Selected Topics in BSDEs Theory
Lecture V: A first Look in Quadratic BSDEs

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What is a quadratic BSDE?

- Still with our BSDE, $Y \in \mathbf{R}$!

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_s \cdot dB_s, \quad 0 \le t \le T \qquad (\mathrm{E}_{\xi,f})$$

- $f : [0, T] \times \mathbf{R} \times \mathbf{R}^d \longrightarrow \mathbf{R}$ continuous generator in (y, z)
- Quadratic BSDE means quadratic w.r.t. z

$$|f(t, y, z)| \le \alpha + \beta|y| + \frac{\gamma}{2}|z|^2$$

  - ⋆ α, β, γ nonnegative real numbers

Theorem (M. Kobylanski, 2000)

If ξ is bounded, BSDE $(\mathrm{E}_{\xi,f})$ has a bounded solution.

- She also proves a comparison result
- Her approach is roughly speaking a PDE approach

What can we hope?

- For the well known equation:

$$Y_t = \xi + \frac{1}{2}\int_t^T |Z_s|^2\, ds - \int_t^T Z_s\, dB_s,$$

- The change of variable $P_t = e^{Y_t}$, $Q_t = e^{Y_t} Z_t$, leads to the equation

$$P_t = e^{\xi} - \int_t^T Q_s\, dB_s$$

- The solution is

$$Y_t = \ln \mathbb{E}\left(e^{\xi} \,\middle|\, \mathcal{F}_t \right)$$

Theorem (Ph. B. & Y. Hu 2006)

Assume that

$$\mathbb{E}\left[\exp\left(\gamma e^{\beta T} |\xi|\right)\right] < +\infty.$$

α, β, γ

Then, $(\mathrm{E}_{\xi,f})$ *has a solution s.t.*

$$|Y_t| \leq \alpha T\, e^{\beta T} + \frac{1}{\gamma} \log \mathbb{E}\left(\exp\left(\gamma e^{\beta T}|\xi|\right) \,\middle|\, \mathcal{F}_t\right).$$

Goal of the lecture

- Probabilistic proof of Kobilanski's result
  - ⋆ with the terminal condition ξ bounded
- Method based on Girsanov's theorem
  - ⋆ with BMO martingales
  - ⋆ Get rid of the dependence in z of the generator
- Get some results when ξ is not bounded

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Un elementary result

- $f : [0, T] \times \mathbf{R} \times \mathbf{R}^d \longrightarrow \mathbf{R}$ s.t.
  - $\star$ $|f(t, 0, 0)| \leq \alpha$
  - $\star$ $|f(t, y, z) - f(t, y', z)| \leq \beta|y - y'|$
  - $\star$ $|f(t, y, z) - f(t, y, z')| \leq \gamma|z - z'|$
- ξ bounded

Proposition

Let (Y, Z) be a solution to $(\mathrm{E}_{\xi,f})$.
Then Y is bounded and the bound does not depend on γ :

$$|Y_t| \leq (\|\xi\|_\infty + \alpha T)\, e^{\beta T}.$$

Proof by linearization

- Let us recall that we write the BSDE as alinear one

$$Y_t = \xi + \int_t^T (f(s,0,0) + a_s Y_s + b_s \cdot Z_s)\, ds - \int_t^T Z_s \cdot dB_s,$$

avec

$$a_s = \frac{f(s,Y_s,Z_s) - f(s,0,Z_s)}{Y_s} \mathbf{1}_{|Y_s|>0}, \quad |a_s| \leq \beta$$

$$b_s = \frac{f(s,0,Z_s) - f(s,0,0)}{|Z_s|^2} Z_s \mathbf{1}_{|Z_s|>0}, \quad |b_s| \leq \gamma$$

- Set $B_s^* = B_s - \int_0^s b_r dr$

$$Y_t = \xi + \int_t^T (f(s,0,0) + a_s Y_s)\, ds - \int_t^T Z_s \cdot dB_s^*$$

Proof by linearization

- $\left\{ M_t = \int_0^t b_s \cdot dB_s : 0 \leq t \leq T \right\}$ is a martingale and

$$Y_t = e_t^{-1} \mathbb{E}^* \left(\xi e_T + \int_t^T e_s f(s,0,0) ds \,\middle|\, \mathcal{F}_t \right), \; e_s = \exp\left(\int_0^s a_r \, dr \right)$$

avec

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \mathcal{E}(M)_T = \exp\left(\int_0^T b_s \cdot dB_s - \frac{1}{2} \int_0^T |b_s|^2 ds \right)$$

- $|Y_t| \leq \left(\|\xi\|_\infty + \alpha T \right) e^{\beta T}$

Bound on Z

Proposition

If the Malliavin derivative of ξ is bounded, then Z is bounded and the bound does not depend on γ:

$$|Z_t| \leq e^{\beta T} \, \|D\xi\|_\infty .$$

- For $h \in \mathrm{L}^2(0, T; \mathbf{R}^d)$, let $B(h) = \int_0^T h(s) \cdot dB_s$.
- If $\xi = \Phi(B(h^1), \ldots, B(h^k))$, où $\Phi \in \mathcal{C}_b^\infty$,

$$D_\theta \xi = \sum_{j=1}^{k} \partial_j \Phi(B(h^1), \ldots, B(h^k)) h^j(\theta)$$

- Chain rule

$$D_\theta \Phi(F) = \Phi'(F) D_\theta F$$

Malliavin Calculus and BSDEs

- We use only two points:

1. If f is smooth and ξ is differentiable in the Malliavin sense, then (Y, Z) is also differentiable in the Malliavin sense and

$$D_\theta Y_t = 0, \quad D_\theta Z_t = 0, \quad 0 \leq t < \theta \leq T,$$

$$D_\theta Y_t = D_\theta \xi + \int_t^T \left(\partial_y f(s, Y_s, Z_s) D_\theta Y_s + \partial_z f(s, Y_s, Z_s) D_\theta Z_s \right) ds$$
$$- \int_t^T D_\theta Z_s dB_s, \qquad \theta \leq t \leq T.$$

2. $\{D_t Y_t : 0 \leq t \leq T\}$ is a version of $\{Z_t : 0 \leq t \leq T\}$

Bound on Z

- Let us assume first that f is $\mathcal{C}^1$.
- (Y, Z) is differentiable in the Malliavin sense
- As we said before, for $0 \leq \theta \leq t \leq T$

$$D^i_\theta Y_t = D^i_\theta \xi + \int_t^T \left(\partial_y f(s, Y_s, Z_s) D^i_\theta Y_s + \partial_z f(s, Y_s, Z_s) D^i_\theta Z_s \right) ds - \int_t^T D^i_\theta Z_s \cdot dB_s$$

- Previous result: $\left| D^i_\theta Y_t \right| \leq e^{\beta T} \left\| D^i_\theta \xi \right\|_\infty$
- For $\theta = t$: $\left| Z^i_t = D^i_t Y_t \right| \leq e^{\beta T} \left\| D^i_t \xi \right\|_\infty$.
- The general case is obtained by regularization

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Framework

- A bounded terminal condition : $\xi \in \mathrm{L}^\infty$
- A quadratic generator $f : [0, T] \times \mathbf{R} \times \mathbf{R}^d \longrightarrow \mathbf{R}$ deterministic:

$$|f(t, y, z)| \leq \alpha + \beta|y| + \frac{\gamma}{2}|z|^2$$

- Some régularity:
  - ⋆ $|f(t, y, z) - f(t, y', z)| \leq \beta|y - y'|$
  - ⋆ $|f(t, y, z) - f(t, y, z')| \leq \rho(1 + |z| + |z'|)|z - z'|$
  - ⋆ $|f(t, 0, 0)| \leq \delta$
  - ⋆ $\gamma = 3\rho$, $\alpha = \delta + \rho/2$

M. Kobylanski's result

Theorem

The BSDE $(\mathrm{E}_{\xi,f})$ *has a unique solution* (Y, Z) *s.t.* Y *is a bounded process.*

Proof by Girsanov

Let $(\xi^n)_{n\geq 1}$ converging in probability to ξ with

$$\xi^n = \Phi^n(B_{t_1^n}, \ldots, B_{t_{p^n}^n}), \quad \Phi^n \in \mathcal{C}_b^\infty, \quad \|\Phi^n\|_\infty \leq \|\xi\|_\infty$$

Step 1

- In this step, n is fixed.
- Let, for $k \geq 1$, $q_k(z) = z\frac{|z| \wedge k}{|z|}$, $f_k(t, y, z) = f(t, y, q_k(z))$.
- f_k is β-Lipschitz en y and $\rho(1 + 2k)$-Lipschitz en z since

$$|f(t, y, z) - f(t, y', z')| \leq \beta|y - y'| + \rho(1 + |z| + |z'|)|z - z'|$$

- Let $(Y^{n,k}, Z^{n,k})$ be the solution to the BSDE

$$Y_t^{n,k} = \xi^n + \int_t^T f_k(s, Y_s^{n,k}, Z_s^{n,k})ds - \int_t^T Z_s^{n,k} \cdot dB_s.$$

- By the first proposition,

$$|Y_t^{n,k}| \leq (\|\xi\|_\infty + \alpha T)\, e^{\beta T}.$$

Step 1

- ξ^n is chosen so that $D_\theta \xi^n$ is bounded
- From the second proposition, $Z^{n,k}$ is bounded independently of k:

$$|Z_t^{n,k}| \leq e^{\beta T} \|D_t \xi^n\|_\infty$$

- It follows that, for k large enough, $q_k(Z^{n,k}) = Z^{n,k}$
- We get a solution (Y^n, Z^n) to the BSDE

$$Y_t^n = \xi^n + \int_t^T f(s, Y_s^n, Z_s^n) ds - \int_t^T Z_s^n \cdot dB_s, \quad 0 \leq t \leq T.$$

- It remains to send $n \to \infty$.

BMO martingales

Definition

$\left\{ M_t = \int_0^t Z_s \cdot dB_s : 0 \leq t \leq T \right\}$ is a BMO martingale if there exists a constant C s.t. for each stopping time $\tau \leq T$:

$$\mathbb{E}\left(|M_T - M_\tau|^2 \mid \mathcal{F}_\tau \right) = \mathbb{E}\left(\int_\tau^T |Z_s|^2 ds \,\Big|\, \mathcal{F}_\tau \right) \leq C.$$

- If M is a BMO martingale, the best constant C in the previous inequality defines $\|M\|_{\mathrm{BMO}}^2$

Properties of BMO martingales (Kazamaki)

- Let M be a BMO martingale and let us denote $N = \|M\|_{\mathrm{BMO}}$
- $\{\mathcal{E}(M)_t\}_{t\in[0,T]}$ is a uniformly integrable martingale where

$$\mathcal{E}(M)_t = \exp(M_t - \langle M\rangle_t/2)$$

- Reverse Hölder inequality : there exists $q_* > 1$ s.t., for $\tau \le T$,

$$\forall 1 < q < q_*, \quad \mathbb{E}\left(\mathcal{E}(M)_T^q \,\middle|\, \mathcal{F}_\tau\right) \le C(q,N)\,\mathcal{E}(M)_\tau^q$$

  - ⋆ $q_* = \phi^{-1}(N)$ with $\phi(p) = \left(1 + \frac{1}{p^2}\log\frac{2p-1}{2(p-1)}\right)^{1/2} - 1$
  - ⋆ $C(q,N) = \dfrac{2}{1 - 2(q-1)(2q-1)^{-1}\exp(q^2(N^2+2N))}$

Back to the proof of the theorem

- (Y^n, Z^n) solves the BSDE

$$Y_t^n = \xi^n + \int_t^T f(s, Y_s^n, Z_s^n)ds - \int_t^T Z_s^n \cdot dB_s, \quad 0 \leq t \leq T.$$

Proposition

$\left\{ M_t^n = \int_0^t Z_s^n \cdot dB_s : t \in [0,T] \right\}$ *is a BMO martingale. Moreover,*

$$\sup_{n \geq 1} \left\| M^n \right\|_{BMO} < +\infty.$$

Proof

- We use Itô's formula with $u(|x|)$ where the function u is defined by

$$\forall x \geq 0, \qquad u(x) = \frac{e^{\gamma x} - 1 - \gamma x}{\gamma^2}$$

Computation

- We denote $\operatorname{sgn}(x) = -\mathbf{1}_{x\leq 0} + \mathbf{1}_{x>0}$,

$$u(|Y_t|) = u(|Y_T|) + \int_t^T \left(u'(|Y_s|)\operatorname{sgn}(Y_s) f(s, Y_s, Z_s) - \frac{1}{2} u''(|Y_s|)|Z_s|^2 \right) ds$$
$$- \int_t^T u'(|Y_s|)\operatorname{sgn}(Y_s) Z_s \cdot dB_s.$$

- Since $u'(x) \geq 0$ for $x \geq 0$

$$u(|Y_t|) + \frac{1}{2}\int_t^T \left(u''(|Y_s|) - \gamma\, u'(|Y_s|)\right)|Z_s|^2 ds \leq u(|Y_T|) +$$
$$\int_t^T u'(|Y_s|)\left(\alpha + \beta|Y_s|\right) ds - \int_t^T u'(|Y_s|)\operatorname{sgn}(Y_s) Z_s \cdot dB_s$$

- u is construct s.t. $(u'' - \gamma u')(x) = 1$ and $u(x) \geq 0$ for $x \geq 0$,

$$\frac{1}{2}\mathbb{E}\left[\int_t^T |Z_s|^2\, ds \,\Big|\, \mathcal{F}_t\right] \leq C(\alpha, \beta, \gamma, T, \|Y^n\|_\infty) = C(\alpha, \beta, \gamma, T)$$

Convergence of (Y^n, Z^n)

Proposition (Ph. B. and F. Confortola, 08)

There exists $p > 1$ s.t. for $r > p$,

$$\mathbb{E}\left[\sup_{0\le t\le T} |Y_t^m - Y_t^n|^r + \left(\int_0^T |Z_t^m - Z_t^n|^2 dt\right)^{r/2}\right] \le C(r,p)\mathbb{E}\left[|\xi^m - \xi^n|^r\right].$$

- The idea is to linearize the BSDE satisfied by $y_s = Y^m - Y^n$

$$y_t = \xi^m - \xi^n + \int_t^T \left(a^{n,m} y_s + b_s^{n,m} \cdot z_s\right) ds - \int_t^T z_s \cdot dB_s$$

$$a_s^{n,m} = \frac{f(s, Y_s^m, Z_s^m) - f(s, Y^n, Z_s^m)}{y_s} \mathbf{1}_{|y_s|>0}$$

$$b_s^{n,m} = \frac{f(s, Y^n, Z_s^m) - f(s, Y_s^n, Z_s^n)}{|z_s|^2} \mathbf{1}_{|z_s|>0} z_s$$

Proof of the estimate

- We have $|a_s^{n,m}| \leq \beta$ and $|b_s^{n,m}| \leq \rho(1 + |Z_s^m| + |Z_s^n|)$.
- $M_t^{n,m} = \int_0^t b_s^{n,m} \cdot dB_s$ is a BMO martingale and

$$N = \sup_{n,m} \|M^{n,m}\|_{\text{BMO}} < +\infty.$$

- There exists $q_* = q_*(N) > 1$ (independent of m and n) s.t. for $1 < q < q_*$

$$\mathbb{E}\left(\left(\mathcal{E}_T^{n,m}\right)^q \mid \mathcal{F}_t\right) \leq C(q,N)\left(\mathcal{E}_t^{n,m}\right)^q$$

- We easily get from the previous linear BSDE

$$|y_t| \leq e^{\beta(T-t)}\left(\mathcal{E}_t^{n,m}\right)^{-1} \mathbb{E}\left(|\xi^m - \xi^n|\mathcal{E}_T^{n,m} \mid \mathcal{F}_t\right)$$

Proof of the estimate

- Let us pick $1 < q < q_*$ et denote by p the conjugate exponent of q.
- We have

$$|y_t| \leq e^{\beta(T-t)} \left(\mathcal{E}_t^{n,m}\right)^{-1} \mathbb{E}\left(|\xi^m - \xi^n|^p \mid \mathcal{F}_t\right)^{1/p} \mathbb{E}\left(\left(\mathcal{E}_T^{n,m}\right)^q \mid \mathcal{F}_t\right)^{1/q}$$

- With the reverse Hölder inequality

$$|y_t| \leq e^{\beta(T-t)} C(q,N) \, \mathbb{E}\left(|\xi^m - \xi^n|^p \mid \mathcal{F}_t\right)^{1/p}$$

- To conclude, we have just to use Doob's maximal inequality
- We deduce the estimate for Z from the bound on Y

End of Proof

- We know that (Y^n, Z^n) is a Cauchy sequence.
- It is easy to check that the limit (Y, Z) solves our BSDE
- Uniqueness is proved by linearization in the same way

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Framework

- There exist $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$ s.t.
  - f is Lipschitz w.r.t. y: for any t, z,
  $$\left|f(t,y,z) - f(t,y',z)\right| \leq \beta\,|y - y'|$$
  - quadratic growth in z:
  $$|f(t,y,z)| \leq \alpha + \beta|y| + \frac{\gamma}{2}|z|^2$$
  - ξ is $\mathcal{F}_T$–measurable, not necessarily bounded,
  $$\forall\lambda > 0, \qquad \mathbb{E}\left[\exp\left(\lambda|\xi|\right)\right] < +\infty.$$
  - for any t, y, $z \longmapsto f(t,y,z)$ is a convex function;
- We want to study BSDE ($E_{\xi,f}$) in this setting
- The first we have to do is to get a tractable a priori estimate on the solution

Exponential change

If (Y, Z) is a solution to

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_S \cdot dB_s,$$

where ξ is bounded

then $P_t = e^{\gamma Y_t}$, $Q_t = \gamma e^{\gamma Y_t} Z_t$, (P, Q) solves the BSDE

$$P_t = e^{\gamma \xi} + \int_t^T F(s, P_s, Q_s)\, ds - \int_t^T Q_s \cdot dB_s$$

with the function F defined by

$$F(s, p, q) = \mathbf{1}_{p>0} \left(\gamma p\, f\left(s, \frac{\ln p}{\gamma}, \frac{q}{\gamma p} \right) - \frac{1}{2} \frac{|q|^2}{p} \right).$$

Upper Bound

This exponential change "kills the quadratic term" since, from the growth of f,

$$F(s,p,q) \leq G(p) := p\left(\alpha\gamma + \beta|\ln p|\right)\mathbf{1}_{(0,+\infty)}(p).$$

This leads to the known estimate $P_t \leq \phi_t$ with

$$\phi_t = e^{\gamma\|\xi\|_\infty} + \int_t^T G(\phi_s)\,ds.$$

This is useless if ξ is unbounded

- We have also

$$F(s,p,q) \leq H(p) := p\,(\alpha\gamma + \beta \ln p)\,\mathbf{1}_{[1,+\infty)}(p) + \gamma\alpha\mathbf{1}_{(-\infty,1)}(p).$$

- The difference between G and H is that

$$H \text{ is convex} \quad (\gamma\alpha \geq \beta).$$

- It allows to compare P_t with the solution to a differential equation without using $\|\xi\|_\infty$.

If $\{\phi_t(x)\}_{0\leq t\leq T}$ stands for the solution to

Formula

$$\phi_t = e^{\gamma x} + \int_t^T H(\phi_s)\,ds,$$

$$P_t \leq \mathbb{E}\left(\phi_t(\xi) \mid \mathcal{F}_t\right), \qquad Y_t \leq \frac{1}{\gamma}\log \mathbb{E}\left(\phi_t(\xi) \mid \mathcal{F}_t\right).$$

First Result

Lemma

If (Y, Z) is solution to BSDE(ξ, f) with Y bounded and $Z \in \mathrm{L}^2$,

$$-\frac{1}{\gamma}\log\mathbb{E}\left(\phi_t(-\xi) \mid \mathcal{F}_t\right) \leq Y_t \leq \frac{1}{\gamma}\log\mathbb{E}\left(\phi_t(\xi) \mid \mathcal{F}_t\right).$$

- This implies

$$|Y_t| \leq \alpha T\, e^{\beta T} + \frac{1}{\gamma}\,\log\mathbb{E}\left(\exp\left(\gamma e^{\beta T}|\xi|\right) \mid \mathcal{F}_t\right).$$

- Actually, it explains the assumption on ξ to get existence

$$\mathbb{E}\left[e^{\gamma e^{\beta T}|\xi|}\right] < \infty$$

which is nothing but

$\phi_0(|\xi|)$ integrable.

Proof of the lemma

Since ϕ_t solves the equation

$$\phi_t = e^{\gamma\xi} + \int_t^T H(\phi_s)\, ds,$$

we have, setting $\Phi_t = \mathbb{E}\left(\phi_t \mid \mathcal{F}_t\right)$,

$$\Phi_t = \mathbb{E}\left(e^{\gamma\xi} + \int_t^T \mathbb{E}\left(H(\phi_s) \mid \mathcal{F}_s\right)\, ds \,\middle|\, \mathcal{F}_t \right).$$

Proof of the lemma

But H is convex:

$$\Phi_t \geq \mathbb{E}\left(e^{\gamma\xi} + \int_t^T H\left(\Phi_s\right)\, ds \,\middle|\, \mathcal{F}_t\right).$$

On the other hand

$$\begin{aligned} P_t &= \mathbb{E}\left(e^{\gamma\xi} + \int_t^T F(s, P_s, Q_s)\, ds \,\middle|\, \mathcal{F}_t\right) \\ &\leq \mathbb{E}\left(e^{\gamma\xi} + \int_t^T H(P_s)\, ds \,\middle|\, \mathcal{F}_t\right). \end{aligned}$$

So, looking at $\Phi_t - P_t$ as the solution to a BSDE, the comparison theorem gives $P_t \leq \Phi_t$.

Comparison theorem

Theorem (Ph. B. and Y. Hu, 08)

Let (Y, Z) and (Y', Z') be solution to $(\mathrm{E}_{\xi,f})$ and $(\mathrm{E}_{\xi',f'})$ where (ξ, f) satisfies (H) and Y, Y' belongs to $\mathcal{E}$ ($\mathcal{E}$:= exponential moment of all order).
If $\xi \le \xi'$ and $f \le f'$ then

$$\forall t \in [0, T], \qquad Y_t \le Y'_t$$

In particular, $(\mathrm{E}_{\xi,f})$ has a unique solution in the class $\mathcal{E}$.

Main idea

Estimate of $Y_t - \mu Y'_t$ for $\mu \in (0, 1)$.

Proof: f independent of y

Set, for $\mu \in (0,1)$, $U_t = Y_t - \mu Y'_t$, $V_t = Z_t - \mu Z'_t$.

$$U_t = U_T + \int_t^T F_s \, ds - \int_t^T V_s \, dB_s, \qquad F_s = f(s, Z_s) - \mu f'\left(s, Z'_s\right)$$

$$F_t = \left[f(t, Z_t) - \mu f\left(t, Z'_t\right)\right] + \mu\left[f\left(t, Z'_t\right) - f'\left(t, Z'_t\right)\right]$$

and $\delta f(t) := f\left(t, Z'_t\right) - f'\left(t, Z'_t\right) \leq 0$.

$$Z_t = \mu Z'_t + (1-\mu)\frac{Z_t - \mu Z'_t}{1-\mu}$$

$$\begin{aligned} f(t, Z_t) &= f\left(t, \mu Z'_t + (1-\mu)\frac{Z_t - \mu Z'_t}{1-\mu}\right) \\ \text{Convexity} \quad &\leq \mu f\left(t, Z'_t\right) + (1-\mu) f\left(t, \frac{Z_t - \mu Z'_t}{1-\mu}\right) \end{aligned}$$

$$f(t,Z_t)-\mu f\left(t,Z'_t\right)\le(1-\mu)f\left(t,\frac{V_t}{1-\mu}\right)\le(1-\mu)\alpha+\frac{\gamma}{2(1-\mu)}|V_t|^2$$

$$F_t\le\mu\delta f(t)+(1-\mu)\alpha+\frac{\gamma}{2(1-\mu)}|V_t|^2$$

Second step

An exponential change of variable to remove the quadratic term

$$P_t=e^{cU_t},\qquad Q_t=cP_tV_t,\qquad c\ge 0$$

$$P_t=P_T+c\int_t^T P_s\left(F_s-\frac{c}{2}|V_s|^2\right)ds-\int_t^T Q_s\,dB_s$$

$c=\frac{\gamma}{1-\mu}$ yield

$$P_t\le P_T+\gamma\int_t^T\left(\alpha+(1-\mu)^{-1}\mu\delta f(s)\right)P_s\,ds-\int_t^T Q_s\,dB_s$$

$$P_t \leq \mathbb{E}\left(\exp\left[\gamma \int_t^T \left(\alpha + (1-\mu)^{-1}\mu\delta f(s)\right)\, ds\right] P_T \,\Big|\, \mathcal{F}_t\right)$$

$$P_T = \exp\left(\frac{\gamma}{1-\mu}(\xi - \mu\xi')\right) = \exp\left(\gamma\left(\xi + \frac{\mu}{1-\mu}\delta\xi\right)\right)$$

$$P_t \leq \mathbb{E}\left(\exp\left[\gamma\left(\xi + \alpha T\right) + \gamma\frac{\mu}{1-\mu}\left(\delta\xi + \int_t^T \delta f(s)\, ds\right)\right] \,\Big|\, \mathcal{F}_t\right)$$

In particular,

$$Y_t - \mu Y_t' \leq \frac{1-\mu}{\gamma}\log\mathbb{E}\left(\exp\left[\gamma\left(\xi + \alpha T\right)\right] \,\Big|\, \mathcal{F}_t\right)$$

and sending μ to 1, we get

$$Y_t - Y_t' \leq 0.$$

Existence

- We had the extra assumption

$$\left|f(t,y,z) - f(t,y,z')\right| \leq \rho\left(1 + |z| + |z'|\right)|z - z'|$$

  - ⋆ This assumption is not needed
  - ⋆ But we prove the result in the bounded case under this assumption!

- Let (Y^n, Z^n) be the solution to the quadratic BSDE

$$Y_t^n = \xi_n + \int_t^T f(s, Y_s^n, Z_s^n)ds - \int_t^T Z_s^n dB_s, \quad 0 \leq t \leq T$$

  - ⋆ $\xi_n = q_n(\xi)$ is bounded!

- From the a priori estimate

$$|Y_t^n| \leq \frac{1}{\gamma} \log \mathbb{E}\left(\exp\left(\gamma e^{\beta T}(|\xi| + \alpha T)\right) \mid \mathcal{F}_t\right).$$

Existence

- We have to prove that (Y^n, Z^n) is a Cauchy sequence.
- Arguing as in the proof of Comparison Theorem, we get

$$Y_t^m - \mu Y_t^n \leq \frac{1-\mu}{\gamma} \log \mathbb{E}\left(\exp\left[\gamma\left(\xi^m + \alpha T\right) + \gamma \frac{\mu}{1-\mu}\left(\xi^m - \xi^n\right)\right] \,\Big|\, \mathcal{F}_t\right)$$

$$Y_t^n - \mu Y_t^m \leq \frac{1-\mu}{\gamma} \log \mathbb{E}\left(\exp\left[\gamma\left(\xi^n + \alpha T\right) + \gamma \frac{\mu}{1-\mu}\left(\xi^n - \xi^m\right)\right] \,\Big|\, \mathcal{F}_t\right)$$

- Taking into account the a priori estimate, we get, for f independent of y,

$$\left|Y_t^m - Y_t^n\right| \leq \frac{1-\mu}{\gamma} \log \mathbb{E}\left(\exp\left[\gamma(|\xi| + \alpha T)\right] \,\big|\, \mathcal{F}_t\right) + \frac{1-\mu}{\gamma} \log \mathbb{E}\left(\exp\left[\gamma\left(|\xi| + \alpha T\right) + \gamma \frac{\mu}{1-\mu}\left|\xi^m - \xi^n\right|\right] \,\Big|\, \mathcal{F}_t\right)$$

Existence

- Using the fact that $\log x \leq x$, we have

$$\left|Y_t^m - Y_t^n\right| \leq \frac{1-\mu}{\gamma}\,\mathbb{E}\left(\exp\left[\gamma(|\xi| + \alpha T)\right] \,\middle|\, \mathcal{F}_t\right) + \frac{1-\mu}{\gamma}\mathbb{E}\left(\exp\left[\gamma\left(|\xi| + \alpha T\right) + \gamma\frac{\mu}{1-\mu}\left|\xi^m - \xi^n\right|\right] \,\middle|\, \mathcal{F}_t\right)$$

- We deduce from Doob's inequality that

$$\mathbb{P}\left(\sup_t \left|Y_t^m - Y_t^n\right| > \varepsilon\right) \leq \frac{2(1-\mu)}{\gamma\varepsilon}\,\mathbb{E}\left(\exp\left[\gamma(|\xi| + \alpha T)\right]\right) + \frac{2(1-\mu)}{\gamma\varepsilon}\,\mathbb{E}\left(\exp\left[\gamma\left(|\xi| + \alpha T\right) + \gamma\frac{\mu}{1-\mu}\left|\xi^m - \xi^n\right|\right]\right)$$

Existence

- It follows that

$$\begin{aligned}\limsup_{n,m} \mathbb{P}\left(\sup_t \left|Y_t^m - Y_t^n\right| > \varepsilon\right) &\leq \frac{2(1-\mu)}{\gamma\varepsilon}\,\mathbb{E}\left(\exp\left[\gamma(|\xi| + \alpha T)\right]\right) \\ &\quad + \frac{2(1-\mu)}{\gamma\varepsilon}\,\mathbb{E}\left(\exp\left[\gamma\left(|\xi| + \alpha T\right)\right]\right) \\ &= \frac{4(1-\mu)}{\gamma\varepsilon}\,\mathbb{E}\left(\exp\left[\gamma(|\xi| + \alpha T)\right]\right)\end{aligned}$$

- It remains to send μ to 1 to show that Y^n is a Cauchy sequence
- From this we construct a solution

Contents

Application to PDEs

- Probabilistic representation for

$$\partial_t u(t,x) + \mathcal{L}u(t,x) + f(t,x,u(t,x),\nabla_x u\sigma(t,x)) = 0, \quad u(T,.) = g,$$

$$\mathcal{L}u(t,x) = \frac{1}{2}\mathrm{trace}(\sigma\sigma^*\nabla_x^2 u(t,x)) + b(t,x)\cdot\nabla_x u(t,x).$$

- The SDE: X^{t_0,x_0} solution to

$$X_t = x_0 + \int_{t_0}^t b(s,X_s)\,ds + \int_{t_0}^t \sigma(s,X_s)\,dB_s$$

- The BSDE: $\left(Y^{t_0,x_0}, Z^{t_0,x_0}\right)$ solution to

$$Y_t = g\left(X_T^{t_0,x_0}\right) + \int_t^T f\left(s, X_s^{t_0,x_0}, Y_s, Z_s\right)\,ds - \int_t^T Z_s\,dB_s$$

- Nonlinear Feynman-Kac's formula: $u(t,x) := Y_t^{t,x}$ is a viscosity solution

Assumptions

- b, σ, f and g are continuous;
- b, σ Lipschitz w.r.t. x

$$\big|b(t,x)-b(t,x')\big|+\big|\sigma(t,x)-\sigma(t,x')\big|\leq\beta|x-x'|;$$

- restriction: σ is bounded;
- f is Lipschitz w.r.t. y

$$\big|f(t,x,y,z)-f(t,x,y',z)\big|\leq\beta|y-y'|;$$

- $z\longmapsto f(t,x,y,z)$ is convex;
- $\exists p<2$ s.t.

$$|g(x)|+|f(t,x,y,z)|\leq C\left(1+|x|^p+|y|+|z|^2\right).$$

u solves the PDE

Proposition

$u(t,x) := Y_t^{t,x}$ is continuous and

$$|u(t,x)| \leq C\left(1 + |x|^p\right).$$

Proposition

$u(t,x) := Y_x^{t,x}$ is a viscosity solution to the PDE.

Without convexity

- In the bounded case, uniquess can be proved with the assumption

$$\left|f(t,y,z)-f(t,y',z')\right| \le C\left(|y-y'|+(1+|z|+|z'|)|z-z'|\right).$$

- and without convexity
- Can we do the same in the non bounded case?
- Very particular result

$$X_t = x + \int_0^t b(s,X_s)\,ds + \sigma B_t,$$
$$Y_t = g(X_T) + \int_t^T f(s,X_s,Y_s,Z_s)\,ds - \int_t^T Z_s\,dB_s,$$

with the assumption, $\lim_{t\to 0^+}\omega(t)=0$,

$$|g(x)-g(x')| + |f(s,x,y,z)-f(s,x',y',z')| \le \omega(|x-x'|) + C\left(|y-y'|+(1+|z|+|z'|)|z-z'|\right),$$

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Explicit formula for ϕ

$$\forall x \geq 0, \qquad \phi_t(x) = \exp\left(\gamma\alpha \frac{e^{\beta(T-t)} - 1}{\beta}\right) \exp\left(x\gamma e^{\beta(T-t)}\right).$$

For $x < 0$:

- if $e^{\gamma x} + \alpha\gamma T \leq 1$,
$$\phi_t(x) = e^{\gamma x} + \alpha\gamma(T-t),$$
- else, $e^{\gamma x} + \alpha\gamma(T-S) = 1$ for some $S \in [0, T]$, and
$$\phi_t(x) = \left[e^{\gamma x} + \alpha\gamma(T-t)\right] \mathbf{1}_{t \geq S} + \exp\left[\gamma\alpha\left(e^{\beta(S-t)} - 1\right)/\beta\right] \mathbf{1}_{t<S}.$$

$t \mapsto \phi_t(x)$ is decreasing and $x \mapsto \phi_t(x)$ is increasing and continuous.

Return