooth function i.e. $f \in \mathscr{C}^{1,2}$, then

$$f(t, X_t) = f(0, X_0) + \int_0^t \partial_s f(s, X_s)\, ds + \sum_{i=1}^n \int_0^t \partial_{x_i} f(s, X_s)\, dX_s^i$$
$$+ \frac{1}{2}\sum_{i,j=1}^n \int_0^t \partial_{x_i, x_j}^2 f(s, X_s)\, d\langle X^i, X^j\rangle_s,$$

  where $dX_s^i = K_s^i\, ds + \sum_{k=1}^d H_s^{i,k}\, dB_s^k$ and $d\langle X^i, X^j\rangle_s = \sum_{k=1}^d H_s^{i,k} H_s^{j,k}\, ds$.

- The formula is simpler using vectors notations: H is an $n\times d$ matrix, X, K columns of length n, B a column of size d,

$$X_t = X_0 + \int_0^t K_s\,ds + \int_0^t H_s\,dB_s, \quad t \geq 0$$

- Itô's formula reads

$$\begin{aligned} f(t,X_t) &= f(0,X_0) + \int_0^t \partial_s f(s,X_s)\,ds + \int_0^t \nabla f(s,X_s)\cdot dX_s \\ &\quad + \frac{1}{2}\int_0^t \operatorname{trace}\left(\mathrm{D}^2 f(s,X_s) H_s H_s^*\right) ds \\ &= f(0,X_0) + \int_0^t \left(\partial_s f(s,X_s) + \nabla f(s,X_s)\cdot K_s\right) ds \\ &\quad + \frac{1}{2}\int_0^t \operatorname{trace}\left(\mathrm{D}^2 f(s,X_s) H_s H_s^*\right) ds + \int_0^t \mathrm{D} f(s,X_s) H_s\,dB_s \end{aligned}$$

  ⋆ Observe that $\operatorname{trace}\left(H_s H_s^*\right) = |H_s|^2$.

3. Important Results

Theorem 9 (Paul Lévy)**.** *Let X be a continuous $\{\mathscr{F}_t\}_{t\geq 0}$–local martingale, with $X_0 = 0$. We assume that, for $i, j \in \{1,\dots,d\}$, $\langle X^i, X^j\rangle_t = \delta_{i,j}\, t$.*

Then X is a $\{\mathscr{F}_t\}_{t\geq 0}$–BM in $\mathbf{R}^d$.

Proof. • We have to prove that

$$\forall 0 \leq s \leq t \leq T, \quad \forall u \in \mathbf{R}^d, \qquad \mathbb{E}\left(e^{iu\cdot(X_t - X_s)} \mid \mathscr{F}_s\right) = \exp\left\{-|u|^2(t-s)/2\right\}.$$

- By Itô's formula applied to $x \longmapsto e^{iu\cdot x}$, we get

$$e^{iu\cdot X_t} = e^{iu\cdot X_s} + \int_s^t i e^{iu\cdot X_r} u\cdot dX_r - \frac{|u|^2}{2}\int_s^t e^{iu\cdot X_r}\,dr.$$

- By BDG inequality, since $\langle X\rangle_t = t$, X is a square integrable martingale

  ⋆ Thus, the same is true for the previous stochastic integral

- Taking conditional expectation w.r.t. $\mathscr{F}_s$, we obtain

$$\mathbb{E}\left(e^{iu\cdot X_t} \mid \mathscr{F}_s\right) = e^{iu\cdot X_s} - \frac{|u|^2}{2}\int_s^t \mathbb{E}\left(e^{iu\cdot X_r} \mid \mathscr{F}_s\right) dr$$

- Thus, we have, for all $t \geq s$,

$$\mathbb{E}\left(e^{iu\cdot(X_t - X_s} \mid \mathscr{F}_s\right) = 1 - \frac{|u|^2}{2}\int_s^t \mathbb{E}\left(e^{iu\cdot(X_r - X_s)} \mid \mathscr{F}_s\right) dr.$$

  ⋆ This gives the result.

□

Theorem 10 (Girsanov)**.** *Let $(h_t)_{0\le t\le T}$ be a stochastic process in $\mathcal{M}^2_{loc}$ taking values in $\mathbf{R}^d$. We consider the process $(D_t)_{0\le t\le T}$ defined by*

$$D_t = \exp\left\{\int_0^t h_s \cdot dB_s - \frac{1}{2}\int_0^t |h_s|^2\, ds\right\}, \quad 0 \le t \le T.$$

If D is a martingale then the stochastic process B^* given by

$$B^*_t = B_t - \int_0^t h_s\, ds, \quad 0 \le t \le T,$$

is a BM w.r.t. $\mathbb{P}^*$ where $d\mathbb{P}^* = D_T \cdot d\mathbb{P}$ on $\mathcal{F}_T$.

- Novikov criterium: If
$$\mathbb{E}\left[\exp\left\{1/2\int_0^T |h_s|^2\, ds\right\}\right] < +\infty$$
then $\{D_t\}_{0\le t\le T}$ is a martingale.

Proof. • B^* is continuous and $\langle B^*\rangle_t = t$

- In view of Lévy theorem, we have to prove that B^* is a $\mathbb{P}^*$–local martingale
- Since, $dD_t = h_t D_t\, dB_t$ and $dB^*_t = -h_t\, dt + dB_t$, we have
$$\begin{aligned} d(D_t B^*_t) &= D_t dB^*_t + B^*_t\, dD_t + h_t D_t\, dt, \\ &= -h_t D_t dt + D_t dB_t + B^*_t h_t D_t\, dB_t + h_t D_t\, dt, \\ &= D_t(1 + h_t B^*_t)\, dB_t \end{aligned}$$
- Thus, DB^* is a local martingale under $\mathbb{P}$ as a stochastic integral
- This gives the result since
$$\mathbb{E}^*(B^*_t \mid \mathcal{F}_s) = D_s^{-1}\mathbb{E}(D_t B^*_t \mid \mathcal{F}_s) = B^*_s.$$

□

Theorem 11 (Brownian martingales)**.** *Let M be a square integrable martingale w.r.t. the Brownian filtration $\{\mathcal{F}^B_t\}_{t\in[0,T]}$.*

Then, there exists a unique process $(H_t)_{t\in[0,T]} \in \mathrm{M}^2(\mathbf{R}^k)$, s.t.

$$\mathbb{P}\text{–a.s.} \qquad \forall\, t \in [0,T], \quad M_t = M_0 + \int_0^t H_s \cdot dB_s.$$

- In particular, every Brownian martingale is continuous
- If ξ is a square integrable r.v., $\mathcal{F}^B_T$–measurable, then
$$\xi = \mathbb{E}[\xi] + \int_0^T H_s \cdot dB_s$$
for a unique $(H_t)_{t\in[0,T]} \in \mathrm{M}^2(\mathbf{R}^k)$.
  - ⋆ This follows from the previous result applied to $M_t = \mathbb{E}\left(\xi \mid \mathcal{F}^B_t\right)$.
- In these results, the process H can be chosen predictable
  - ⋆ The sigma algebra of predictable sets is generated by continuous and adapted processes

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Lecture II. Basic Properties of BSDEs

1. Review of Previous Lecture

- Let X be an Itô process in $\mathbf{R}^n$

$$dX_t = K_t\,dt + H_t\,dB_t, \quad t \geq 0$$

- From Itô's formula, for $0 \leq t \leq T$,

$$|X_T|^2 = |X_t|^2 + \int_t^T \left(2X_s \cdot K_s + |H_s|^2\right) ds + 2\int_t^T X_s \cdot H_s dB_s$$

  and, for any $\alpha \in \mathbf{R}$,

$$e^{\alpha T}|X_T|^2 = e^{\alpha t}|X_t|^2 + \int_t^T e^{\alpha s}\left(2X_s \cdot K_s + |H_s|^2 + \alpha|X_s|^2\right) ds + 2\int_t^T e^{\alpha s} X_s \cdot H_s dB_s$$

- If $\xi \in \mathrm{L}^2\left(\mathscr{F}_T^B\right)$, then, there exists a unique $H \in \mathrm{M}^2(\mathbf{R}^k)$, s.t.

$$\mathbb{E}\left(\xi \mid \mathscr{F}_t^B\right) = \mathbb{E}[\xi] + \int_0^T H_s \cdot dB_s, \quad 0 \leq t \leq T$$

- The process H can be chosen predictable
  - ⋆ The sigma algebra of predictable sets is generated by continuous and adapted processes

2. Notations

- $(\Omega, \mathscr{F}, \mathbb{P})$ complete probability space

- B is a standard Brownian motion in $\mathbf{R}^d$

  ⋆ $\mathscr{F}_t = \mathscr{F}_t^B \vee \mathscr{N}$

- $f : [0,T] \times \Omega \times \mathbf{R}^k \times \mathbf{R}^{k\times d} \longrightarrow \mathbf{R}^k$ a measurable map w.r.t. $\mathscr{P} \otimes \mathscr{B}\left(\mathbf{R}^k\right) \otimes \mathscr{B}\left(\mathbf{R}^{k\times d}\right)$ and $\mathscr{B}\left(\mathbf{R}^k\right)$ where $\mathscr{P}$ is the sigma algebra of the progressive sets over $[0,T] \times \Omega$.

- ξ a random variable in $\mathbf{R}^k$, $\mathscr{F}_T$–measurable.

- We consider the following BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dB_s, \quad 0 \le t \le T, \tag{$\mathrm{E}_{\xi,f}$}$$

  ⋆ In differential form

$$\begin{aligned} +dY_t &= -f(t, Y_t, Z_t)\, dt + Z_t\, dB_t, \quad 0 \le t \le T, \qquad Y_T = \xi, \\ -dY_t &= +f(t, Y_t, Z_t)\, dt - Z_t\, dB_t, \quad 0 \le t \le T, \qquad Y_T = \xi. \end{aligned}$$

Definition 7. A solution to the BSDE $(\mathrm{E}_{\xi,f})$ is a pair of processes (Y, Z) with values in $\mathbf{R}^k \times \mathbf{R}^{k\times d}$ such that Y is continuous and adapted, Z is predictable and, $\mathbb{P}$–a.s., $t \longmapsto Z_t$ belongs to $\mathrm{L}^2(0,T)$, $t \longmapsto f(t, Y_t, Z_t)$ belongs to $\mathrm{L}^1(0,T)$ $\mathbb{P}$–a.s. and

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)\, dr - \int_t^T Z_r\, dB_r, \qquad 0 \le t \le T.$$

- Two sets of processes

$$\begin{aligned} \mathscr{S}^2\left(\mathbf{R}^k\right) &= \left\{ Y \in \mathbf{R}^k : Y \text{ continuous and adapted } \mathbb{E}\left[\sup_{0\le t\le T} |Y_t|^2\right] < +\infty \right\} \\ \mathrm{M}^2\left(\mathbf{R}^{k\times d}\right) &= \left\{ Z \in \mathbf{R}^{k\times d} : Z \text{ predictable } \mathbb{E}\left[\int_0^T |Z_t|^2\, dt\right] < +\infty \right\} \end{aligned}$$

- $\mathscr{B}^2 := \mathscr{S}^2 \times \mathrm{M}^2$

- Is there any chance to solve the problem?

  ⋆ Yes we can! Easy case: $f(t, y, z) = f(t)$

3. Pardoux-Peng's result

- We will denote by (L) the following assumption :
  - There exists $\lambda \geq 0$, such that $\mathbb{P}$–a.s., for all $t \in [0,T]$,

$$\forall (y,y'), \quad \forall (z,z'), \quad \left|f(t,y,z) - f\left(t,y',z'\right)\right| \leq \lambda\left(\left|y-y'\right| + \left|z-z'\right|\right);$$

  - ξ and $\{f(t,0,0)\}_{0\leq t\leq T}$ are square integrable:

$$\mathbb{E}\left[|\xi|^2 + \int_0^T |f(t,0,0)|^2\, dt\right] < +\infty.$$

Theorem 1 (Pardoux-Peng, 1990)**.** *Let (L) holds. The BSDE* $(\mathrm{E}_{\xi,f})$ *has a unique solution* $(Y,Z) \in \mathscr{B}^2$*. Moreover*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |Y_t|^2 + \int_0^T |Z_t|^2\, dt\right] \leq C(\lambda,T)\,\mathbb{E}\left[|\xi|^2 + \int_0^T |f(t,0,0)|^2\, dt\right],$$

$C(\lambda,T) = C\, e^{(2\lambda^2+2\lambda+1)T}$.

Remark. Under (L), if (Y,Z) solves $(\mathrm{E}_{\xi,f})$ with $Z \in \mathrm{M}^2$ then $Y \in \mathscr{S}^2$. In Pardoux-Peng's theorem, we get a unique solution s.t. $Z \in \mathrm{M}^2$.

- For $t \in [0,T]$,

$$Y_t = Y_0 - \int_0^t f(r,Y_r,Z_r)\, dr + \int_0^t Z_r\, dB_r,$$

- Using the Lipschitz assumption on f,

$$|Y_t| \leq |Y_0| + \int_0^T \left(|f(r,0,0)| + \lambda |Z_r|\right) dr + \sup_{0\leq t\leq T}\left|\int_0^t Z_r\, dB_r\right| + \lambda \int_0^t |Y_r|\, dr.$$

- Let us introduce

$$\zeta = |Y_0| + \int_0^T \left(|f(r,0,0)| + \lambda |Z_r|\right) dr + \sup_{0\leq t\leq T}\left|\int_0^t Z_r\, dB_r\right|.$$

  ⋆ $\zeta \in \mathrm{L}^2$

- Gronwall's lemma gives

$$\sup_{0\leq t\leq T} |Y_t| \leq \zeta e^{\lambda T}.$$

- Still true if f has a linear growth

$$\left|f(t,y,z)\right| \leq f_t + \lambda\left(|y| + |z|\right).$$

Lemma 2. *If* $Y \in \mathscr{S}^2$ *and* $Z \in \mathrm{M}^2$*, then* $M_t = 2\int_0^t Y_s \cdot Z_s\, dB_s$ *is a uniformly integrable martingale and, there exists a constant* c *(*$c = 3$*) s.t., for* $\eta > 0$*,*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |M_t|\right] \leq \eta\, \mathbb{E}\left[\sup_{0\leq t\leq T} |Y_t|^2\right] + \frac{c^2}{\eta}\mathbb{E}\left[\int_0^T |Z_t|^2\, dt\right].$$

Proof.

- From BDG inequality, $(c = 3)$

$$\mathbb{E}\left[\sup_{0\le t\le T}|M_t|\right] \le c\,\mathbb{E}\left[\langle M\rangle_T^{1/2}\right] \le 2c\,\mathbb{E}\left[\left(\int_0^T |Y_s|^2|Z_s|^2\,ds\right)^{1/2}\right]$$
$$\le 2c\,\mathbb{E}\left[\sup_{0\le t\le T}|Y_t|\left(\int_0^T |Z_s|^2\,ds\right)^{1/2}\right]$$

- Use $2ab \le \eta a^2 + b^2/\eta$

□

Proposition 3 (A priori estimate)**.** *Let (Y,Z) be a solution to $(\mathrm{E}_{\xi,f})$ with $Z \in \mathrm{M}^2$. Then, for $\varepsilon > 0$,*

$$\mathbb{E}\left[\sup_{0\le t\le T} e^{2\alpha t}|Y_t|^2 + \int_0^T e^{2\alpha t}|Z_t|^2\right] \le 4\left(1+8c^2\right)\mathbb{E}\left[e^{2\alpha T}|\xi|^2 + \varepsilon\int_0^T e^{2\alpha t}|f(t,0,0)|^2\,dt\right],$$

as soon as $\alpha \ge \alpha_\varepsilon := \lambda^2 + \lambda + 1/(2\varepsilon)$ ($c = 3$ works!).

- For the estimate of Pardoux-Peng's theorem, use $\varepsilon = 1$!

Proof.

- Itô's formula to $e^{2\alpha t}|Y_t|^2$, $\alpha \in \mathbf{R}$.
- Compute $-\int_t^T d\left(e^{2\alpha s}|Y_s|^2\right)$ and, for $0 \le t \le T$,

$$e^{2\alpha t}|Y_t|^2 + \int_t^T e^{2\alpha s}|Z_s|^2 ds$$
$$= e^{2\alpha T}|\xi|^2 + \int_t^T e^{2\alpha s}\left(2Y_s\cdot f(s,Y_s,Z_s) - 2\alpha|Y_s|^2\right)ds - (M_T - M_t),$$

where $M_t = 2\int_0^t e^{2\alpha s}Y_s Z_s dB_s$.

- f is Lipschitz and $2\,a\,b \le \varepsilon|a|^2 + |b|^2/\varepsilon$

$$2y\cdot f(s,y,z) \le 2|y|\,|f(s,y,z)| \le 2|y||f(s,0,0)| + 2\lambda|y|^2 + 2\lambda|y|\,|z|$$
$$\le \varepsilon|f(s,0,0)|^2 + |z|^2/2 + (1/\varepsilon + 2\lambda + 2\lambda^2)|y|^2$$

- If $\alpha \ge (1/(2\varepsilon) + \lambda + \lambda^2)$, for all $0 \le t \le T$,

$$e^{2\alpha t}|Y_t|^2 + \frac{1}{2}\int_t^T e^{2\alpha s}|Z_s|^2 ds \le e^{2\alpha T}|\xi|^2 + \varepsilon\int_t^T e^{2\alpha s}|f(s,0,0)|^2 ds - (M_T - M_t), \tag{1}$$
$$\le X_T - (M_T - M_t), \tag{2}$$

where we have set $X_T = e^{2\alpha T}|\xi|^2 + \varepsilon\int_0^T e^{2\alpha s}|f(s,0,0)|^2 ds$.

- Taking the conditional expectation of (1), we deduce immediately

$$e^{2\alpha t}|Y_t|^2+\frac{1}{2}\mathbb{E}\left(\int_t^T e^{2\alpha s}|Z_s|^2 ds\,\middle|\,\mathscr{F}_t\right)\le\mathbb{E}\left(e^{2\alpha T}|\xi|^2+\varepsilon\int_t^T e^{2\alpha s}|f(s,0,0)|^2 ds\,|\,\mathscr{F}_t\right). \tag{3}$$

- $t=0$, we have, taking the expectation of (2),

$$\frac{1}{2}\mathbb{E}\left[\int_0^T e^{2\alpha s}|Z_s|^2 ds\right]\le\mathbb{E}\left[X_T\right], \tag{4}$$

- Using the inequality of the lemma, coming back to (2)

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}e^{2\alpha t}|Y_t|^2\right]&\le\mathbb{E}\left[X_T\right]+2\mathbb{E}\left[\sup_{t\in[0,T]}|M_t|\right]\\&\le\mathbb{E}[X_T]+2\eta\,\mathbb{E}\left[\sup_{t\in[0,T]}e^{2\alpha t}|Y_t|^2\right]+\frac{2c^2}{\eta}\mathbb{E}\left[\int_0^T e^{2\alpha s}|Z_s|^2 ds\right]\end{aligned}$$

- Choose $\eta=1/4$ to get, taking the inequality (4)

$$\frac{1}{2}\mathbb{E}\left[\sup e^{2\alpha t}|Y_t|^2\right]\le\mathbb{E}[X_T]+\frac{16c^2}{2}\mathbb{E}\left[\int_0^T e^{2\alpha s}|Z_s|^2 ds\right]\le(1+16c^2)\,\mathbb{E}[X_T]$$

- Finally,

$$\mathbb{E}\left[\sup e^{2\alpha t}|Y_t|^2\right]+\mathbb{E}\left[\int_0^T e^{2\alpha s}|Z_s|^2 ds\right]\le 4(1+8c^2)\,\mathbb{E}[X_T]$$

□

Remark.

- Actually, we prove that if ξ and $f(t,0,0)$ are bounded, then Y is a bounded process.
- Indeed, (3) gives, for $\varepsilon=1$, $\alpha=\lambda^2+\lambda+1/2$

$$\begin{aligned}e^{2\alpha t}|Y_t|^2&\le\mathbb{E}\left(e^{2\alpha T}|\xi|^2+\int_t^T e^{2\alpha s}|f(s,0,0)|^2 ds\,|\,\mathscr{F}_t\right),\\|Y_t|^2&\le\mathbb{E}\left(e^{2\alpha(T-t)}|\xi|^2+\int_t^T e^{2\alpha(s-t)}|f(s,0,0)|^2 ds\,|\,\mathscr{F}_t\right),\\&\le e^{(2\lambda^2+2\lambda+1)T}\left(\|\xi\|_\infty^2+T\|f(\cdot,0,0)\|_\infty^2\right)\end{aligned}$$

Corollary 4. *If (Y^1,Z^1), (Y^2,Z^2) solves the BSDEs associated to (ξ^1,f^1) and (ξ^2,f^2) then, for $\varepsilon>0$,*

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le t\le T}e^{2\alpha t}|\delta Y_t|^2+\int_0^T e^{2\alpha t}|\delta Z_t|^2\,dt\right]\\&\qquad\le 4\left(1+8c^2\right)\mathbb{E}\left[e^{2\alpha T}|\delta\xi|^2+\varepsilon\int_0^T e^{2\alpha t}|\delta f|^2\left(t,Y_t^2,Z_t^2\right)dt\right],\end{aligned}$$

where $\alpha\ge\alpha_\varepsilon:=\lambda_1^2+\lambda_1+1/(2\varepsilon)$, $c\ge 3$ and $\delta BlaBla=BlaBla^1-BlaBla^2$.

- λ is the Lipschitz constant of f^1.

Proof of Pardoux–Peng's theorem.

- Uniqueness is a direct consequence of the a priori estimate see Corollory 4.
- Existence by a fixed point argument.
- If $(U,V) \in \mathscr{B}^2$, let us solve the BSDE

$$Y_t = \xi + \int_t^T f(s,U_s,V_s)\,ds - \int_t^T Z_s\,dB_S, \quad 0 \le t \le T.$$

- The solution is given by

$$\begin{aligned} Y_t &= \mathbb{E}\left(\xi + \int_t^T f(s,U_s,V_s)\,ds \,\middle|\, \mathscr{F}_t\right) \\ &= \mathbb{E}\left(\xi + \int_0^T f(s,U_s,V_s)\,ds \,\middle|\, \mathscr{F}_t\right) - \int_0^t f(s,U_s,V_s)\,ds \\ &= \mathbb{E}\left[\xi + \int_0^T f(s,U_s,V_s)\,ds\right] + \int_0^t Z_s\,dB_s - \int_0^t f(s,U_s,V_s)\,ds. \end{aligned}$$

- By Corollary 4, for $\varepsilon > 0$ and $\alpha \ge 1/(2\varepsilon)$,

$$\mathbb{E}\left[\sup_{0\le t\le T} e^{2\alpha t}|\delta Y_t|^2 + \int_0^T e^{2\alpha t}|\delta Z_t|^2\,dt\right] \\ \le 4\left(1+8c^2\right)\varepsilon\,\mathbb{E}\left[\int_0^T e^{2\alpha t}|f(t,U_t,V_t) - f(t,U'_t,V'_t)|^2\,dt\right]$$

- Using the Lipschitz assumption,

$$|f(t,U_t,V_t) - f(t,U'_t,V'_t)|^2 \le 2\lambda^2\left(|\delta U_t|^2 + |\delta V_t|^2\right)$$

- We finally get

$$\mathbb{E}\left[\sup_{0\le t\le T} e^{2\alpha t}|\delta Y_t|^2 + \int_0^T e^{2\alpha t}|\delta Z_t|^2\,dt\right] \\ \le 4\left(1+8c^2\right)2(1\vee T)\lambda^2\varepsilon\,\mathbb{E}\left[\sup_{0\le t\le T} e^{2\alpha t}|\delta U_t|^2 + \int_0^T e^{2\alpha t}|\delta V_t|^2\,dt\right]$$

- Choose ε s.t. $4\left(1+8c^2\right)2(1\vee T)\lambda^2\varepsilon = 1/2$! α is now fixed
- The map is a contraction w.r.t. the norm on $\mathscr{B}^2$

$$\|(Y,Z)\|_\alpha^2 := \mathbb{E}\left[\sup_{0\le t\le T} e^{2\alpha t}|Y_t|^2 + \int_0^T e^{2\alpha t}|Z_t|^2\,dt\right].$$

□

- What is really used in the proof is

$$2\left(y-y'\right)\cdot\left(f(t,y,z)-f(t,y',z')\right)\le 2\lambda_y\,|y-y'|^2+2\lambda_z\,|y-y'|\,|z-z'|.$$

Exercise (For next lecture). Prove that under (L), one has

$$\mathbb{E}\left[e^{2\alpha t}|Y_t|^2+\int_0^T e^{2\alpha s}|Z_s|^2ds\right]\le C\,\mathbb{E}\left[e^{2\alpha T}|\xi|^2+\left(\int_0^T e^{\alpha s}|f(s,0,0)|\,ds\right)^2\right],$$

C universal constant, $\alpha\ge\lambda^2+\lambda$.

4. Linear BSDEs and Comparison Theorem

- In this section, we consider only real-valued BSDEs: $k=1$
- We will see an explicit formula for linear BSDE

$$Y_t=\xi+\int_t^T(a_sY_s+Z_sb_s+c_s)\,ds-\int_t^T Z_s\,dB_s,\quad 0\le t\le T.$$

  - ⋆ $f(t,y,z)=c_t+a_ty+zb_t$.
- Let us start with $c\equiv 0$ and $a\equiv 0$:

$$\begin{aligned}Y_t&=\xi+\int_t^T Z_sb_sds-\int_t^T Z_sdB_s\\&=\xi-\int_t^T Z_sdB_s^*,\qquad B_s^*=B_s-\int_0^t b_sds.\end{aligned}$$

- Girsavov's theorem

$$\begin{aligned}Y_t&=\mathbb{E}^*\left(\xi\,|\,\mathscr{F}_t\right),\quad d\mathbb{P}^*=D_T\,d\mathbb{P}\\D_t&=\exp\left(\int_0^t b_s\cdot dB_s-\frac{1}{2}\int_0^t|b_s|^2ds\right)\end{aligned}$$

- In the general case

$$\begin{aligned}Y_t&=D_t^{-1}\,\mathbb{E}\left(D_T\left(\xi e^{\int_t^T a_r\,dr}+\int_t^T c_se^{\int_t^s a_r\,dr}\,ds\right)\,\Big|\,\mathscr{F}_t\right)\\&=\mathbb{E}^*\left(\xi e^{\int_t^T a_r\,dr}+\int_t^T c_se^{\int_t^s a_r\,dr}\,ds\,\Big|\,\mathscr{F}_t\right).\end{aligned}$$

Proposition 5 (Linear BSDE)**.** *Let a, b and c be progressively measurable processes in* $\mathbf{R}$, $\mathbf{R}^{1\times d}$ *and* $\mathbf{R}$ *s.t. a and b are bounded and $c\in\mathrm{M}^2$. Let $\xi\in\mathrm{L}^2(\mathscr{F}_T)$. Then the solution to the BSDE*

$$Y_t=\xi+\int_t^T(a_sY_s+Z_sb_s+c_s)\,ds-\int_t^T Z_s\,dB_s,\quad 0\le t\le T.$$

is given by

$$\begin{aligned}Y_t&=D_t^{-1}\,\mathbb{E}\left(D_T\left(\xi e^{\int_t^T a_r\,dr}+\int_t^T c_se^{\int_t^s a_r\,dr}\,ds\right)\,\Big|\,\mathscr{F}_t\right)\\&=\left(D_te^{\int_0^t a_s\,ds}\right)^{-1}\,\mathbb{E}\left(D_T\xi e^{\int_0^T a_r\,dr}+\int_t^T c_se^{\int_0^s a_r\,dr}D_s\,ds\,\Big|\,\mathscr{F}_t\right).\end{aligned}$$

Proof.

- The assumption (L) is satisfied.
- Set $\Gamma_t = e^{\int_0^t a_s ds} D_t$

$$
\begin{aligned}
d\Gamma_t &= \Gamma_t (a_t dt + b_t \cdot dB_t) \\
dY_t &= -(a_t Y_t + Z_t b_t + c_t) dt + Z_t dB_t
\end{aligned}
$$

- Integration by parts formula gives

$$
\begin{aligned}
d(Y_t \Gamma_t) &= \Gamma_t dY_t + Y_t d\Gamma_t + d\langle Y, \Gamma \rangle_t \\
&= -\Gamma_t c_t dt + \Gamma_t Z_t dB_t + \Gamma_t Y_t b_t \cdot dB_t
\end{aligned}
$$

- Γ, Y in $\mathscr{S}^2$ and $Z \in \mathrm{M}^2$, $Y_t \Gamma_t + \int_0^t c_s \Gamma_s \, ds$ is a martingale and

$$
\begin{aligned}
Y_t \Gamma_t + \int_0^t c_s \Gamma_s ds &= \mathbb{E}\left(\xi \Gamma_T + \int_0^T c_s \Gamma_s \, ds \,\middle|\, \mathscr{F}_t \right) \\
Y_t \Gamma_t &= \mathbb{E}\left(\xi \Gamma_T + \int_t^T c_s \Gamma_s \, ds \,\middle|\, \mathscr{F}_t \right)
\end{aligned}
$$

□

- **Fundamental remark:** If $\xi \geq 0$ and c is a nonnegative process then $Y_t \geq 0$.

Theorem 6 (Comparison theorem)**.** *Let (L) holds for* (ξ, f) *and* (ξ', f').

Let us assume that $\mathbb{P}$*–a.s.* $\xi \leq \xi'$ *and* $m \otimes \mathbb{P}$*–a.e.* $f(t, Y_t, Z_t) \leq f'(t, Y_t, Z_t)$. *Then,* $\mathbb{P}$*–a.s.,*

$$
\forall 0 \leq t \leq T, \qquad Y_t \leq Y'_t.
$$

If, in addition, $Y_0 = Y'_0$ *then* $\xi = \xi'$ *and* $f(t, Y_t, Z_t) = f'(t, Y_t, Z_t)$.

- The strict comparison theorem is used as follows: if (in addition), $\mathbb{P}(\xi < \xi') > 0$ then $Y_0 < Y'_0$.

Proof.

- Set $U_t = Y'_t - Y_t$, $V_t = Z'_t - Z_t$, $\zeta = \xi' - \xi$. We want to see that $U_t \geq 0$.
- We have

$$
U_t = \zeta + \int_t^T \left(f'(s, Y'_s, Z'_s) - f(s, Y_s, Z_s) \right) ds - \int_t^T V_s dB_s \tag{5}
$$

- The idea is to linearize the generator

$$
\begin{aligned}
f'(s, Y'_s, Z'_s) - f(s, Y_s, Z_s) &= f'(s, Y'_s, Z'_s) - f'(s, Y_s, Z'_s) + f'(s, Y_s, Z'_s) - f'(s, Y_s, Z_s) \\
&\quad + c_s := f'(s, Y_s, Z_s) - f(s, Y_s, Z_s)
\end{aligned}
$$

- Let us define

$$
\begin{aligned}
a_s &= \left(Y'_s - Y_s\right)^{-1}\left(f'\left(s, Y'_s, Z'_s\right) - f'\left(s, Y_s, Z'_s\right)\right)\mathbf{1}_{|U_s|>0} \\
b_s &= \left|Z'_s - Z_s\right|^{-2}\left(f'\left(s, Y_s, Z'_s\right) - f'\left(s, Y_s, Z_s\right)\right)\left(Z'_s - Z_s\right)^* \mathbf{1}_{|V_s|>0}
\end{aligned}
$$

- We can rewrite (5) as

$$
U_t = \zeta + \int_t^T \left(a_s U_s + V_s b_s + c_s\right) ds - \int_t^T V_s dB_s
$$

- It follows that, since $\zeta \geq 0$ and $c \geq 0$

$$
U_t = \Gamma_t^{-1}\mathbb{E}\left(\Gamma_T \zeta + \int_t^T c_s \Gamma_s \, ds \,\middle|\, \mathscr{F}_t\right) \geq 0
$$

- If **moreover** $U_0 = 0$, then

$$
\mathbb{E}\left[\Gamma_T \zeta + \int_0^T c_s \Gamma_s \, ds\right] = 0 \quad \Longrightarrow \quad \zeta = 0, \quad c \equiv 0.
$$

□

Remark.

- For real BSDEs, linearization is a powerful tool
- Roughly speaking, sometimes one can get rid of the dependance in z of the driver.
- If ξ and $f(\cdot, 0, 0)$ are bounded, we saw that Y is bounded see (3).
- In the real case, we can see that the bound does not depend on the Lipschitz constant in z, λ_z.
- This easily seen from the formula

$$
\begin{aligned}
Y_t &= D_t^{-1}\mathbb{E}\left(D_T\left(\xi e^{\int_t^T a_r \, dr} + \int_t^T f(s,0,0) e^{\int_t^s a_r \, dr} \, ds\right) \,\middle|\, \mathscr{F}_t\right) \\
|Y_t| &\leq \left(\|\xi\|_\infty + T\,\|f(\cdot,0,0)\|_\infty\right) e^{\lambda_y T}.
\end{aligned}
$$

References

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Lecture III. Markovian BSDEs and PDEs

1. Review of Previous Lecture

- B Brownian motion in $\mathbf{R}^d$ on a complete probability space
- $f : [0,T] \times \Omega \times \mathbf{R}^k \times \mathbf{R}^{k\times d} \longrightarrow \mathbf{R}^k$ "measurable"
- ξ $\mathscr{F}_T$–measurable

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,ds - \int_t^T Z_s\,dB_s, \quad 0 \le t \le T. \tag{$\mathrm{E}_{\xi,f}$}$$

Theorem 1 (Pardoux-Peng, 1990)**.** *If f is Lipschitz w.r.t. (y,z) (uniformly in (t,ω)) and*

$$\mathbb{E}\left[|\xi|^2 + \int_0^T |f(s,0,0)|^2 ds\right] < \infty$$

the BSDE $(\mathrm{E}_{\xi,f})$ has a unique solution s.t. $Z \in \mathrm{L}^2$

- Main tool: a priori estimate
- If (Y,Z) is a solution to $(\mathrm{E}_{\xi,f})$ and

$$y \cdot f(t,y,z) \le |y|\, f_t + \mu |y|^2 + \lambda |y|\,|z|$$

then, there exists a universal constant C s.t.

$$\mathbb{E}\left[\sup_{0\le t\le T} e^{2\alpha t}|Y_t|^2 + \int_0^T e^{2\alpha t}|Z_t|^2\right] \le C\,\mathbb{E}\left[e^{2\alpha T}|\xi|^2 + \int_0^T e^{2\alpha t} f_t^2\,dt\right],$$

as soon as $\alpha \ge \lambda^2 + \mu + 1/2$.

- Linear BSDEs have an explicit solution in the scalar case ($Y \in \mathbf{R}$)

$$Y_t = \xi + \int_t^T (a_s Y_s + Z_s b_s + c_s)\, ds - \int_t^T Z_s dB_s$$

- Y is given by Girsanov's theorem

$$Y_t = D_t^{-1} \mathbb{E}\left(D_T \left(\xi e^{\int_t^T a_r\, dr} + \int_t^T c_s e^{\int_t^s a_r\, dr}\, ds \right) \Big| \mathscr{F}_t \right)$$
$$D_t = \exp\left(\int_0^t b_s \cdot dB_s - \frac{1}{2} \int_0^t |b_s|^2 ds \right).$$

Theorem 2 (Comparison theorem). *If $\xi \le \xi'$ and $f \le f'$ then*

$$\forall t \in [0,T], \qquad Y_t \le Y'_t.$$

Strict version of this result.

2. Markovian BSDEs

2.1. Framework

- We consider the following SDE

$$X_u^{t,\theta} = \theta + \int_t^u b\left(s, X_s^{t,\theta}\right) ds + \int_t^u \sigma\left(s, X_s^{t,\theta}\right) dB_s, \quad t \le u \le T \tag{1}$$

- θ r.v. $\mathscr{F}_t$–measurable
- If needed, for $0 \le u \le t$, $X_u^{t,\theta} = \mathbb{E}(\theta \mid \mathscr{F}_u)$
- Now we consider the following BSDE

$$Y_u^{t,\theta} = g\left(X_T^{t,\theta}\right) + \int_u^T f\left(s, X_s^{t,\theta}, Y_s^{t,\theta}, Z_s^{t,\theta}\right) ds - \int_u^T Z_s^{t,\theta} dB_s, \quad 0 \le u \le T \tag{2}$$

- The SDE and the BSDE are decoupled
  - ⋆ Fistly, we solve the SDE
  - ⋆ Then, we solve the BSDE
- The generator of the BSDE is given by

$$F(s,\omega,y,z) = f\left(s, X_s^{t,\theta}(\omega), y, z\right)$$

- Main idea: Transfer properties of the SDE to the BSDE
- Very simple framework denoted by (L)

- $b:[0,T]\times\mathbf{R}^n\longrightarrow\mathbf{R}^n$ and $\sigma:[0,T]\times\mathbf{R}^n\longrightarrow\mathbf{R}^{n\times d}$ are continuous and
  1. $\left|b(t,x)-b(t,x')\right|+\left|\sigma(t,x)-\sigma(t,x')\right|\le\lambda|x-x'|$;
  2. $|b(t,x)|+|\sigma(t,x)|\le\lambda(1+|x|)$.
- $g:\mathbf{R}^n\longrightarrow\mathbf{R}^k$ and $f:[0,T]\times\mathbf{R}^n\times\mathbf{R}^k\times\mathbf{R}^{k\times d}\longrightarrow\mathbf{R}^k$ are continuous and
  1. $\left|g(x)-g(x')\right|\le\lambda\,|x-x'|$;
  2. $\left|f(t,x,y,z)-f(t,x',y',z')\right|\le\lambda\left(|x-x'|+|y-y'|+|z-z'|\right)$;
  3. $\left|g(x)\right|+\left|f(t,x,y,z)\right|\le\lambda\left(1+|x|+|y|+|z|\right)$.

2.2. Elementary properties

- For $\theta\in\mathrm{L}^2(\mathscr{F}_t)$, the SDE (1) has a unique strong solution and

$$\mathbb{E}\left[\sup_{0\le u\le T}\left|X_u^{t,\theta}\right|^2\right]\le C\left(1+\mathbb{E}\left[|\theta|^2\right]\right),$$
$$\mathbb{E}\left[\sup_{0\le u\le T}\left|X_u^{t,\theta}-X_u^{t,\theta'}\right|^2\right]\le C\,\mathbb{E}\left[|\theta-\theta'|^2\right],$$
$$\mathbb{E}\left[\sup_{0\le u\le T}\left|X_u^{t,x}-X_u^{t',x'}\right|^2\right]\le C\left\{|x-x'|^2+|t-t'|\left(1+|x|^2+|x'|^2\right)\right\}$$

where C depends on T and λ.

Proposition 3. *For $\theta\in\mathrm{L}^2(\mathscr{F}_t)$, the BSDE (2) has a unique solution and, if $\theta'\in\mathrm{L}^2(\mathscr{F}_t)$,*

$$\mathbb{E}\left[\sup_{0\le u\le T}\left|Y_u^{t,\theta}\right|^2+\int_0^T\left|Z_r^{t,\theta}\right|^2dr\right]\le C\left(1+\mathbb{E}\left[|\theta|^2\right]\right),$$
$$\mathbb{E}\left[\sup_{0\le u\le T}\left|Y_u^{t,\theta}-Y_u^{t,\theta'}\right|^2+\int_0^T\left|Z_r^{t,\theta}-Z_r^{t,\theta'}\right|^2dr\right]\le C\,\mathbb{E}\left[\left|\theta-\theta'\right|^2\right],$$

where C depends on T and λ.

- BSDE (2) is associated to

$$\xi:=g\left(X_T^{t,\theta}\right),\quad F(s,y,z)=f\left(s,X_s^{t,\theta},y,z\right)$$

- We have from (L)

$$|\xi|+|F(s,0,0)|\le\lambda\left(1+\sup_{0\le u\le T}\left|X_u^{t,\theta}\right|\right)\in\mathrm{L}^2$$

- Use Pardoux-Peng's result, the A priori Estimate for BSDEs and the estimate on the SDEs

$$\left|g\left(X_T^{t,\theta}\right)-g\left(X_T^{t,\theta'}\right)\right|+\left|f\left(s,X_s^{t,\theta},Y_s^{t,\theta},Z_s^{t,\theta}\right)-f\left(s,X_s^{t,\theta'},Y_s^{t,\theta},Z_s^{t,\theta}\right)\right|\le\lambda\sup_{0\le u\le T}\left|X_u^{t,\theta}-X_u^{t,\theta'}\right|$$

3. Markov Property

- It is well know that under (L), we have the following flow property

$$X_t^{r,x} = X_t^{s,X_s^{r,x}}, \quad r \le s \le t \tag{3}$$

- We are going to prove that the same is true for Y and Z
- Notation : for $s \le t$,

$$\mathscr{F}_t^s = \sigma(\mathcal{N}, B_u - B_s : s \le u \le t)$$

Proposition 4. *Let* $(t,x) \in [0,T] \times \mathbf{R}^n$. $\{X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x}\}_{t \le u \le T}$ *is adapted w.r.t.* $\{\mathscr{F}_u^t\}_{t \le u \le T}$.
In particular, $Y_t^{t,x}$ *is deterministic.*

- In the sequel, we will denote by u the function defined by

$$\forall (t,x) \in [0,T] \times \mathbf{R}^n, \qquad u(t,x) := Y_t^{t,x}. \tag{4}$$

Proof.

- $W_u = B_{t+u} - B_t$, $\mathscr{F}_u^W = \mathscr{F}_{t+u}^t$.
- Let $\{X_u\}_{0 \le u \le T-t}$ be the solution to the SDE

$$X_u = x + \int_0^u b(t+r, X_r)dr + \int_0^u \sigma(t+r, X_r)dW_r, \qquad 0 \le u \le T-t.$$

  - ⋆ $\{X_u\}_{0 \le u \le T-t}$ is $\{\mathscr{F}_u^W\}_u$–adapted
- For $v \in [t,T]$, we have

$$X_{v-t} = x + \int_0^{v-t} b(t+r, X_r)dr + \int_0^{v-t} \sigma(t+r, X_r)dW_r$$

- Set $s = r + t$; we have

$$\int_0^{v-t} b(t+r, X_r)dr = \int_t^v b(s, X_{s-t})ds, \qquad \int_0^{v-t} \sigma(t+r, X_r)dW_r = \int_t^v \sigma(s, X_{s-t})dB_s,$$

- It follows that

$$X_{v-t} = x + \int_t^v b(s, X_{s-t})ds + \int_t^v \sigma(s, X_{s-t})dB_s, \qquad t \le v \le T$$

and by definition of $X^{t,x}$

$$X_v^{t,x} = x + \int_t^v b(s, X_s^{t,x})ds + \int_t^v \sigma(s, X_s^{t,x})dB_s, \qquad t \le v \le T.$$

- By uniqueness of solutions to the SDE (1), $X_v^{t,x} = X_{v-t} \in \mathscr{F}_{v-t}^W = \mathscr{F}_v^t$.
- For the BSDE, the method is the same
- $\{(Y_u, Z_u)\}_{0 \le u \le T-t}$ solution $\mathscr{F}_u^W$–adapted to

$$Y_u = g(X_{T-t}) + \int_u^{T-t} f(t+r, X_r, Y_r, Z_r)\,dr - \int_u^{T-t} Z_r\,dW_r, \qquad 0 \le u \le T-t,$$

- We write this BSDE as

$$Y_{v-t} = g(X_{T-t}) + \int_{v-t}^{T-t} f(t+r, X_r, Y_r, Z_r)\,dr - \int_{v-t}^{T-t} Z_r\,dW_r, \qquad t \le v \le T$$

and by $s = r + t$

$$= g(X_{T-t}) + \int_v^T f(s, X_{s-t}, Y_{s-t}, Z_{s-t})\,ds - \int_v^T Z_{s-t}\,dB_s, \qquad t \le v \le T$$

and since $X_v^{t,x} = X_{v-t}$

$$= g\left(X_T^{t,x}\right) + \int_v^T f\left(s, X_s^{t,x}, Y_{s-t}, Z_{s-t}\right)ds - \int_v^T Z_{s-t}\,dB_s, \qquad t \le v \le T.$$

- $\{Y_{v-t}, Z_{v-t}\}_{v \in [t,T]}$ and $\{Y_v^{t,x}, Z_v^{t,x}\}_{v \in [t,T]}$ solve the same BSDE
- This gives the result since $\mathscr{F}_{v-t}^W = \mathscr{F}_v^t$. □

Proposition 5. *u is continuous and*

$$|u(t,x)| \le C(1+|x|),$$
$$\left|u(t,x) - u(t',x')\right| \le C\left(\left|x-x'\right| + \left|t-t'\right|^{1/2}(1+|x|+|x'|)\right).$$

Proof.

- The growth of u comes from Proposition 3.
- For the regularity, if $t' \ge t$,

$$u(t',x') - u(t,x) = Y_{t'}^{t',x'} - Y_t^{t,x} = \mathbb{E}\left[Y_{t'}^{t',x'} - Y_t^{t,x}\right] = \mathbb{E}\left[Y_{t'}^{t',x'} - Y_{t'}^{t,x}\right] + \mathbb{E}\left[Y_{t'}^{t,x} - Y_t^{t,x}\right]\ ;$$

- For the second term,

$$Y_t^{t,x} = Y_{t'}^{t,x} + \int_t^{t'} f\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right)dr - \int_t^{t'} Z_r^{t,x}\,dB_r,$$

- With Hölder inequality,

$$\left|\mathbb{E}\left[Y_{t'}^{t,x} - Y_t^{t,x}\right]\right|^2 = \left|\mathbb{E}\left[\int_t^{t'} f\big(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big)\,dr\right]\right|^2$$
$$\le \left|t-t'\right| \mathbb{E}\left[\int_0^T \left|f\big(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big)\right|^2 dr\right].$$

- From the growth of f

$$\mathbb{E}\left[\int_0^T \left|f\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right)\right|^2 dr\right] \leq C\,\mathbb{E}\left[1 + \sup_{0\leq r\leq T}\left\{\left|X_r^{t,x}\right|^2 + \left|Y_r^{t,x}\right|^2\right\} + \int_0^T \left|Z_r^{t,x}\right|^2 dr\right]$$
$$\leq C\left(1 + |x|^2 + |x'|^2\right).$$

- Finally, for the first one, from the apriori estimate,

$$\left|\mathbb{E}\left[Y_{t'}^{t',x'} - Y_{t'}^{t,x}\right]\right|^2 \leq \mathbb{E}\left[\sup_{r\in[0,T]}\left|Y_r^{t',x'} - Y_r^{t,x}\right|^2\right] \leq C\left|x - x'\right|^2$$

□

- A notational ambiguity

Theorem 6. *Let* $t \in [0,T]$ *and* $\theta \in \mathrm{L}^2(\mathscr{F}_t)$. *Then*

$$Y_t^{t,\theta} = u(t,\theta) := Y_t^{t,\cdot} \circ \theta.$$

Proof.

- Suppose first that

$$\theta = \sum_{i=1}^{l} x_i \mathbf{1}_{A_i}, \quad (A_i)_{1\leq i\leq l} \text{ partition of } \Omega,\ A_i \in \mathscr{F}_t,\ x_i \in \mathbf{R}^d$$

- Let us write $\left(X_r^i, Y_r^i, Z_r^i\right)_{0\leq r\leq T}$ instead of $\left(X_r^{t,x_i}, Y_r^{t,x_i}, Z_r^{t,x_i}\right)_{0\leq r\leq T}$.
- For $t \leq r \leq T$, we have

$$X_r^{t,\theta} = \sum_i \mathbf{1}_{A_i} X_r^i, \quad Y_r^{t,\theta} = \sum_i \mathbf{1}_{A_i} Y_r^i, \quad Z_r^{t,\theta} = \sum_i \mathbf{1}_{A_i} Z_r^i.$$

- Indeed, for each i and $r \geq t$,

$$X_r^i = x_i + \int_t^r b(u, X_u^i) du + \int_t^r \sigma(u, X_u^i) dB_u$$

- Multiplying by $\mathbf{1}_{A_i}$ and summing in i, we get since $A_i \in \mathscr{F}_t$,

$$\sum_i \mathbf{1}_{A_i} X_r^i = \theta + \int_t^r \sum_i \mathbf{1}_{A_i} b(u, X_u^i) du + \int_t^r \sum_i \mathbf{1}_{A_i} \sigma(u, X_u^i) dB_u$$

- But $\sum_i \mathbf{1}_{A_i} H(BlaBla_i) = H\left(\sum_i \mathbf{1}_{A_i} BlaBla_i\right)$ and

$$\sum_i \mathbf{1}_{A_i} X_r^i = \theta + \int_t^r b\left(u, \sum_i \mathbf{1}_{A_i} X_u^i\right) du + \int_t^r \sigma\left(u, \sum_i \mathbf{1}_{A_i} X_u^i\right) dB_u$$

and by definition of $X^{t,\theta}$

$$X_r^{t,\theta} = \theta + \int_t^r b\left(u, X_u^{t,\theta}\right) du + \int_t^r \sigma\left(u, X_u^{t,\theta}\right) dB_u$$

- By uniqueness, we get the flow property

$$\forall t \le r \le T, \qquad X_r^{t,\theta} = \sum_i \mathbf{1}_{A_i} X_r^i = \sum_i \mathbf{1}_{A_i} X_r^{t,x_i} = X_r^{t,\cdot} \circ \theta.$$

- Arguing in the same way, for each i,

$$Y_r^i = g\left(X_T^i\right) + \int_r^T f\left(u, X_u^i, Y_u^i, Z_u^i\right) du - \int_r^T Z_u^i dB_u.$$

- It follows that $\left(\sum_i \mathbf{1}_{A_i} Y_r^i, \sum_i \mathbf{1}_{A_i} Z_r^i\right)$ solves the following BSDE on $[t,T]$

$$\begin{aligned} Y_r' &= g\left(\sum_i \mathbf{1}_{A_i} X_T^i\right) + \int_r^T f\left(u, \sum_i \mathbf{1}_{A_i} X_u^i, Y_u', Z_u'\right) du - \int_r^T Z_u' \, dB_u \\ &= g\left(X_T^{t,\theta}\right) + \int_r^T f\left(u, X_u^{t,\theta}, Y_u', Z_u'\right) du - \int_r^T Z_u' \, dB_u \end{aligned}$$

- By uniqueness

$$Y_r^{t,\theta} = \sum_i \mathbf{1}_{A_i} Y_r^i, \qquad Z_r^{t,\theta} = \sum_i \mathbf{1}_{A_i} Z_r^i,$$

- In particular, for $r = t$,

$$Y_t^{t,\theta} = \sum_i \mathbf{1}_{A_i} Y_t^i = \sum_i \mathbf{1}_{A_i} Y_t^{t,x_i} = \sum_i \mathbf{1}_{A_i} u(t,x_i) = u\left(t, \sum_i \mathbf{1}_{A_i} x_i\right) = u(t,\theta).$$

- For $\theta \in \mathrm{L}^2(\mathscr{F}_t)$, let $\theta_n \longrightarrow \theta$ with θ_n of the previous form

$$\mathbb{E}\left[\left|Y_t^{t,\theta_n} - Y_t^{t,\theta}\right|^2\right] \le C\,\mathbb{E}\left[|\theta_n - \theta|^2\right]$$

$$\mathbb{E}\left[|u(t,\theta_n) - u(t,\theta)|^2\right] \le C\,\mathbb{E}\left[|\theta_n - \theta|^2\right].$$

- Since $u(t,\theta_n) = Y_t^{t,\theta_n}$, $u(t,\theta) = Y_t^{t,\theta}$. □

Corollary 7. *Let $t \in [0,T]$ and $\theta \in \mathrm{L}^2(\mathscr{F}_t)$. Then*

$$\forall s \in [t,T], \qquad Y_s^{t,\theta} = u\left(s, X_s^{t,\theta}\right).$$

Proof.

- By the previous result

$$u(s, X_s^{t,\theta}) = Y_s^{s, X_s^{t,\theta}}$$

- But by definition $\left\{\left(Y_r^{s,X_s^{t,\theta}}, Z_r^{s,X_s^{t,\theta}}\right)\right\}_r$ solves the BSDE

$$Y_u = g\left(X_T^{s,X_s^{t,\theta}}\right) + \int_u^T f\left(r, X_r^{s,X_s^{t,\theta}}, Y_r, Z_r\right) dr - \int_u^T Z_r \, dB_r, \qquad s \le u \le T.$$

- By construction, $X_r^{s,X_s^{t,\theta}}$ and $X_r^{t,\theta}$ are both solution to the SDE

$$X_r = X_s^{t,\theta} + \int_s^r b(u, X_u)du + \int_s^r \sigma(u, X_u)dBu, \qquad s \le r \le T$$

- By uniqueness

$$\forall r \in [s,T], \qquad X_r^{s,X_s^{t,\theta}} = X_r^{t,\theta}.$$

- We deduce that $\left\{\left(Y_r^{s,X_s^{t,\theta}}, Z_r^{s,X_s^{t,\theta}}\right)\right\}_r$ and $\left\{\left(Y_r^{t,\theta}, Z_r^{t,\theta}\right)\right\}_r$ solve the BSDE

$$Y_u = g\left(X_T^{t,\theta}\right) + \int_u^T f\left(r, X_r^{t,\theta}, Y_r, Z_r\right)dr - \int_u^T Z_r\, dB_r, \qquad s \le u \le T.$$

- It follows that

$$Y_s^{t,\theta} = Y_s^{s,X_s^{t,\theta}} = u(s, X_s^{t,\theta}).$$

□

4. Nonlinear Feynman-Kac's Formula

- In this section, Y is real-valued, $k = 1$!
- Let u is a smooth solution to the semilinear PDE

$$\partial_t u(t,x) + \mathscr{L}u(t,x) + f(t,x,u(t,x), \nabla_x u \cdot \sigma(t,x)) = 0, \qquad u(T,.) = g, \tag{5}$$

where $\mathscr{L}$ is the linear differential operator

$$\mathscr{L}u(t,x) = \frac{1}{2}\mathrm{trace}(\sigma\sigma^* \nabla_x^2 u(t,x)) + b(t,x) \cdot \nabla_x u(t,x)$$

- Verification theorem: by Itô's formula

$$\left(u\left(s, X_s^{t,x}\right), \nabla_x u \cdot \sigma\left(s, X_s^{t,x}\right)\right)$$

solves the BSDE (2)

$$Y_r^{t,x} = g\left(X_T^{t,x}\right) + \int_r^T f\left(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}\right)ds - \int_r^T Z_s^{t,x} dB_s, \quad t \le r \le T,$$

where $X^{t,x}$ stands for the solution to the SDE (1)

$$X_s^{t,x} = x + \int_t^s b\left(r, X_r^{t,x}\right)dr + \int_t^s \sigma\left(r, X_r^{t,x}\right)dB_r, \quad t \le s \le T.$$

- A more probabilistic point of view is to construct the solution u to the PDE from the BSDE

Theorem 8. *Under (L), the function u defined by*

$$\forall (t,x) \in [0,T] \times \mathbf{R}^n, \qquad u(t,x) := Y_t^{t,x}$$

is a viscosity solution to the PDE (5).

- In the linear case, $f(t,x,u) = a(t,x)u + c(t,x)$, we get (linear BSDE)

$$\begin{aligned} Y_t^{t,x} &= \mathbb{E}\left[g\left(X_T^{t,x}\right) e^{\int_t^T a(r,X_r^{t,x})dr} + \int_t^T c\left(s, X_s^{t,x}\right) e^{\int_t^s a(r,X_r^{t,x})dr} \,\middle|\, \mathscr{F}_t \right] \\ &= \mathbb{E}\left[g\left(X_T^{t,x}\right) e^{\int_t^T a(r,X_r^{t,x})dr} + \int_t^T c\left(s, X_s^{t,x}\right) e^{\int_t^s a(r,X_r^{t,x})dr} \right] \end{aligned}$$

which is the usual Feynman-Kac formula.

- Let us recall the definition of viscosity solution

Definition 8. A continuous function u, with $u(T,\cdot) = g$, is a viscosity subsolution (supersolution) if, whenever $u - \varphi$ has a local maximum (minimum) at (t,x) where φ is $\mathscr{C}^{1,2}$,

$$\partial_t \varphi(t,x) + \mathscr{L}\varphi(t,x) + f(t,x,u(t,x), \nabla\varphi \cdot \sigma(t,x)) \geq 0, \quad (\leq 0)$$

A solution is both a sub and a supersolution.

Proof.

- By construction u is continuous and $u(T,\cdot) = g$.
- Let us show that u is a subsolution.
  - ⋆ Let $(t,x) \in [0,T[\times\mathbf{R}^n$ be a local maximum of $u - \varphi$
  - ⋆ Without loss of generality, we assume that $\varphi(t,x) = u(t,x)$
  - ⋆ We have to prove that

$$\partial_t \varphi(t,x) + \mathscr{L}\varphi(t,x) + f\left(t,x,u(t,x), \nabla_x\varphi \cdot \sigma(t,x)\right) \geq 0.$$

- If not, there exist $\delta > 0$ and $0 < \alpha \leq T - t$ such that

$$u(s,y) \leq \varphi(s,y), \quad \partial_t \varphi(s,y) + \mathscr{L}\varphi(s,y) + f\left(s,y,u(s,y), \nabla_x\varphi \cdot \sigma(s,y)\right) \leq -\delta$$

as soon as $t \leq s \leq t + \alpha$ and $|x - y| \leq \alpha$.

- Consider the stopping time

$$\tau = \inf\left\{s \geq t : \left|X_s^{t,x} - x\right| \geq \alpha\right\} \wedge (t+\alpha).$$

- $\left(Y_s', Z_s'\right) := \left(\varphi\left(s \wedge \tau, X_{s\wedge\tau}^{t,x}\right), \mathbf{1}_{s\leq\tau} \nabla_x \varphi\sigma\left(s, X_s^{t,x}\right)\right)$ solves

$$Y_s' = \varphi\left(\tau, X_\tau^{t,x}\right) + \int_s^{t+\alpha} -\mathbf{1}_{r\leq\tau}\left\{\partial_t\varphi + \mathscr{L}\varphi\right\}\left(r, X_r^{t,x}\right) dr - \int_s^{t+\alpha} Z_r' \, dB_r$$

- $\left(Y^{t,x}_{s\wedge\tau}, \mathbf{1}_{s\le\tau} Z^{t,x}_s\right)$ solves the BSDE

$$Y_s = Y_{t+\alpha} + \int_s^{t+\alpha} \mathbf{1}_{r\le\tau} f\left(r, X^{t,x}_r, Y_r, Z_r\right) dr - \int_s^{t+\alpha} Z_r dB_r$$

- By the Markov property $Y^{t,x}_s = u\left(s, X^{t,x}_s\right)$

$$Y_s = u\left(\tau, X^{t,x}_\tau\right) + \int_s^{t+\alpha} \mathbf{1}_{r\le\tau} f\left(r, X^{t,x}_r, u\left(r, X^{t,x}_r\right), Z_r\right) dr - \int_s^{t+\alpha} Z_r dB_r$$

- By definition of τ, $u\left(\tau, X^{t,x}_\tau\right) \le \varphi\left(\tau, X^{t,x}_\tau\right)$ and

$$f\left(s, X^{t,x}_s, u\left(s, X^{t,x}_s\right), \nabla_x\varphi\cdot\sigma(s, X^{t,x}_s)\right) + \left\{\partial_t\varphi + \mathscr{L}\varphi\right\}\left(s, X^{t,x}_s\right) \le -\delta$$

- Strict comparison: $u(t,x) = Y_t < Y'_t = \varphi(t,x)$
- But $u(t,x) = \varphi(t,x)$! □

Exercise (For next lecture). Prove the nonlinear Feynman-Kac formula in the following setting:

- $b : [0,T]\times\mathbf{R}^n \longrightarrow \mathbf{R}^n$ and $\sigma : [0,T]\times\mathbf{R}^n \longrightarrow \mathbf{R}^{n\times d}$ are continuous and
  1. $\left|b(t,x) - b(t,x')\right| + \left|\sigma(t,x) - \sigma(t,x')\right| \le \lambda|x - x'|$;
  2. $|b(t,x)| + |\sigma(t,x)| \le \lambda(1 + |x|)$.
- $g : \mathbf{R}^n \longrightarrow \mathbf{R}^k$ and $f : [0,T]\times\mathbf{R}^n\times\mathbf{R}^k\times\mathbf{R}^{k\times d} \longrightarrow \mathbf{R}^k$ are continuous and
  1. $\left|f(t,x,y,z) - f(t,x,y',z')\right| \le \lambda\left(|y - y'| + |z - z'|\right)$;
  2. $\left|g(x)\right| + \left|f(t,x,y,z)\right| \le \lambda\left(1 + |x|^p + |y| + |z|\right)$.

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Lecture IV. Additional results on BSDEs

1. Review of the previous lecture

- $\{X_s^{t,x}\}_{t\le s\le T}$ solution to the SDE

$$X_s^{t,x} = x + \int_t^s b\left(r, X_r^{t,x}\right) dr - \int_t^s \sigma\left(r, X_r^{t,x}\right) dB_r$$

- $\{(Y_s^{t,x}, Z_s^{t,x})\}_{t\le s\le T}$ solution to the BSDE

$$Y_s^{t,x} = g\left(X_T^{t,x}\right) + \int_s^T f\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) dr - \int_s^T Z_r^{t,x}\, dB_r$$

- Define the function u by $u(t,x) := Y_t^{t,x}$
- $Y_s^{t,x} = u(s, X_s^{t,x})$
- u is a viscosity solution to

$$\partial_t u(t,x) + \mathscr{L} u(t,x) + f(t,x,u(t,x), \nabla_x u \cdot \sigma(t,x)) = 0, \qquad u(T,.) = g,$$

where $\mathscr{L}$ is the linear differential operator

$$\mathscr{L} u(t,x) = \frac{1}{2}\mathrm{trace}(\sigma\sigma^* \nabla_x^2 u(t,x)) + b(t,x)\cdot \nabla_x u(t,x)$$

2. The monotonicity condition

- Still working with our BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dB_s, \quad 0 \le t \le T. \qquad (\mathrm{E}_{\xi,f})$$

- As already said, what is needed to get a priori estimate is

$$(y-y')\cdot\big(f(t,y,z)-f(t,y',z')\big)\leq \mu|y-y'|^2+\lambda|y-y'|\,|z-z'|$$

  - ⋆ Existence and uniqueness under this assumption?

- What about the growth of f w.r.t. y?

$$|f(t,y,z)|\leq f_t+\lambda|z|+\varphi(|y|)$$

  - ⋆ φ linear, then polynomial, then arbitrary

Remark.

- If φ has not a linear growth, $Z\in \mathrm{L}^2$ does not necessarily imply $Y\in\mathscr{S}^2$!
- Uniqueness will be for $(Y,Z)\in\mathscr{B}^2$ not for $Z\in\mathrm{L}^2$.

- We will work with the following set of assumptions called (M): there exist $\lambda\geq 0$ and $\mu\in\mathbf{R}$ s.t.

  - $y\longmapsto f(t,y,z)$ is continuous
  - $(y-y')\cdot\big(f(t,y,z)-f(t,y',z)\big)\leq\mu|y-y'|^2$
  - $|f(t,y,z)-f(t,y,z')\ \leq\lambda|z-z'|$
  - $\forall r>0$,

$$\psi_r(t)=\sup_{|y|\leq r}|f(t,y,0)-f(t,0,0)|\in\mathrm{L}^1((0,T)\times\Omega)$$

  - Integrability:

$$\mathbb{E}\left[|\xi|^2+\int_0^T|f(t,0,0)|^2\right]<\infty$$

- There is no growth condition on y!
- If f is Lipschitz, then $\mu=\lambda$ and $\psi_r(t)=\lambda r$.

Theorem 1 (B., Delyon, Hu, Pardoux and Stoica, 2003)**.** *Under (M), BSDE* $(\mathrm{E}_{\xi,f})$ *has a unique solution* $(Y,Z)\in\mathscr{B}^2$ *and*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}e^{2\alpha t}|Y_t|^2+\int_0^T e^{2\alpha t}|Z_t|^2\right]\leq C\,\mathbb{E}\left[e^{2\alpha T}|\xi|^2+\int_0^T e^{2\alpha t}|f(t,0,0)|^2\,dt\right],$$

as soon as $\alpha\geq\lambda^2+\mu+1/2$.

- Uniqueness follows directly from the a priori estimate.
- The proof of existence is divided into three steps

Proof of Step 1.

- Let us assume that ξ is bounded and f is bounded

$$|\xi| + |f(t,y,z)| \le M$$

- We will first prove the result when f does not depend on z.
- More precisely, let V be a given process in M^2, we construct a solution to

$$Y_t = \xi + \int_t^T f(s, Y_s, V_s)\,ds - \int_t^T Z_s\,dB_s, \quad 0 \le t \le T$$

  - ⋆ We set $h(t,y) = f(t,y,V_t)$; h is bounded.
- Let $\rho : \mathbf{R}^k \longrightarrow \mathbf{R}_+$ be a smooth nonnegative function with support in the unit ball and s.t.

$$\int \rho(u)\,du = 1.$$

  - ⋆ For $n \in \mathbf{N}^*$, we set $\rho_n(u) = n^k \rho(nu)$.
- Let h_n defined by

$$h_n(t,y) := \rho_n \star h(t,\cdot)(y) = \int_{\mathbf{R}^k} \rho_n(y-u)h(t,u)\,du = \int_{\mathbf{R}^k} \rho_n(u)h(t,y-u)\,du.$$

  - ⋆ h_n is bounded by M
  - ⋆ h_n is Lipschitz w.r.t. y

$$\left\|\nabla_y h_n(t,y,z)\right\| \le \left|\int \nabla\rho_n(u) \otimes h(t,y-u)\,du\right| \le M \int \left|\nabla\rho_n(u)\right| du \le C\,n.$$

- By Pardoux-Peng's theorem, let $(Y^n, Z^n) \in \mathscr{B}^2$ solution to the BSDE

$$Y_t^n = \xi + \int_t^T h_n(r, Y_r^n)\,dr - \int_t^T Z_r^n\,dW_r, \qquad 0 \le t \le T.$$

  - ⋆ Since h_n and ξ are bounded by M, Y^n is bounded:

$$\sup_n \sup_{0\le t\le T} |Y_t^n| \le M(1+T) := a$$

- Let us see that (Y^n, Z^n) is a Cauchy sequence.
  - ⋆ We can not use the Lipschitz constant in y!
  - ⋆ But since $y - y' = y - u - (y' - u)$

$$\begin{aligned}(y-y')\cdot\big(h_n(t,y,z) - h_n(t,y',z)\big) &= \int \rho_n(u)(y-y')\cdot\big(h(t,y-u) - h(t,y'-u)\big)\,du \\ &\le \mu|y-y'|^2.\end{aligned}$$

- We can apply the a priori estimate, $\alpha = 1/2 + 2\mu$

$$\mathbb{E}\left[\sup_{0\le t\le T} e^{2\alpha t}|\delta Y_t|^2 + \int_0^T e^{2\alpha r}|\delta Z_r|^2\,dr\right] \le C\mathbb{E}\left[\int_0^T e^{2\alpha t}|h_m - h_n|^2(t, Y_t^n)\right]$$
$$\le C\mathbb{E}\left[\int_0^T e^{2\alpha t}\sup_{|y|\le a}|h_m(t,y) - h_n(t,y)|\,dt\right].$$

- But $y \longmapsto h(t,y)$ is continuous and $h_n(t,\cdot)$ converges to $h(t,\cdot)$ uniformly on compact sets and

$$\sup_{|y|\le a}\left|h_m(t,y) - h_n(t,y)\right| \le 2M$$

- This shows that (Y^n, Z^n) is a Cauchy sequence in $\mathscr{B}^2$.
- It is easy to prove that the limit (Y,Z) is a solution!
  - ⋆ First

$$\mathbb{E}\left[|Y_t^n - Y_t|^2\right] \le \mathbb{E}\left[\sup_t |Y_t^n - Y_t|^2\right], \quad \mathbb{E}\left[\left|\int_t^T (Z_r^n - Z_r)\,dB_r\right|^2\right] \le 4\mathbb{E}\left[\int_0^T \|Z_r^n - Z_r\|^2\,dr\right].$$

  - ⋆ and for the nonlinear term

$$\mathbb{E}\left[\sup_t\left|\int_t^T \left\{h_n(r,Y_r^n) - h(r,Y_r)\right\}dr\right|^2\right]$$
$$\le \quad 2T\mathbb{E}\left[\int_0^T \left|h_n(r,Y_r^n) - h(r,Y_r^n)\right|^2\,dr\right] + 2T\mathbb{E}\left[\int_0^T \left|h(r,Y_r^n) - h(r,Y_r)\right|^2\,dr\right] \;;$$

  - ⋆ $\left|h_n(r,Y_r^n) - h(r,Y_r^n)\right| \le \sup_{|y|\le a}|h_n(r,y) - h(r,y)|$.
  - ⋆ Since $h(t,\cdot)$ is continuous $h(t,Y_t^n) \longrightarrow h(t,Y_t)$.
- Let us prove the result in the general case by showing that the map $(U,V) \longrightarrow (Y,Z)$ where

$$Y_t = \xi + \int_t^T f(s,Y_s,V_s)ds - \int_t^T Z_s dB_s, \quad 0 \le t \le T$$

is a contraction.

- This is very easy since f is Lipschitz w.r.t. to z. By the a priori estimate $(\alpha = 1/(2\varepsilon) + 2\mu)$

$$\mathbb{E}\left[\sup_{0\le t\le T} e^{2\alpha t}|\delta Y_t|^2 + \int_0^T e^{2\alpha r}|\delta Z_r|^2\,dr\right] \le C\varepsilon\mathbb{E}\left[\int_0^T e^{2\alpha t}|f(t,Y_t,V_t) - f(t,Y_t,V_t')|^2 dt\right]$$
$$\le C\varepsilon\lambda^2\mathbb{E}\left[\int_0^T e^{2\alpha t}|V_t - V_t'|^2\,dt\right]$$
$$\le C\varepsilon\lambda^2\mathbb{E}\left[\sup_{0\le t\le T} e^{2\alpha t}|\delta U_t|^2 + \int_0^T e^{2\alpha t}|\delta V_t|^2\,dt\right].$$

□

- For the last two steps, we assume that $\mu = 0$
- If not, set $Y_t^\mu = e^{\mu t} Y_t$ and $Z_t^\mu = e^{\mu t} Z_t$
- (Y^μ, Z^μ) solves the BSDE

$$Y_t^\mu = \xi^\mu + \int_t^T f^\mu(s, Y_s^\mu, Z_s^\mu) ds - \int_t^T Z_s^\mu dB_s, \quad 0 \le t \le T,$$

where $\xi^\mu = \xi e^{\mu T}$ and

$$f^\mu(t,y,z) = e^{\mu t} f\left(t, e^{-\mu t} y, e^{-\mu t} z\right) - \mu y$$

- f^μ satisfies (M) with $\mu = 0$!

Proof of Step 2.

- We assume that ξ and $\sup_t |f_t^0 := f(t,0,0)|$ are bounded random variables.
- Let r be a positive real such that

$$e^{(1+2\lambda^2)T} \left(\|\xi\|_\infty^2 + T \left\| f^0 \right\|_\infty^2 \right) < r^2.$$

- Let θ_r be a smooth function such that $0 \le \theta_r \le 1$, $\theta_r(y) = 1$ for $|y| \le r$ and $\theta_r(y) = 0$ as soon as $|y| \ge r+1$.
- For each $n \in \mathbf{N}^*$, we denote $q_n(z) = z \frac{n}{|z| \vee n}$ and set

$$h_n(t,y,z) = \theta_r(y) \left(f(t,y,q_n(z)) - f_t^0 \right) \frac{n}{\psi_{r+1}(t) \vee n} + f_t^0.$$

- h_n is bounded

$$|h_n(t,y,z)| \le (1+\lambda) n + \|f^0\|_\infty$$

- h_n is λ-Lipschitz w.r.t. z
- h_n satisfies (M) with a positive constant.
  - ⋆ It is trivial If $|y| > r+1$ and $|y'| > r+1$.
  - ⋆ If $|y'| \le r+1$. We write

$$\begin{aligned}\left\langle y-y', h_n(t,y,z) - h_n(t,y',z) \right\rangle &= \theta_r(y) \frac{n}{n \vee \psi_{r+1}(t)} \left\langle y-y', f(t,y,q_n(z)) - f(t,y',q_n(z)) \right\rangle \\ &\quad + \frac{n}{n \vee \psi_{r+1}(t)} \left(\theta_r(y) - \theta_r(y') \right) \left\langle y-y', \left[f(t,y',q_n(z)) - f_t^0 \right] \right\rangle.\end{aligned}$$

⋆ The first term of the right hand side is non positive since (M) is in force for f with $\mu = 0$.

⋆ For the second term, we use the fact that θ_r is $C(r)$–Lipschitz, to get, since $|y'| \le r+1$,

$$\begin{aligned}\left(\theta_r(y) - \theta_r(y') \right) \left\langle y-y', \left[f(t,y',q_n(z)) - f_t^0 \right] \right\rangle &\le C(r) \left| y-y' \right|^2 \left| f(t,y',q_n(z)) - f_t^0 \right| \\ &\le C(r)(\lambda n + \psi_{r+1}(t)) \left| y-y' \right|^2,\end{aligned}$$

and thus

$$\frac{n}{n\vee\psi_{r+1}(t)}\left(\theta_r(y)-\theta_r(y')\right)\left\langle y-y',\left[f(t,y',q_n(z))-f_t^0\right]\right\rangle\leq C(r)(\lambda+1)n\left|y-y'\right|^2.$$

- The pair (ξ,h_n) satisfies the assumptions of Step 1.
- Let (Y^n,Z^n) be the solution to the BSDE associated to (ξ,h_n)
- Let us notice that ξ is bounded and that

$$\left\langle y,h_n(t,y,z)\right\rangle\leq|y|\,\left\|f^0\right\|_\infty+\lambda|y|\,|z|.$$

- Y^n is bounded and more precisely,

$$\forall n\in\mathbf{N}^*,\quad\forall t,\qquad|Y_t^n|\leq r.$$

- We have also from the a priori estimate

$$\sup_n\left\|Z^n\right\|_{\mathrm{M}^2}<\infty\tag{1}$$

- Thus (Y^n,Z^n) is a solution to the BSDE associated to (ξ,f_n) where

$$f_n(t,y,z)=\left(f(t,y,q_n(z))-f_t^0\right)\frac{n}{\psi_{r+1}(t)\vee n}+f_t^0;$$

- We made some progress since f_n satisfies (M) with $\mu=0$!
- Setting $U=Y^{n+i}-Y^n$, $V=Z^{n+i}-Z^n$ and using the assumptions on f_{n+i} we have

$$\begin{aligned}&e^{2\lambda^2t}|U_t|^2+\frac{1}{2}\int_t^Te^{2\lambda^2s}|V_s|^2\,ds\\ &\leq\ 2\int_t^Te^{2\lambda^2s}\left\langle U_s,f_{n+i}\left(s,Y_s^n,Z_s^n\right)-f_n\left(s,Y_s^n,Z_s^n\right)\right\rangle ds-2\int_t^Te^{2\lambda^2s}\left\langle U_s,V_sdB_s\right\rangle.\end{aligned}$$

- But $\|U\|_\infty\leq 2r$ so that

$$\begin{aligned}&e^{2\lambda^2t}|U_t|^2+\frac{1}{2}\int_t^Te^{2\lambda^2s}|V_s|^2\,ds\\ &\leq\ 4r\int_0^Te^{2\lambda^2s}\left|f_{n+i}\left(s,Y_s^n,Z_s^n\right)-f_n\left(s,Y_s^n,Z_s^n\right)\right|ds-2\int_t^Te^{2\lambda^2s}\left\langle U_s,V_sdB_s\right\rangle,\end{aligned}$$

- Using the BDG inequality, we get, for a constant C depending only on λ and T,

$$\mathbb{E}\left[\sup_t|U_t|^2+\int_0^T|V_s|^2\,ds\right]\leq Cr\,\mathbb{E}\left[\int_0^T\left|f_{n+i}\left(s,Y_s^n,Z_s^n\right)-f_n\left(s,Y_s^n,Z_s^n\right)\right|ds\right].$$

- Finally, since $\|Y^n\|_\infty\leq r$, we have

$$\left|f_{n+i}\left(s,Y_s^n,Z_s^n\right)-f_n\left(s,Y_s^n,Z_s^n\right)\right|\leq2\lambda\left|Z_s^n\right|\mathbf{1}_{|Z_s^n|>n}+2\lambda\left|Z_s^n\right|\mathbf{1}_{\psi_{r+1}(s)>n}+2\psi_{r+1}(s)\mathbf{1}_{\psi_{r+1}(s)>n},$$

- The conclusion is the following: the integrability of ψ_r is enough to show that (Y^n,Z^n) is a Cauchy sequence!
- It is easy to check that the limit is a solution. □

Proof of the third Step.

- For each $n \in \mathbf{N}^*$,

$$\xi_n = q_n(\xi), \qquad f_n(t,y,z) = f(t,y,z) - f_t^0 + q_n\left(f_t^0\right).$$

- $\left(\xi_n, f_n\right)$ satisfies the assumptions of Step 2.
- By the a priori estimate

$$\mathbb{E}\left[\sup_t \left|Y_t^{n+i} - Y_t^n\right|^2 + \left(\int_0^T \left|Z_s^{n+i} - Z_s^n\right|^2 ds\right)\right] \\ \leq C\mathbb{E}\left[|\xi_{n+i} - \xi_n|^2 + \int_0^T \left|q_{n+i}\left(f_t^0\right) - q_n\left(f_t^0\right)\right|^2 dt\right],$$

where C depends on T and λ.

- (Y^n, Z^n) is a Cauchy sequence and the limit is a solution. □

- Actually, the fact that ξ and $f(t,0,0)$ are square integrable is not really needed

Theorem 2. *Under (M) (without the integrability), if for some $p > 1$,*

$$\mathbb{E}\left[|\xi|^p + \left(\int_0^T |f(s,0,0)|ds\right)^p\right] < \infty$$

then BSDE $(\mathrm{E}_{\xi,f})$ *has a unique solution* $(Y,Z) \in \mathscr{B}^p$ *i.e. s.t.*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |Y_t|^p + \left(\int_0^T |Z_s|^2 ds\right)^{p/2}\right] < \infty$$

3. Infinite horizon BSDEs

- Let us consider the BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_s dB_s$$

- We want to replace the deterministic terminal time T by a stopping time τ
  - ⋆ τ not necessarily bounded !
- In the talk, I will consider only the case $\tau \equiv +\infty$.
  - ⋆ This related to elliptic PDEs in the whole space.
- Roughly speaking, we want to deal with

$$Y_t = \int_t^\infty f(s, Y_s, Z_s) ds - \int_t^\infty Z_s dB_s, \quad t \geq 0. \tag{2}$$

- A solution is a couple of progressively measurable processes s.t.,

$$\forall t \leq T, \qquad Y_t = Y_T + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_s dB_s$$

  - ⋆ I will keep the non correct writing!

- The assumption on the generator are the following : $f : [0,T] \times \Omega \times \mathbf{R}^k \times \mathbf{R}^{k\times d} \longrightarrow \mathbf{R}^k$
  - $y \longrightarrow f(t,y,z)$ is coninuous
  - Lipschitz in z:
$$|f(t,y,z) - f(t,y,z')| \leq \lambda |z - z'|$$
  - Monotone in y
$$(y - y') \cdot \big(f(t,y,z) - f(t,y,z')\big) \leq \mu |y - y'|^2$$
  - For the integrability, we assume that
$$|f(t,0,0)| \leq M$$

Theorem 3 (Darling and Pardoux, 97)**.** *If $\lambda^2 + 2\mu < 0$, BSDE* (2) *has a unique solution s.t.*

$$\mathbb{E}\left[\int_0^\infty e^{(\lambda^2+2\mu)s}\left(|Y_s|^2 + |Z_s|^2\right) ds\right] < \infty$$

For each $\varepsilon > 0$,

$$\mathbb{E}\left[\sup_{t\geq 0} e^{-\varepsilon s}|Y_s|^2 + \int_0^\infty e^{-\varepsilon s}\left(|Y_s|^2 + |Z_s|^2\right) ds\right] < \infty$$

- Advantage: multidimensional result
- Drawback: $\mu < -\lambda^2/2$!
- Proof: a priori estimate

Theorem 4 (B. and Y. Hu, 98 — M. Royer, 04)**.** *In the one dimensional case, if $\mu < 0$, BSDE* (2) *has a unique solution s.t. Y is bounded and $Z \in \mathrm{L}^2((0,T)\times\Omega)$ for all T.*

- Advantage: $\mu < 0$ which is reasonable from the PDE point of view
- Drawback: one dimensional

Proof.

- The main argument is to get rid of z by linearization.
- Roughly speaking, we will study

$$\begin{aligned} Y_t &= Y_T + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s \\ &= Y_T + \int_t^T \big(f(s, Y_s, 0) + Z_s b_s\big)\, ds - \int_t^T Z_s dB_s \\ &= Y_T + \int_t^T f(s, Y_s, 0) ds - \int_t^T Z_s dB_s^* \end{aligned}$$

- And apply Girsanov's theorem
- Let us start with uniqueness.
- (Y,Z) and (Y',Z') are two solutions with Y and Y' bounded.
- Itô-Tanaka formula to compute $de^{\mu s}|\delta Y_s|$ gives with $\operatorname{sgn}(y) = -\mathbf{1}_{y\leq 0} + \mathbf{1}_{y>0}$

$$d\left(e^{\mu t}|\delta Y_t|\right) = e^{\mu t}\left(\mu|\delta Y_t| - \operatorname{sgn}(\delta Y_t)F_t + \operatorname{sgn}(\delta Y_t)Z_t dB_t + dL_t\right),$$

where L is the local time at 0 of δY and where we have set

$$F_t = f(t, Y_t, Z_t) - f(t, Y'_t, Z'_t)$$

- Remember that we compute $-\int_t^T$ so that

$$\begin{aligned} e^{\mu t}|\delta Y_t| &= e^{\mu T}|\delta Y_T| + \int_t^T e^{\mu s}\left(\operatorname{sgn}(\delta Y_s)F_s - \mu|\delta Y_s|\right)ds - \int_t^T e^{\mu s}\operatorname{sgn}(\delta Y_s)\delta Z_s dB_s - \int_t^T e^{\mu s}dL_s \\ &\leq e^{\mu T}|\delta Y_T| + \int_t^T e^{\mu s}\left(\operatorname{sgn}(\delta Y_s)F_s - \mu|\delta Y_s|\right)ds - \int_t^T e^{\mu s}\operatorname{sgn}(\delta Y_s)\delta Z_s dB_s \end{aligned}$$

- We write F_s as the sum

$$F_s = \left(f(s, Y_s, Z_s) - f(s, Y'_s, Z_s)\right) + \left(f(s, Y'_s, Z_s) - f(s, Y'_s, Z'_s)\right)$$

- Since $\delta Y_s\left(f(s, Y_s, Z_s) - f(s, Y'_s, Z_s)\right) \leq \mu|\delta Y_s|^2$, we have

$$\operatorname{sgn}(\delta Y_s)\left(f(s, Y_s, Z_s) - f(s, Y'_s, Z_s)\right) \leq \mu|\delta Y_s|$$

- Moreover, we define

$$b_s = \frac{f(s, Y'_s, Z_s) - f(s, Y'_s, Z'_s)}{|\delta Z_s|^2}\delta Z_s^* \mathbf{1}_{|\delta Z_s|>0}$$

so that

$$Z_s b_s = f(s, Y'_s, Z_s) - f(s, Y'_s, Z'_s)$$

- Putting things together, we get

$$\begin{aligned} e^{\mu t}|\delta Y_t| &\leq e^{\mu T}|\delta Y_T| + \int_t^T e^{\mu s}\operatorname{sgn}(\delta Y_s)\delta Z_s b_s ds - \int_t^T e^{\mu s}\operatorname{sgn}(\delta Y_s)\delta Z_s dB_s \\ &\leq e^{\mu T}|\delta Y_T| + \int_t^T e^{\mu s}\operatorname{sgn}(\delta Y_s)\delta Z_s dB_s^* \end{aligned}$$

where $B_s^* = B_s - \int_O^s b_r dr$

- By Girsavov's theorem (on $[0,T]$), b is bounded

$$|\delta Y_t| \leq e^{\mu(T-t)}\mathbb{E}^*\left(|\delta Y_T|\,|\,\mathscr{F}_t\right) \leq e^{\mu(T-t)}2M, \qquad |\delta Y_t| \leq 0 = \lim_{T\to\infty} e^{\mu(T-t)}2M$$

- Itô's formula gives $\delta Z \equiv 0$.
- Existence: same approach
- Let (Y^n, Z^n) be the solution to the BSDE

$$Y_t^n = 0 + \int_t^n f(s, Y_s^n, Z_s^n)ds - \int_t^n Z_s^n dB_s, \quad 0 \le t \le n.$$

- For $t \ge n$, $Y_t^n = 0$, $Z_t^n = 0$.
- Let us prove that Y_t^n is bounded. Arguing as before,

$$e^{\mu t}|Y_t^n| \le \int_t^n e^{\mu s}\left(\text{sgn}(Y_s^n) f(s, Y_s^n, Z_s^n) - \mu|Y_s^n|\right) - \int_t^n e^{\mu s}\text{sgn}(Y_s^n) Z_s^n dB_s$$

- Splitting

$$\begin{aligned} f(s, Y_s^n, Z_s^n) &= f(s,0,0) + f(s, Y_s^n, 0) - f(s,0,0) + f(s, Y_s^n, Z_s^n) - f(s, Y_s^n, 0) \\ &= f(s,0,0) + f(s, Y_s^n, 0) - f(s,0,0) + Z_s^n b_s^n \end{aligned}$$

- We have, since $\text{sgn}(Y_s^n)\left(f(s, Y_s^n, 0) - f(s,0,0)\right) \le \mu|Y_s^n|$,

$$\begin{aligned} e^{\mu t}|Y_s^n| &\le \int_t^n e^{\mu s}|f(s,0,0)|ds - \int_t^n e^{\mu s}\text{sgn}(Y_s^n) Z_s^n dB_s^n \\ &\le \frac{M}{\mu}\left(e^{\mu n} - e^{\mu t}\right) - \int_t^n e^{\mu s}\text{sgn}(Y_s^n) Z_s^n dB_s^n \end{aligned}$$

- Taking the conditional expectation, we get

$$|Y_t^n| \le \frac{M}{|\mu|}.$$

- In the same way, for $t \le n \le m$,

$$\begin{aligned} e^{\mu t}|Y_t^m - Y_t^n| &\le \int_n^m e^{\mu s}|f(s,0,0)|ds - \int_t^n e^{\mu s}\text{sgn}(Y_s^m - Y_s^n)\left(Z_s^m - Z_s^n\right) dB_s^{m,n} \\ |Y_t^m - Y_t^n| &\le \frac{M}{|\mu|} e^{\mu(n-t)}. \end{aligned}$$

- Y^n is a Cauchy sequence and ... we get a solution. □

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