The uniform rugosity effect

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Abstract

Relying on the effect of microscopic asperities, one can mathematically justify that viscous fluids adhere completely on the boundary of an impermeable domain. The rugosity effect accounts asymptotically for the transformation of complete slip boundary conditions on a rough surface in total adherence boundary conditions, as the amplitude of the rugosities vanishes. The decreasing rate (average velocity divided by the amplitude of the rugosities) computed on close flat layers is definitely influenced by the geometry. Recent results prove that this ratio has a uniform upper bound for certain geometries, like periodical and "almost Lipschitz" boundaries. The purpose of this paper is to prove that such a result holds for arbitrary (non-periodical) crystalline boundaries and general (non-smooth) periodical boundaries.

Keywords: rugosity effect, non-periodic boundaries, uniform decreasing of the velocity

1 Introduction

It is commonly accepted that viscous fluids adhere to rough surfaces. One mathematical explanation is based on the so called rugosity effect. For the Navier-Stokes equation, complete slip boundary conditions on a rough surface transform asymptotically in no-slip conditions as the amplitutde of the rugosities vanishes, provided that the energy of the solutions is uniformly bounded and the there is "enough roughness" of the oscillating boundaries. We refer the reader to the pioneering paper of Casado-Díaz, Fernández-Cara and Simon where this result is proved in the case of periodic, self-similar C^2- boundaries [9]. In their proof, the authors prove implicitly that the rugosity effect has a uniform character, in the sense that the decreasing rate of the average velocity on a flat layer close to the boundary can be estimated uniformly.

Recent results obtained in [5, 8, 7] (see also [11, 12]) give a quite complete understanding of the rugosity effect for arbitrary boundaries. For equi-Lipschitz domains, if Ω_ε is a geometric perturbation of Ω (in the sense that the Hausdorff distance vanishes, $d_H(\Omega_\varepsilon, \Omega) \to 0$), the solutions $\mathbf{u}_\varepsilon$ of a Stokes equation with complete slip boundary conditions in Ω_ε converge to the solution $\mathbf{u}$ of the same Stokes equation complemented by the so called friction-driven boundary conditions (see [8]): there exists a suitable trio $\{\mu, A, V\}$ such that

- μ *is a capacitary measure* concentrated on $\partial\Omega$
- $\{V(x)\}_{x\in\Gamma}$ is a family of *vector subspaces in* $\mathbb{R}^{N-1}$
- A is a *positive symmetric matrix function* A defined on $\partial\Omega$

and formally the boundary conditions read

$$\left\{ \begin{array}{rcl} \mathbf{u}(x) & \in & V(x) \text{ for q.e. } x \in \partial\Omega \\ \Big[\mathbf{D}[\mathbf{u}]\cdot\mathbf{n} + \mu A\mathbf{u}\Big]\cdot\mathbf{v} & = & 0 \text{ for any } \mathbf{v}\in V(x),\ x\in\Gamma \end{array} \right. \tag{1}$$

The rugosity effect holds provided that above $V(x) = \{0\}$ for a.e. $x \in \partial\Omega$, which implies that $\mathbf{u} = 0$ on $\partial\Omega$. However, assuming that the energies of the solutions $\|u_\varepsilon\|_{\mathbf{H}^1(\Omega_\varepsilon)}$ are uniformly bounded, the estimate of the average velocity decay rate is more delicate and relies on finding on an upper bound for

$$\limsup_{\varepsilon\to 0} \frac{\displaystyle\int_{\partial\Omega_\varepsilon} |\mathbf{u}_\varepsilon|^2 dS}{d_H(\Omega_\varepsilon, \Omega)}. \tag{2}$$

For general situations one can not expect this number to be finite. If finite, the rugosity effect is said to be uniform (we refer to [6] for applications). All previous results in the literature providing an upper bound for (2) were given for periodic self-similar boundaries in the framework of [9], under more or less regularity assumptions. We refer to the recent paper of Březina [4], where the C^2-regularity is weakened to *almost Lipschitz* boundaries (see the precise sense in [4]). Nevertheless, at least intuitively, less regularity of the boundary should enforce the physical rugosity effect but, of course, new technical difficulties arise.

The main purpose of the paper is to analyse general geometric perturbations of flat domains for which an upper bound can be found in (2). On the one hand, we remove any smoothness hypothesis and prove that the rugosity effect is uniform for general, continuous, periodic self-similar boundaries which are not riblets. This first result is only a technical improvement of previous results of [9, 6, 4] and involves a weak interpretation of the riblets associated to continuous boundaries (gradientless formulation). In a second step, we remove the periodicity assumption and prove that for arbitrary crystalline boundaries the rugosity effect is again uniform. This second result relies on a fine use of the Young measures to understand second order pointwise oscillations of the boundaries, i.e. the oscillations of the normal fields of *locally rescaled* domains. Uniform behaviours are in general difficult to capture for non-periodic structures. In our case, this is possible due to the very specific crystalline structure. We also refer to the paper [3] for a different result involving uniformity in a random geometric framework.

2 General settings and main results

Throughout the paper, we fix the dimension of the space $N = 3$. For simplicity, and without losing generality, we shall assume that Ω is the cube $(0,1)^2 \times (-1,0)$. Let us denote

$\varphi_\varepsilon : (0,1)^2 \to [0,1]$ a family of lower semi-continuous functions such that $(\varphi_\varepsilon)_\varepsilon$ converges uniformly to zero as $\varepsilon \to 0$. We introduce the geometric perturbations of Ω,

$$\Omega_\varepsilon = \left\{(x_1, x_2, x_3) \mid (x_1, x_2) \in (0,1)^2,\ -1 < x_3 < \varphi_\varepsilon(x_1, x_2)\right\}, \tag{3}$$

which are open sets approaching in a certain geometric sense Ω, as $\varepsilon \to 0$. In this case, it is convenient to replace in (2) the Hausdorff distance with the amplitude of the rugosities $\|\varphi_\varepsilon\|_\infty$. Note that discontinuity points on φ_ε are *a priori* admissible and correspond to severe "roughness points" of the boundary. Define $D = (0,1)^2 \times (-1,2)$ so that

$$\Omega_\varepsilon \subset D \quad \forall \varepsilon > 0.$$

The non penetration condition. We denote $\mathbf{H}^1(\Omega_\varepsilon)$ the usual Sobolev space $H^1(\Omega_\varepsilon, \mathbb{R}^3)$. For a function $\mathbf{v} \in \mathbf{H}^1(\Omega_\varepsilon)$, the non-penetration condition $\mathbf{v} \cdot \mathbf{n}_\varepsilon = 0$ on

$$\Gamma_\varepsilon = \{x \in \partial\Omega_\varepsilon : x_3 \geqslant 0\},$$

has to be understood in a weak sense, as soon as the normal vector field $\mathbf{n}_\varepsilon$ is not properly defined on Γ_ε. Precisely, if φ_ε is Lipschitz, the trace of $\mathbf{v}$ on Γ_ε and the normal vector field $\mathbf{n}_\varepsilon$ are pointwise defined a.e. If φ_ε is only continuous, we say that $\mathbf{v} \in \mathbf{H}^1(\Omega_\varepsilon)$ satisfies the *non-penetration condition* provided that

$$\forall \psi \in C_c^1(D) \qquad \int_{\Omega_\varepsilon} [(\operatorname{div} \mathbf{v})\psi + \mathbf{v} \cdot \nabla\psi] \mathrm{d}x = 0. \tag{4}$$

If φ_ε is lower semi-continuous, the above form of the non-penetration condition is incomplete, and special attention has to be given to discontinuity points. The following form of the non-penetration condition turns out to be equivalent to (4) as soon as the boundary is represented by a continuous graph, but captures also information at discontinuity points:

$$\forall \psi \in H^1(\Omega_\varepsilon) \qquad \int_{\Omega_\varepsilon} [(\operatorname{div} \mathbf{v})\psi + \mathbf{v} \cdot \nabla\psi] \mathrm{d}x = 0. \tag{5}$$

Rugosity effect and uniform decay rates. The rugosity effect reads: let $\mathbf{v}_\varepsilon \in \mathbf{H}^1(\Omega_\varepsilon)$ satisfy the non-penetration condition on Γ_ε, such that $(1_{\Omega_\varepsilon}\mathbf{v}_\varepsilon, 1_{\Omega_\varepsilon} \cdot \nabla\mathbf{v}_\varepsilon)$ converge weakly in $L^2(D, \mathbb{R}^{12})$ to $(1_\Omega \mathbf{v}, 1_\Omega \cdot \nabla\mathbf{v})$. Then $\mathbf{v} = 0$ on $\Gamma = (0,1)^2 \times \{0\}$.

The rugosity effect above is said to be *uniform* if the following estimate holds:

$$\exists C > 0, \exists \varepsilon_0 > 0, \forall \varepsilon_0 > \varepsilon > 0, \forall \mathbf{v} \in \mathbf{H}^1(\Omega_\varepsilon), \mathbf{v} \cdot \mathbf{n}_\varepsilon = 0 \text{ on } \Gamma_\varepsilon \tag{6}$$

$$\int_\Gamma |\mathbf{v}|^2 \mathrm{d}S \leqslant C\|\varphi_\varepsilon\|_\infty \int_{\Omega_\varepsilon} |\nabla\mathbf{v}|^2 \mathrm{d}x. \tag{7}$$

Notice that in this inequality, one does not require $\mathbf{v}$ to vanish on some part of the boundary. As a consequence of the continuity of the trace operator, estimate (7) proves that the rugosity effect holds, when $\varepsilon \to 0$.

Provided that φ_ε are equi-Lipschitz, one can replace in (6)-(7) the sum on Γ by the sum on Γ_ε

$$\int_{\Gamma_\varepsilon} |\mathbf{v}|^2 \mathrm{d}S \leqslant C' \|\varphi_\varepsilon\|_\infty \int_{\Omega_\varepsilon} |\nabla \mathbf{v}|^2 \mathrm{d}x, \tag{8}$$

which provides an upper bound in (2). If φ_ε were not smooth, the trace on Γ_ε is not defined. In this case, one can expect only estimates of type (7).

Main results of the paper. We first consider periodic structures generated by

$$\varphi_\varepsilon(x_1, x_2) = \varepsilon\varphi(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}), \tag{9}$$

where φ is a continous function on the two-dimensional torus in $\mathcal{T}^2 = [0,1]^2|_{\{0,1\}}$ extended by periodicity in $\mathbb{R}^2$. We give a characterization of the continuous functions φ such that (7) holds, removing the "almost Lipschitz" hypothesis considered in [4]. Loosely speaking, non-smooth boundaries should provide a stronger rugosity effect than the smooth ones! For technical reasons we restrict our study only to continuous boundaries but in the final section we shall discuss briefly boundaries with singularities generated by lower-semicontinuous functions φ.

We say φ is a riblet if there exists $(c_1, c_2) \in \mathbb{R}^2 \setminus \{0\}$ such that for every $(x_1, x_2) \in \mathbb{R}^2$ and every $h \in \mathbb{R}$,

$$\varphi(x_1 + hc_1, x_2 + hc_2) = \varphi(x_1, x_2). \tag{10}$$

If φ were differentiable, this would correspond to $\nabla\varphi \cdot (c_1, c_2) = 0$.

Theorem 2.1 (Characterization of the uniform rugosity effect) *Assume that φ is a continuous strictly positive function on $\mathcal{T}^2$. The following statements are equivalent:*

(i) there exist $k > 0$ and $\varepsilon_0 > 0$ such that for every $\varepsilon_0 > \varepsilon > 0$ and every $\mathbf{v} \in \mathbf{H}^1(\Omega_\varepsilon)$ satisfying the non-penetration condition on Γ_ε (in its weak integral form), the following inequality holds:

$$\int_\Gamma |\mathbf{v}|^2 \mathrm{d}S \leqslant \varepsilon k \|\nabla \mathbf{v}\|^2_{L^2(\Omega_\varepsilon)}, \tag{11}$$

(ii) φ is not a riblet.

In the case of lower-semicontinuous boundaries the characterization Theorem 2.1 can be rephrased, but provides a less clear geometric criterion for the uniform rugosity effect.

The second result of the paper is concerned with arbitrary crystalline boundaries (the periodicity assumption is removed). We prove that estimate (7) holds under a mild non-degeneracy assumption, similar to the one introduced in [5]. Let us consider a finite set $K \subset \mathbb{R}^2$ which satisfies the non-degeneracy assumption

$$\forall y_1, y_2 \in K, \quad 0 \notin [y_1, y_2]. \tag{12}$$

This condition simply avoids the creation of riblets on a crystalline structure. A function $\varphi \in W^{1,\infty}((0,1)^2)$ is admissible provided that

$$\text{for a.e. } y \in (0,1)^2 \quad \varphi(y) \in [0,1] \text{ and } \nabla\varphi(y) \in K.$$

For every admissible function φ we define the crystalline boundary

$$\Gamma_\varphi = \left\{(x_1, x_2, \varphi(x_1, x_2)) \mid (x_1, x_2) \in (0,1)^2\right\},$$

and the corresponding domain

$$\Omega_\varphi = \left\{(x_1, x_2, x_3) \mid (x_1, x_2) \in (0,1)^2,\ -1 < x_3 < \varphi(x_1, x_2)\right\}.$$

Theorem 2.2 (Crystalline boundaries) *There exist $k > 0$ and $\varepsilon_0 > 0$ such that inequality*

$$\int_\Gamma |\mathbf{v}|^2 \mathrm{d}S + \int_{\Gamma_\varphi} |\mathbf{v}|^2 \mathrm{d}S \leqslant k\|\varphi\|_\infty \|\nabla \mathbf{v}\|^2_{L^2(\Omega_\varphi)} \tag{13}$$

holds for every admissible function φ verifying $\varepsilon_0 > \|\varphi\|_\infty > 0$, and every $\mathbf{v} \in \mathbf{H}^1(\Omega_\varphi)$ satisfying the non-penetration condition on Γ_φ.

From a technical point of view, the uniformity of the rugosity effect is related to the analysis of the spectral abscissa of an elliptic operator in a family of infinite domains with "wildly" moving boundaries. The rugosity (Young) measures introduced in [5] provide a useful tool for dealing with the crystalline case. For our purpose, their use has to be refined in order to understand the local oscillations of the rescaled boundaries.

All results of the paper are given in $\mathbf{H}^1$ but they can be extended easily to $\mathbf{W}^{1,p}$ spaces. For instance, inequality (13) becomes

$$\|\mathbf{v}\|_{L^q(\Gamma)} + \|\mathbf{v}\|_{L^q(\Gamma_\varphi)} \leqslant k\|\varphi\|^\alpha_\infty \|\nabla \mathbf{v}\|_{L^p(\Omega_\varphi)}, \tag{14}$$

which holds for every for every admissible function $\varphi \in W^{1,\infty}((0,1)^2)$ such that $\varepsilon_0 > \|\varphi\|_\infty > 0$ and every $\mathbf{v} \in W^{1,p}(\Omega_\varphi, \mathbb{R}^3)$ satisfying the non-penetration condition on Γ_φ, where

$$1 < p < 3,\ 1 < q < \frac{2}{3-p},\ \alpha = 1 - \frac{3}{p} + \frac{2}{q}.$$

3 Pointwise roughness and Young measures

A fine tool allowing to understand the rugosity effect is given by the general theory of Young measures (see for instance [10, 13]). For the sake of clarity, we recall the fundamental theorem of Young measures.

Theorem 3.1 (Fundamental theorem of Young measures) *Let $U \subset \mathbb{R}^n$ be an open set and $K \subset \mathbb{R}^m$ be a compact set. Consider a sequence $(f_k)_{k\in\mathbb{N}} \subset L^\infty(U, K)$. There exists a subsequence (f_{k_j}) and for a.e. $y \in U$ a Borel probability measure $\mathcal{R}_y$ on $\mathbb{R}^m$ such that for each $F \in C(\mathbb{R}^m)$ we have*

$$F(f_{k_j}) \rightharpoonup \overline{F} \quad \textit{weakly-}* L^\infty(U),$$

where

$$\overline{F}(y) = \int_{\mathbb{R}^m} F(Z)\mathrm{d}\mathcal{R}_y(Z) \quad \textit{a.e. } y \in U.$$

We call $\{\mathcal{R}_y\}_{y\in U}$ a family of Young measures *associated with the subsequence (f_{k_j}).*

Remark 3.2 We recall that the family of Young measures $\{\mathcal{R}_y\}_{y\in U}$ may be not unique for a given sequence $(f_k)_{k\in\mathbb{N}}$ and satisfies the following property:

$$\mathrm{spt}(\mathcal{R}_y) \subset K \quad \text{a.e. } y \in U.$$

In particular, for the crystalline framework when K is finite, the support of the measures $\mathcal{R}_y$ is discrete.

Remark 3.3 Let $(\varphi_k)_k$ be a bounded sequence in $W^{1,\infty}((0,1)^2,[0,1])$. We assume, up to renaming the indices, that there exists $\varphi \in W^{1,\infty}((0,1)^2,[0,1])$ such that

$$\varphi_k \to \varphi \quad \text{uniformly on } (0,1)^2, \quad \nabla\varphi_k \rightharpoonup \nabla\varphi \quad \text{weakly-}* \ L^\infty((0,1)^2).$$

We denote by $\{\mathcal{R}_y\}_{y\in(0,1)^2}$ a family of Young measures associated with a subsequence of $(\nabla\varphi_k)_{k\in\mathbb{N}}$. Then

$$\int_{\mathbb{R}^2} Z \mathrm{d}\mathcal{R}_y(Z) = \nabla\varphi(y) \quad \text{a.e. } y \in (0,1)^2.$$

This is a direct consequence of the fundamental theorem of Young measures, taking for F the identity function.

Lemma 3.4 *With the notations of Remark 3.3, let us denote*

$$\Omega_k = \left\{(x_1,x_2,x_3) \mid (x_1,x_2) \in (0,1)^2,\ -1 < x_3 < \varphi_k(x_1,x_2)\right\},$$

and consider $\mathbf{u}_k \in \mathbf{H}^1(\Omega_k)$ *satisfying the non-penetration condition*

$$\mathbf{u}_k \cdot \mathbf{n}_k = 0 \quad \text{on } \Gamma_k = \{x \in \partial\Omega_k : x_3 \geqslant 0\}$$

converging weakly in $\mathbf{H}^1(D)$ *to* $\mathbf{c} = (c_1,c_2,c_3) \in \mathbb{R}^3$.

Then for every $g \in C(\mathbb{R}^2,\mathbb{R})$ *we have*

$$(c_1,c_2)\cdot\int_{\mathbb{R}^2} g(Z) Z \mathrm{d}\mathcal{R}_y(Z) = c_3 \int_{\mathbb{R}^2} g(Z)\mathrm{d}\mathcal{R}_y(Z) \quad \text{a.e. } y \in (0,1)^2.$$

Proof Let $\xi \in C(\mathbb{R}^2,\mathbb{R})$. We prove that for every $\psi \in C_c^\infty((0,1)^2,\mathbb{R})$,

$$\int_{(0,1)^2} \psi(y)\left[(c_1,c_2)\cdot\int_{\mathbb{R}^2}\sqrt{1+Z^2}\xi(Z)Z\mathrm{d}\mathcal{R}_y(Z) - c_3\int_{\mathbb{R}^2}\sqrt{1+Z^2}\xi(Z)\mathrm{d}\mathcal{R}_y(Z)\right]\mathrm{d}y = 0. \tag{15}$$

Taking into account that $\mathbf{n}_k(y,\varphi_k(y))$ is co-linear with $(-\nabla\varphi_k(y),1)$, the non penetration condition on Γ_k yields

$$\int_{\Gamma_k}\psi(y)\xi(\nabla\varphi_k(y))(-\nabla\varphi_k(y),1)\cdot\mathbf{u}_k(y,\varphi_k(y))\mathrm{d}S = 0.$$

Performing a change of variables from Γ_k to Γ, we get

$$\int_{(0,1)^2}\psi(y)\xi(\nabla\varphi_k(y))\nabla\varphi_k(y)\cdot(u_k^1(y,\varphi_k(y)),u_k^2(y,\varphi_k(y)))\sqrt{1+\|\nabla\varphi_k(y)\|^2}\mathrm{d}y \tag{16}$$
$$= \int_{(0,1)^2}\psi(y)\xi(\nabla\varphi_k(y))u_k^3(y,\varphi_k(y))\sqrt{1+\|\nabla\varphi_k(y)\|^2}\mathrm{d}y$$

Without loss of generality, we may assume that u_k belongs to $\mathbf{H}^1(D) \cap C^\infty(D, \mathbb{R}^3)$. Then integrating on vertical lines and using the Cauchy-Schwartz inequality, we get

$$\int_{(0,1)^2} |\mathbf{u}_k(y, \varphi_k(y)) - \mathbf{u}_k(y, \varphi(y))| \,\mathrm{d}y \leqslant \|\varphi_k - \varphi\|_\infty \|\nabla \mathbf{u}_k\|_{L^2(D)} \to 0 \qquad \text{as } k \to +\infty.$$

Passing to the limit in (16), we get (15). Since ψ is arbitrary, we finish the proof choosing $\xi(Z) = \frac{g(Z)}{\sqrt{1+Z^2}}$. □

Lemma 3.5 *Let $(\varphi_k)_k$ be a bounded sequence in $W^{1,\infty}((0,1)^2)$. Let $(\mathcal{R}_y)_{y \in (0,1)^2}$ be a family of Young measures associated to a subsequence of $(\nabla \varphi_k)_k$, and $\mathbf{N} \in K$. If*

$$\liminf_{k \to +\infty} \left| \left\{ y \in (0,1)^2 \mid \nabla \varphi_k(y) = \mathbf{N} \right\} \right| > 0. \tag{17}$$

then

$$| \left\{ y \in (0,1)^2 \mid \mathbf{N} \in spt(\mathcal{R}_y) \right\} | > 0.$$

Proof Assume for contradiction that

$$| \left\{ y \in (0,1)^2 \mid \mathbf{N} \in \mathrm{spt}(\mathcal{R}_y) \right\} | = 0.$$

Since $N \notin \mathrm{spt}(\mathcal{R}_y)$ for a.e. $y \in (0,1)^2$, there exists $r > 0$ such that

$$\mathcal{R}_y(B(\mathbf{N}, r)) = 0 \qquad \text{for a.e. } y \in G.$$

Let $\psi \in C_c^\infty(\mathbb{R}^2)$ such that $\mathrm{spt}(\psi) \subset B(\mathbf{N}, r)$,

$$\psi(\mathbf{N}) = 1, \quad \psi(n) = 0 \quad \forall n \in K \setminus \{N\}.$$

By definition of the Young measures, up to a subsequence,

$$\psi(\nabla \varphi_k) \rightharpoonup 0 \quad \text{weakly} * \text{in} \quad L^\infty((0,1)^2),$$

which yields

$$\int_{(0,1)^2} \psi(\nabla \varphi_k) \mathrm{d}x \to 0,$$

that is,

$$\lim_{k \to +\infty} \left| \left\{ y \in (0,1)^2 \mid \nabla \tilde{\varphi}_\varepsilon(y) = \mathbf{N} \right\} \right| = 0$$

in contradiction with hypothesis (17).

□

4 Proof Theorem 2.1

In order to prove Theorem 2.1, we start with two general results involving lower semicontinuous boundaries.

First, we recall the following Poincaré Lemma, which does not require smoothness over all the boundary. The proof of this result is standard, summing on vertical lines.

Lemma 4.1 (Poincaré Lemma for l.s.c. boundaries) *Let $\varphi : (0,1)^2 \to [0,1]$ be lower semicontinuous. Then there exists a constant $K > 0$ depending only on φ, such that*

$$\int_{\Omega_\varphi} |\mathbf{u}|^2 \mathrm{d}x \leqslant K \left(\int_{\Omega_\varphi} |\nabla \mathbf{u}|^2 \mathrm{d}x + \int_\Gamma |\mathbf{u}|^2 \ \mathrm{d}S \right) \qquad \forall \mathbf{u} \in \mathbf{H}^1(\Omega_\varphi) \tag{18}$$

wher Ω_φ is the open set defined by

$$\Omega_\varphi = \left\{ (x_1, x_2, x_3) \mid (x_1, x_2) \in (0,1)^2, \ -1 < x_3 < \varphi(x_1, x_2) \right\}.$$

Lemma 4.2 *Assume that $\varphi : \mathcal{T}^2 \to [0,1]$ is a continuous function and define φ_ε by (9). The following statements are equivalent:*

(i) the only constant $\mathbf{c} \in \mathbb{R}^3$ satisfying

$$\forall \psi \in C_c^1(D) \qquad \int_{\Omega_1} \mathbf{c} \cdot \nabla \psi \mathrm{d}x = 0 \tag{19}$$

is $\mathbf{c} = 0$;

(ii) there exists $k, \varepsilon_0 > 0$ such that for every $\varepsilon_0 > \varepsilon > 0$ and every $\mathbf{v} \in \mathbf{H}^1(\Omega_\varepsilon)$ satisfying the weak form of the non-penetration condition (4), the following inequality holds:

$$\int_\Gamma |\mathbf{v}|^2 \mathrm{d}S \leqslant \varepsilon k \|\nabla \mathbf{v}\|^2_{L^2(\Omega_\varepsilon)}. \tag{20}$$

Proof (i)⇒(ii) Considering a subdivision of $(0,1)^2$ in squares $(S_i^\varepsilon)_i$ of size $\varepsilon \times \varepsilon$ parallel to the axes, it is enough to prove the following:

$$\int_{S_i^\varepsilon \times \{0\}} |\mathbf{v}|^2 \mathrm{d}S \leqslant \varepsilon k \|\nabla \mathbf{v}\|^2_{L^2(V_i^\varepsilon)} \tag{21}$$

where

$$V_i^\varepsilon \ = \ \{(x_1, x_2, x_3) \mid (x_1, x_2) \in S_i^\varepsilon, \ -\varepsilon < x_3 < \varphi_\varepsilon(x_1, x_2)\}.$$

Performing a translation if necessary, we may assume that the square S_i^ε coincides with $(0,\varepsilon) \times (0,\varepsilon)$. Using the scaling function $H_\varepsilon : (x_1, x_2, x_3) \in \mathbb{R}^3 \to (\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}) \in \mathbb{R}^3$, (21) is equivalent to

$$\int_\Gamma |\mathbf{v}|^2 \mathrm{d}S \leqslant k \|\nabla \mathbf{v}\|^2_{L^2(\Omega_1)}, \tag{22}$$

where $\Omega_1 = H_\varepsilon(V_i^\varepsilon)$ is not depending on ε.

Assume for contradiction that there exists a sequence $\mathbf{v}_k \in \mathbf{H}^1(\Omega_1)$ satisfying the non-penetration condition (4) and such that $\int_\Gamma |\mathbf{v}_k|^2 = 1$ and $\|\nabla \mathbf{v}_k\|_{L^2(\Omega_1)} \to 0$. From Lemma 4.1 we get that $\|\mathbf{v}_k\|_{\mathbf{H}^1(\Omega_1)}$ is bounded, so there exists $\mathbf{v}^*$ in $\mathbf{H}^1(\Omega_1)$ such that up to a subsequence,

$$\mathbf{v}_k \rightharpoonup \mathbf{v}^* \qquad \text{weakly in } \mathbf{H}^1(\Omega_1).$$

As a result, $\|\nabla v^*\|_{L^2(\Omega_1)} = 0$ so there exists $\mathbf{c} \in \mathbb{R}^3$ such that

$$\mathbf{v}^* = \mathbf{c} \qquad \text{a.e. in } \Omega_1.$$

Using the compactness of the trace operator from $H^1(\Omega_1)$ to $L^2(\Gamma)$, we get

$$\int_\Gamma |\mathbf{v}^*|^2 dS = 1$$

which implies that $\mathbf{c} \neq 0$. This contradicts hypothesis (i) passing to the limit the non-penetration condition (4) for $\mathbf{v}_k$.

(ii)⇒(i) Assume for contradiction that there exists $\mathbf{c} \in \mathbb{R}^3 \setminus \{0\}$ such that (19) holds. Then the function $v \in \mathbf{H}^1(\Omega_1)$ defined by $\mathbf{v} = \mathbf{c}$ satisfies the non-penetration condition in the weak form (4), but contradicts inequality (11) for every $k > 0$.

□

Proof of Theorem 2.1. **(i)⇒(ii)** Assume for contradiction that φ is a riblet in the sense of (10). From Lemma 4.2, it is enough to show that a constant vector field of the form $\mathbf{v} = (c_1, c_2, 0)$, with $(c_1, c_2) \in \mathbb{R}^2 \setminus \{0\}$, satisfies (19) (or (4)). This would readily imply that hypothesis (i) does not hold.

Since φ is a riblet, one may consider a sequence of smooth functions θ_n approaching uniformly φ on $\mathbb{R} \times \{0\} \times \{0\}$ and construct a riblet from θ_n in the direction $(c_1, c_2, 0)$. Then, the weak form of the non-penetration condition can be written on the domain defined by the smooth riblet associated to θ_n

$$\Theta_n = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in (0,1)^2,\ -1 < x_3 < \theta_n(x_1, x_2)\}.$$

Since $1_{\Theta_n} \to 1_{\Omega_1}$ in $L^1(\mathbb{R}^3)$, the passage to the limit of the non-penetration condition concludes the proof.

(ii)⇒(i) We need the following result concerning the uniqueness of riblet structures associated to a non constant, periodic and continuous function.

Lemma 4.3 *Let $\varphi \in C(\mathcal{T}^2)$ be a non-constant function, extended on $\mathbb{R}^2$ by periodicity. Assume there exist (c_1, c_2, c_3), (c_1^*, c_2^*, c_3) in $\mathbb{R}^3 \setminus \{0\}$ such that for every $(x_1, x_2) \in (0,1)^2$ and every $h \in \mathbb{R}$,*

$$\begin{aligned} \varphi(x_1 + hc_1, x_2 + hc_2) &= \varphi(x_1, x_2) + hc_3, \\ \varphi(x_1 + hc_1^*, x_2 + hc_2^*) &= \varphi(x_1, x_2) + hc_3. \end{aligned} \tag{23}$$

Then either $c_3 = 0$ and (c_1, c_2), (c_1^*, c_2^*) are co-linear, or $(c_1, c_2) = (c_1^*, c_2^*)$.

Proof Assume that $(c_1, c_2) = \alpha(c_1^*, c_2^*)$, for some $\alpha \in \mathbb{R}$. Then, relations (23) give directly $hc_3 = h\alpha c_3$ for every $h \in \mathbb{R}$. Consequently, either $c_3 = 0$, or $\alpha = 1$.

Assume now that (c_1, c_2) and (c_1^*, c_2^*) are not co-linear. We will prove that φ is constant, in contradiction with our hypothesis. Indeed, let $(x_1, x_2) \in (0,1)^2$. For every $\alpha, \beta > 0$ we define

$$(d_1, d_2) = \alpha(c_1, c_2) + \beta(c_1^*, c_2^*). \tag{24}$$

By formula (23) we get

$$\varphi(x_1 + d_1, x_2 + d_2) = \varphi(x_1, x_2) + (\alpha + \beta)c_3. \tag{25}$$

Since φ is continuous and periodic, and since (d_1, d_2) is arbitrary, (25) implies that $c_3 = 0$, i.e. φ is constant. □

Proof of Theorem 2.1 (continuation). Assume for contradiction that condition (i) in Theorem 2.1 does not hold.

If φ is constant then (10) holds for every $(c_1, c_2) \in \mathbb{R}^2 \backslash \{0\}$. Assume that φ is not constant. From the proof of Lemma 4.2, there exists $\mathbf{c} = (c_1, c_2, c_3) \in \mathbb{R}^3 \backslash \{0\}$ such that (19) holds. We prove that for every $(x_1, x_2) \in (0,1)^2$ and every $h \in \mathbb{R}$ such that $(x_1 + hc_1, x_2 + hc_2) \in (0,1)^2$,

$$\varphi(x_1 + hc_1, x_2 + hc_2) = \varphi(x_1, x_2) + hc_3. \tag{26}$$

Let ρ be the standard mollifier in $\mathbb{R}^3$. For every $\varepsilon > 0$ we define ρ_ε by

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^3}\rho\left(\frac{x}{\varepsilon}\right) \qquad \forall x \in \mathbb{R}^3.$$

Let $\eta \in (0,1)$ be a given constant. We introduce the following open subset of D:

$$D_\eta = \{x \in D \mid \mathrm{dist}(x, \partial D) > \eta\}.$$

We will prove that for every $\varepsilon \in (0, \eta)$, every $x \in D_\eta$ and every $h \in \mathbb{R}$ such that $x + h\mathbf{c} \in D_\eta$, the following equality holds:

$$\mathbf{1}_{\Omega_1} * \rho_\varepsilon(x + h\mathbf{c}) = \mathbf{1}_{\Omega_1} * \rho_\varepsilon(x). \tag{27}$$

Let $\psi \in C_c^1(D)$ such that $\mathrm{supp}(\psi) \subset D_\eta$, and let $\varepsilon \in (0, \eta)$. Applying (19) to $\psi * \rho_\varepsilon \in C_c^1(D)$ we get

$$\int_D \mathbf{1}_{\Omega_1} \mathbf{c} \cdot \nabla(\psi * \rho_\varepsilon) \mathrm{d}x = 0$$

which yields

$$\int_D (\mathbf{1}_{\Omega_1} * \rho_\varepsilon) \mathbf{c} \cdot \nabla\psi \mathrm{d}x = 0.$$

Integrating by part we get

$$\int_D \frac{\partial(\mathbf{1}_{\Omega_1} * \rho_\varepsilon)}{\partial \mathbf{c}} \psi \mathrm{d}x = 0. \tag{28}$$

Since (28) holds for every $\psi \in C_c^1(D_\eta)$ and since $\mathbf{1}_{\Omega_1} * \rho_\varepsilon \in C^\infty(D_\eta)$, we obtain

$$\frac{\partial(\mathbf{1}_{\Omega_1} * \rho_\varepsilon)}{\partial \mathbf{c}}(x) = 0 \qquad \forall x \in D_\eta.$$

Consequently (27) is proved.

Let us denote Γ_1 the upper part of $\partial\Omega_1$, namely

$$\Gamma_1 = \left\{(x_1, x_2, x_3) \mid (x_1, x_2) \in (0,1)^2, x_3 = \varphi(x_1, x_2)\right\}.$$

We need the following result.

Lemma 4.4 *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function satisfying the following property:*

$$\exists b \in \mathbb{R},\ \forall x \in \mathbb{R},\ \forall y \in \mathbb{R} \quad \phi(x) > y \Longrightarrow \forall h \in \mathbb{R} \quad \phi(x+h) \geqslant y + hb. \tag{29}$$

Then $\phi(h) = \phi(0) + bh$ for every $h \in \mathbb{R}$.

Proof On the one hand, for every $\varepsilon > 0$, we have $\phi(0) > \phi(0) - \varepsilon$, so that

$$\forall h \in \mathbb{R},\ \phi(h) \geqslant \phi(0) - \varepsilon + hb.$$

Making $\varepsilon \to 0$, we get

$$\forall h \in \mathbb{R},\ \phi(h) \geqslant \phi(0) + hb.$$

Assume for contradiction that there exists $x_0 \in \mathbb{R}$ such that

$$\phi(h_0) > \phi(0) + h_0 b.$$

Then, using property (29), we get for a suitable $\varepsilon > 0$

$$\phi(h_0 + h) \geqslant \phi(0) + h_0 b + \varepsilon + hb.$$

Taking $h = -h_0$ we get a contradiction. □

We shall prove in the sequel that for every $x \in D$ and for every $h \in \mathbb{R}$ such that $x + hc \in D$, we have

$$x \in \Omega_1 \Rightarrow x + h\mathbf{c} \in \Omega_1 \cup \Gamma_1,$$

or equivalently that property (29) is satisfied by the restriction of φ on any line parallel to (c_1, c_2).

Let $x \in \Omega_1$ and $h \in \mathbb{R}$ such that $x + h\mathbf{c} \in D$. There exists $\eta > 0$ such that $x, x + h\mathbf{c} \in D_\eta$ so (27) holds for every $\varepsilon \in (0, \eta)$. Since φ is continuous, Ω_1 is open so there exists $\eta_1 > 0$ such that $B(x, \eta_1) \subset \Omega_1$. In particular, for every $\varepsilon \in (0, \min(\eta, \eta_1))$,

$$\mathbf{1}_{\Omega_1} * \rho_\varepsilon(x) = \mathbf{1}_{\Omega_1}(x) = 1.$$

To prove that $x + h\mathbf{c} \in \Omega_1 \cup \Gamma_1$ we argue by contradiction. Suppose that $x + h\mathbf{c} \in D \setminus (\Omega_1 \cup \Gamma_1)$. Since φ is continuous, $D \setminus (\Omega_1 \cup \Gamma_1)$ is open, so by the same argument as above we obtain that there exists $\eta_2 > 0$ such that

$$\mathbf{1}_{\Omega_1} * \rho_\varepsilon(x + h\mathbf{c}) = 0 \qquad \forall \varepsilon \in (0, \eta_2).$$

This is a contradiction with (27) and (4).

Applying Lemma 4.4 to the function

$$h \in \mathbb{R} \to \varphi(x_1 + hc_1, x_2 + hc_2)$$

we get (26).

In order to prove that $c_3 = 0$, we reproduce the previous argument for p periods of φ. Let us fix $p \in \mathbb{N}^*$. We define the following subsets of $\mathbb{R}^3$:

$$\begin{aligned} \Omega_1' &= \left\{(x_1, x_2, x_3) \mid (x_1, x_2) \in (0,p)^2, -1 < x_3 < \varphi(x_1, x_2)\right\}, \\ D' &= \left\{(x_1, x_2, x_3) \mid (x_1, x_2) \in (0,p)^2, -1 < x_3 < 2\right\}, \\ \Gamma' &= \left\{(x_1, x_2, x_3) \mid (x_1, x_2) \in (0,p)^2, x_3 = 0\right\}. \end{aligned}$$

As above, there exists $c' = (c_1', c_2', c_3')$ such that

$$\int_{\Omega_1'} c' \cdot \nabla \psi \mathrm{d}x = 0 \qquad \forall \psi \in C_c^1(D') \tag{30}$$

and

$$(c_1')^2 + (c_2')^2 + (c_3')^2 = 1. \tag{31}$$

We first prove that $c' = \pm c$. By (30) we get that for every $(x_1, x_2) \in (0,p)^2$ and every $h \in \mathbb{R}$ such that $(x_1 + hc_1', x_2 + hc_2') \in (0,p)^2$,

$$\varphi(x_1 + hc_1', x_2 + hc_2') = \varphi(x_1, x_2) + hc_3'. \tag{32}$$

Since $|c'| = |c|$ it is enough to prove that c' and c are colinear. Recall the following relations:

$$\varphi(x_1 + hc_1, x_2 + hc_2) = \varphi(x_1, x_2) + hc_3, \tag{33}$$

$$\varphi(x_1 + h^* c_1', x_2 + h^* c_2') = \varphi(x_1, x_2) + h^* c_3', \tag{34}$$

which hold for every $(x_1, x_2) \in (0,1)^2$ and every $h, h^* \in \mathbb{R}$ such that $(x_1 + hc_1, x_2 + hc_2), (x_1 + h^* c_1', x_2 + h^* c_2') \in (0,1)^2$.

If $c_3' = 0$ then immediately $c_3 = 0$. Assume that $c_3' \neq 0$. Setting $h^* = h\frac{c_3}{c_3'}$ in (34), we get that for every $(x_1, x_2) \in (0,1)^2$ and for h small enough,

$$\varphi(x_1 + h\frac{c_3}{c_3'} c_1', x_2 + h\frac{c_3}{c_3'} c_2') = \varphi(x_1, x_2) + hc_3. \tag{35}$$

Lemma 4.3 and relations (33)-(35) imply that $(\frac{c_3}{c_3'} c_1', \frac{c_3}{c_3'} c_2')$ and (c_1, c_2) are colinear, which yields

$$c_3 \left(c_1 c_2' - c_1' c_2\right) = 0.$$

If $c_3 \neq 0$ by (33)-(34) we get that $(c_1, c_2) \neq 0$ and $(c_1', c_2') \neq 0$, so there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that

$$(c_1, c_2) = \lambda(c_1', c_2').$$

Denoting $h^* = \lambda h$ in (34) and using (33) we get

$$c_3 = \lambda c_3'.$$

Thus $c = \lambda c'$, with $\lambda = \pm 1$.

Finally, for every $p \in \mathbb{N}^*$ relation (32) holds with the same constant vector $\mathbf{c}$. Since φ is bounded, we get that $c_3 = 0$ hence φ is a riblet. □

5 Proof of Theorem 2.2

Proof of Theorem 2.2. Since Γ_φ is a $W^{1,\infty}$- parametrization of Γ, with bounded norm independent on φ, it is enough to prove that there exists $\varepsilon_0 > 0$ such that

$$\int_\Gamma |\mathbf{v}|^2 \mathrm{d}S \leqslant k\|\varphi\|_\infty \|\nabla \mathbf{v}\|^2_{L^2(\Omega_\varphi)} \tag{36}$$

for every admissible function $\varphi \in W^{1,\infty}((0,1)^2)$ such that $\varepsilon_0 > \|\varphi\|_\infty > 0$ and every $\mathbf{v} \in \mathbf{H}^1(\Omega_\varphi)$ satisfying the non-penetration condition on Γ_φ.

We fix $M > 0$. The value of M depends only on K, and will be fixed later in the proof. Let $\Gamma_M = (0,M)^2 \times \{0\}$,

$$\tilde{\varphi}(x_1, x_2) = \frac{1}{\|\varphi\|_\infty} \varphi(\|\varphi\|_\infty x_1, \|\varphi\|_\infty x_2) \qquad \forall (x_1, x_2) \in (0, M)^2.$$

and

$$U_\varphi = \left\{(x_1, x_2, x_3) \mid (x_1, x_2) \in (0,M)^2,\ -1 < x_3 < \tilde{\varphi}(x_1, x_2)\right\}.$$

Using the same scaling method as in the proof of Theorem 2.1 (formally for $\varepsilon = \|\varphi\|_\infty$), it is enough to prove the existence of a positive constant k such that

$$\int_{\Gamma_M} |\mathbf{v}|^2 \mathrm{d}S \leqslant k \|\nabla \mathbf{v}\|^2_{L^2(U_\varphi)} \tag{37}$$

for every admissible function $\varphi \in W^{1,\infty}((0,1)^2)$ with $\varepsilon_0 > \|\varphi\|_\infty > 0$ and for every $\mathbf{v} \in \mathbf{H}^1(U_\varphi)$ satisfying the non-penetration condition

$$\int_{U_\varphi} [(\mathrm{div}\ \mathbf{v})\psi + \mathbf{v} \cdot \nabla \psi] \mathrm{d}x = 0 \qquad \forall \psi \in C^1_c((0,M)^2 \times (-1,2)). \tag{38}$$

Assume for contradiction that there exists a sequence of admissible functions φ_k and a sequence of functions $\mathbf{v}_k \in \mathbf{H}^1(U_{\varphi_k})$ satisfying (38) such that

$$\int_{\Gamma_M} |\mathbf{v}_k|^2 \mathrm{d}S = 1 \quad \text{and} \quad \|\nabla \mathbf{v}_k\|_{L^2(U_{\varphi_k})} \to 0. \tag{39}$$

Since $(\tilde{\varphi}_k)_k$ is uniformly bounded in $W^{1,\infty}((0,M)^2)$, there exists a Lipschitz-continuous function $\tilde{\varphi} \in C((0,M)^2)$ such that, up to a subsequence,

$$\tilde{\varphi}_k \to \tilde{\varphi}$$

uniformly on $(0,M)^2$ and weakly * in $W^{1,\infty}((0,M)^2)$.

We now define

$$U := \left\{(x_1, x_2, x_3) \mid (x_1, x_2) \in (0,1)^2, \quad -1 < x_3 < \tilde{\varphi}(x_1, x_2)\right\}.$$

Clearly

$$1_{U_{\varphi_k}} \to 1_U \quad \text{in } L^1(\mathbb{R}^3). \tag{40}$$

As a consequence of a lemma 4.1, there exists a constant $C > 0$ such that

$$\forall k, \quad \|\mathbf{v}_k\|_{\mathbf{H}^1(U_{\varphi_k})} \leqslant C.$$

Define $D_M = (0, M)^2 \times (-1, 2)$. Since the family $(U_{\varphi_k})_k$ is equi-Lipschitz, there exists a family of extension operators

$$P_k : \mathbf{H}^1(U_{\varphi_k}) \to \mathbf{H}^1(D_M)$$

and a constant $C > 0$ such that

$$\|P_k\| \leqslant C \qquad \forall k \in \mathbb{N}.$$

As a consequence, v_k (identified to $P_k(\mathbf{v}_k)$) is uniformly bounded in $\mathbf{H}^1(D_M)$, so there exists $\mathbf{v}^* \in \mathbf{H}^1(D_M)$ such that, up to a subsequence,

$$\mathbf{v}_k \rightharpoonup \mathbf{v}^* \qquad \text{weakly in } \mathbf{H}^1(D_M). \tag{41}$$

By (39), there exists $\mathbf{c} = (c_1, c_2, c_3) \in \mathbb{R}^3$ such that

$$\mathbf{v}^* = \mathbf{c} \qquad \text{a.e. in } U_\varphi.$$

Using the compactness of the trace operator from $H^1(U)$ to $L^2(\Gamma_M)$, we obtain that $M^2\|\mathbf{c}\|^2 = 1$, hence $c \neq 0$.

At this point, we get from Lemma 3.4 that for every $g \in C(\mathbb{R}^2, \mathbb{R})$

$$(c_1, c_2) \cdot \int_{\mathbb{R}^2} g(Z) Z \mathrm{d}\mathcal{R}_y(Z) = c_3 \int_{\mathbb{R}^2} g(Z) \mathrm{d}\mathcal{R}_y(Z) \quad \text{a.e. } y \in (0, M)^2.$$

On the other hand, from Remark 3.3 we get

$$\int_{\mathbb{R}^2} Z \mathrm{d}\mathcal{R}_y(Z) = \nabla\tilde{\varphi}(y) \quad \text{a.e. } y \in (0, M)^2,$$

while the non-penetration condition gives (in a purely geometric way, or as a consequence of Lemma 3.4 for $g \equiv 1$)

$$-(c_1, c_2) \cdot \nabla\tilde{\varphi}(y) + c_3 = 0, \quad \text{a.e. } y \in (0, M)^2. \tag{42}$$

Finally, for every $g \in C(\mathbb{R}^2, \mathbb{R})$

$$(c_1, c_2) \cdot \left[\int_{\mathbb{R}^2} g(Z) Z \mathrm{d}\mathcal{R}_y(Z) - \int_{\mathbb{R}^2} Z \mathrm{d}\mathcal{R}_y(Z) \int_{\mathbb{R}^2} g(Z) \mathrm{d}\mathcal{R}_y(Z)\right] = 0, \quad \text{a.e. } y \in (0, M)^2. \tag{43}$$

If for some $y \in (0, M)^2$ the support of $\mathcal{R}_y$ contains two independent vectors of K, say N_1 and N_2, then we get $(c_1, c_2) = 0$. This can be proved by taking test functions g of the form

$$g(Z) = a g_1(Z) + b g_2(Z),$$

where $a, b \in \mathbb{R}$ and g_i are continuous functions equal to 1 on N_i and vanishing on $K \setminus \{N_i\}$, $i = 1, 2$. Consequently, from (42) we also have $c_3 = 0$, in contradiction with $M^2\|c\|^2 = 1$.

We may assume that for a.e. $y \in (0, M)^2$, the support of $\mathcal{R}_y$ contains only co-linear vectors of K. Recall, that the non degeneracy hypothesis implies that all co-linear vectors in K have the same sense. Since $\nabla\tilde{\varphi}(y) = \int_{\mathbb{R}^2} Z \mathrm{d}\mathcal{R}_y(Z)$, there exists $\alpha_y \geqslant 1$ and $N_y \in K$ such that

$$\nabla\tilde{\varphi}(y) = \alpha_y N_y.$$

For simplicity we choose N_y as the shortest vector in each family of co-linear vectors in K.

If $c_3 = 0$ then from (42) all N_y coincide with some fixed vector $\mathbf{v} \in K$, otherwise (c_1, c_2) would vanish. Consequently,

$$\frac{\partial\tilde{\varphi}}{\partial\mathbf{v}}(y) \geqslant \|\mathbf{v}\|^2 \quad \text{a.e. } y \in (0, M)^2.$$

Since K is finite, we can fix $M > 0$ depending only on K to conclude that $\|\tilde{\varphi}\|_\infty > 1$ and get a contradiction.

Assuming now that $c_3 \neq 0$, we get from (42) that

$$-(c_1, c_2)\alpha_y N_y + 1 = 0 \quad \text{for a.e. } y \in (0, M)^2.$$

This implies that there exists a half plane containing almost every N_y. Hence, we can repeat the argument above with a suitable vector v contained in the half plane and such that $v \cdot N_y$ is bounded from below by a positive constant depending only on K. □

6 Further remarks on singular rough boundaries

Assume that the function $\varphi : \mathcal{T}^2 \to [0, 1]$ is only lower semicontinuous and $\varphi_\varepsilon(x_1, x_2) = \varepsilon\varphi(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon})$. Notice that their sub-graphs are open sets. Due to the lower semicontinous character, singular rugosities may occur around discontinuity points. In this case, the non-penetration condition has to be weakened, in the sense of (5). Around discontinuity points of φ, testing only with $C_c^1(D)$-functions in (4) is not enough to capture the non-penetration of a vertical obstacle.

If φ is lower semicontinuous, a suitable version of Lemma 4.2, in which the non-penetration condition (4) is replaced by (5) can be interpreted as a characterisation of the uniform rugosity effect. The main difficulty is to give a correct geometric interpretation of a riblet, similar to the one obtained in Theorem 2.1.

The following geometry, which is not a riblet in the sense of (10), does not produce a uniform rugosity effect:

$$\varphi : \mathcal{T}^2 \to \mathbb{R}, \quad \varphi(x_1, x_2) = 1 - x_1 \frac{1}{2} \delta_{\{x_2 = \frac{1}{2}\}}.$$

In fact, around the discontinuity points of φ, vertical surfaces of positive capacity have a crucial influence on the rugosity effect. This kind of situation falls out from (10), and can not be captured, from a technical point of view, by the proof of Theorem 2.1. Indeed, the convolution method used in Theorem 2.1 ignores sets of zero Lebesgue measure, even though their capacity is strictly positive.

If the uniform rugosity effect does not hold, one we can prove, using the convolution technique of Theorem 2.1, that if $x \in \Omega_1$, there exists $\varepsilon > 0$ and $\mathbf{c} = (c_1, c_2, 0) \neq 0$ such that for every $h \in \mathbb{R}$

$$|\Omega_1^c \cap B(x + h\mathbf{c}, \varepsilon)| = 0.$$

If for every $x_0 \in \mathcal{T}^2$, the value $\varphi(x_0)$ coincides with its approximate lower limit at x_0

$$\varphi(x_0) = \mathrm{ap} \lim_{y \to x_0} \varphi(y),$$

then Lemma 4.4 can be used and φ is a ribblet. So Theorem 2.1 can be extended for lower and approximate lower semicontinuous functions. An example of such a function is

$$\varphi : \mathcal{T}^2 \to \mathbb{R}, \quad \text{generated by} \quad \forall (x_1, x_2) \in [0,1)^2 \quad \varphi(x_1, x_2) = x_1.$$

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