Jyväskylä Summer School, 2019

Selected Topics in BSDEs Theory

Take-Home Exam : to be sent to philippe.briand@univ-smb.fr by 2019-09-15

1. B is a one-dimensional Brownian motion ;
2. ξ is a **nonnegative** square integrable random variable, $\mathcal{F}_T^B$–measurable ;
3. $f : \mathbf{R} \longrightarrow \mathbf{R}$ is defined by $f(y) = -y^2$.

1. For $n \geq 1$, let $\xi^n = \min(\xi, n)$ and $f_n(y) = -\min(y^+, n)^2$ with $y^+ = \max(y, 0)$.
   (a) Explain briefly why the BSDE
   $$Y_t^n = \xi^n + \int_t^T f_n(Y_s^n)\, ds - \int_t^T Z_s^n\, dB_s, \quad 0 \leq t \leq T,$$
   has a unique solution in $\mathcal{B}^2$. This solution is denoted (Y^n, Z^n).
   (b) Solve the BSDE
   $$U_t = 0 + \int_t^T f_n(U_s)\, ds - \int_t^T V_s\, dB_s, \quad 0 \leq t \leq T.$$
   `Hint: The simplest is often the best! Look for a deterministic solution.`
   (c) Use comparison theorem to prove that $0 \leq Y_t^n \leq n$. `Hint: for the upper bound, remark that` $f_n \leq 0$ `and solve the BSDE with` n `as terminal condition and` 0 `as generator.`
   (d) Deduce from the previous result that (Y^n, Z^n) solves the BSDE
   $$Y_t^n = \xi^n + \int_t^T f(Y_s^n)\, ds - \int_t^T Z_s^n\, dB_s, \quad 0 \leq t \leq T.$$
2. Use Itô's formula and the fact that f is nonincreasing on $\mathbf{R}_+$ to see that, for any n and m and for $0 \leq t \leq T$,
   $$|Y_t^m - Y_t^n|^2 + \int_t^T |Z_r^m - Z_r^n|^2\, dr \leq |\xi^m - \xi^n|^2 - 2\int_t^T (Y_r^m - Y_r^n)\,(Z_r^m - Z_r^n)\, dB_s.$$
   `Hint:` $d(Y_t^m - Y_t^n) = -\left(f(Y_t^m) - f(Y_t^n)\right) dt + (Z_t^m - Z_t^n)\, dB_t$
3. Prove that the sequence (Y^n, Z^n) is converging in $\mathcal{B}^2$ toward (Y, Z).
4. (Bonus) Prove that (Y, Z) solves the BSDE
   $$Y_t = \xi + \int_t^T f(Y_s)\, ds - \int_t^T Z_s\, dB_s, \quad 0 \leq t \leq T.$$
5. Do you think that the condition $\xi \geq 0$ is decorative ?