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Is every toric variety an M-variety?

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Abstract. A complex algebraic variety X defined over the real numbers is called an M-variety if the sum of its Betti numbers (for homology with closed supports and coefficients in $\mathbb{Z}/2$) coincides with the corresponding sum for the real part of X. It has been known for a long time that any nonsingular complete toric variety is an M-variety. In this paper we consider whether this remains true for toric varieties that are singular or not complete, and we give a positive answer when the dimension of X is less than or equal to 3 or when X is complete with isolated singularities.

1. Introduction

Let X be a topological space equipped with a continuous involution σ, and let X^σ denote the fixed point set of σ. For simplicity we assume that X is a finite-dimensional cell complex and σ is a cellular involution. The Smith–Thom inequality asserts that the sum of the Betti numbers of X^σ does not exceed the corresponding sum for X,

$$\sum_k b_k\left(X^\sigma\right) \leq \sum_k b_k(X). \tag{1}$$

Here and in the rest of the paper we consider ordinary homology groups with coefficients in $\mathbb{Z}/2$, or in the non-compact case homology groups with closed supports, also known as Borel–Moore homology.

Consider the case where X is a complex algebraic variety defined over the real numbers. Thus X is equipped with an antiholomorphic involution, complex conjugation. The fixed point set of this involution is called the real part of X and will be denoted by $X(\mathbb{R})$. In contrast we shall often denote X by $X(\mathbb{C})$. The variety X is called an *M-variety* (*maximal variety*) if equality occurs in (1). In other words,

$$\sum_k b_k(X(\mathbb{R})) = \sum_k b_k(X(\mathbb{C})). \tag{2}$$

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M-varieties have attracted much attention in the study of topological properties of real algebraic varieties. One of the first results in this domain is due to Harnack in 1876 (see [23] for a survey). He proved an upper bound for the number of connected components of a real algebraic curve in the projective plane; his theorem is a special case of the Smith–Thom inequality. He also showed that the bound is sharp by constructing M-curves. In the same vein, Itenberg and Viro [15] have recently constructed M-hypersurfaces of degree d in n-dimensional projective space for all positive integers n and d.

One of the most familiar M-varieties is projective space; in this case the sums in (2) are equal to $n + 1$, where n is the dimension. Projective space is the simplest example of a nonsingular complete toric variety. Any toric variety is defined over the integers, hence over the reals. There are many ways to see that every nonsingular complete toric variety is an M-variety; see section 4. In fact, there is a degree-halving isomorphism of $\mathbb{Z}/2$-algebras $H^{2*}(X(\mathbb{C})) \to H^*(X(\mathbb{R}))$ and the mod 2 Betti numbers $b_{2k}(X(\mathbb{C})) = b_k(X(\mathbb{R}))$ admit a simple combinatorial description in terms of fan data.

Another example of a toric M-variety is the complex algebraic torus itself, whose real part is a real algebraic torus of the same rank. Any toric variety is the disjoint union of torus orbits, hence of M-varieties. In itself, the existence of such a stratification is not enough for a variety to be an M-variety (see for example Remark 3.1). Nevertheless, our study of numerous examples has led us to the following conjecture, which will be sharpened in section 8.

Conjecture 1.1. Every toric variety is an M-variety for homology with closed support.

Note that for singular or non-complete toric varieties there is no hope of getting a correspondence between individual Betti numbers: In general the odd homology groups of $X(\mathbb{C})$ do not vanish, and in particular the homology is not algebraic. In the nonsingular non-complete case this already happens for the one-dimensional torus; in the complete case such a phenomenon appears in dimension 2 [see Proposition 9.1 below, case (2)].

Systematic studies of the stratification of a toric variety by torus orbits and the associated spectral sequence have been made by Fischli [10] and Jordan [16]. By comparing this spectral sequence with another spectral sequence for the homology of the real points, we arrive at our main results, whose proofs will appear in section 8.

Theorem 1.1. *Let X be a (possibly singular) toric variety such that*

1. *X is complete and has isolated singularities, or*
2. *the dimension of X is not greater than* 3.

Then X is an M-variety for homology with closed supports.

In low dimensions straightforward spectral sequence calculations give the individual Betti numbers of $X(\mathbb{C})$ and $X(\mathbb{R})$; see Proposition 9.1.

2. Preliminaries

All vector spaces are over $\mathbb{Z}/2$ unless otherwise stated. We write $H_*(X)$ for homology with *closed supports* (also known as *Borel–Moore homology*) with coefficients in $\mathbb{Z}/2$. Recall that for compact triangulable spaces, homology with closed supports coincides with singular homology.

We now describe basic properties of toric varieties and review some standard notation, referring to [13] and [20] for details.

Any toric variety can be constructed in the following way: Start with a lattice N of rank n and a rational fan Δ in the real vector space $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. To each cone $\sigma \in \Delta$ corresponds the *affine toric variety* $X_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$, where $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ is the lattice dual to N, $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$, $\sigma^\vee = \{v \in M_{\mathbb{R}} \mid v(u) \geq 0$ $\forall\, u \in \sigma\}$ is the cone dual to σ, and $S_\sigma = \sigma^\vee \cap M$ is the corresponding semigroup. If τ is a face of σ, then X_τ can be identified with a principal open subset of X_σ. The *toric variety* X_Δ is constructed by gluing together the affine toric varieties X_σ along their common open subsets. The (complex algebraic) torus associated with the lattice N is

$$\mathbb{T}_N := \mathrm{Spec}(\mathbb{C}[M]) = \mathrm{Hom}_{\mathbb{Z}}\left(M, \mathbb{C}^*\right) = N \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq \left(\mathbb{C}^*\right)^n.$$

The torus $\mathbb{T}_N$ is open in X_Δ, acts on each X_σ, and this action extends to all of X_Δ via the gluings. The $\mathbb{T}_N$-orbits of X_Δ are in one-to-one correspondence with the cones in Δ via the map $\sigma \mapsto \mathcal{O}_\sigma$, where $\mathcal{O}_\sigma$ is the $\mathbb{T}_N$-orbit of the distinguished point $x_\sigma \in X_\sigma = \mathrm{Hom}_{\mathrm{sg}}(S_\sigma, \mathbb{C})$ defined by $x_\sigma(m) = 1$ if $-m \in S_\sigma$ and $x_\sigma(m) = 0$ otherwise. The Zariski closure $\overline{\mathcal{O}}_\sigma$ is the union of all orbits $\mathcal{O}_\tau$ such that σ is a face of τ. For any cone σ, define the lattices

$$N_\sigma := (\sigma \cap N) + (-\sigma \cap N), \quad N(\sigma) := \frac{N}{N_\sigma}.$$

The lattice N_σ has rank $\dim(\sigma)$ and $N(\sigma)$ has rank $n - \dim(\sigma)$. The dual lattices $M_\sigma = \mathrm{Hom}_{\mathbb{Z}}(N_\sigma, \mathbb{Z})$ and $M(\sigma) = \mathrm{Hom}_{\mathbb{Z}}(N(\sigma), \mathbb{Z})$ are respectively

$$M_\sigma = \frac{M}{\left(\sigma^\perp \cap M\right)}, \quad M(\sigma) = \sigma^\perp \cap M.$$

The isotropy subgroup of the $\mathbb{T}_N$-action on $\mathcal{O}_\sigma$ is the isotropy group of the distinguished point x_σ, which consists of the $t \in \mathbb{T}_N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$ such that $t(m) = 1$ for any $m \in \sigma^\perp \cap M$. This gives the identification

$$\mathcal{O}_\sigma = \mathbb{T}_{N(\sigma)} = \frac{\mathbb{T}_N}{\mathbb{T}_{N_\sigma}}.$$

3. Toric varieties as real varieties

This section is based on chapter 4 of [13]; see also [21]. A toric variety X_Δ is defined by polynomials with integer coefficients. Thus we can consider X_Δ to be a real algebraic variety, by which we mean a complex variety defined over the real numbers. This is the standard real structure on a toric variety, and it will be the only real structure we consider in this paper.

Remark 3.1. An example of a toric variety with a non-standard real structure is the variety $X = \mathbb{C}P^1 \times \mathbb{C}P^1$ equipped with the antiholomorphic involution $(z, w) \mapsto (\bar{w}, \bar{z})$ (where the bar designates the usual complex conjugation on each factor). This type of real structure (compatible with the action of a torus with a nonstandard involution) has been studied in great detail by Delaunay [7, 8]. In this case the real part is homeomorphic to the 2-sphere, and it is easy to check that X is *not* an M-variety. On the other hand, taking the diagonal $D \subset \mathbb{C}P^1 \times \mathbb{C}P^1$, we have that both D and its complement are M-varieties, so X does admit a stratification where the open strata are M-varieties.

As mentioned before, we will denote the real part (i.e., the set of real points) of a real variety X by $X(\mathbb{R})$. For clarity we shall often denote the complex points by $X(\mathbb{C})$. The real part $X_\Delta(\mathbb{R})$ of a toric variety X_Δ is covered by the affine open subsets

$$X_\sigma(\mathbb{R}) = \mathrm{Hom}_{\mathrm{sg}}(S_\sigma, \mathbb{R}),$$

where $\mathbb{R}$ is the multiplicative semigroup $\mathbb{R}^* \cup \{0\}$, and

$$\begin{aligned}\mathbb{T}_N(\mathbb{R}) &= \mathrm{Spec}(\mathbb{R}[M]) = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{R}^*) = N \otimes_{\mathbb{Z}} \mathbb{R}^* \simeq (\mathbb{R}^*)^n,\\ \mathcal{O}_\sigma(\mathbb{R}) &= \mathbb{T}_N(\mathbb{R}) \cdot x_\sigma = \mathbb{T}_{N(\sigma)}(\mathbb{R}) = \mathrm{Hom}_{\mathbb{Z}}(M(\sigma), \mathbb{R}^*).\end{aligned}$$

(Note that $x_\sigma \in X_\sigma(\mathbb{R})$.) The real part $X_\Delta(\mathbb{R})$ of a toric variety has an orbit stratification similar to that of the underlying complex toric variety. $X_\Delta(\mathbb{R})$ is obtained by gluing together the $X_\sigma(\mathbb{R})$ for $\sigma \in \Delta$, it is also the union of the orbits $\mathcal{O}_\sigma(\mathbb{R}) \simeq (\mathbb{R}^*)^{n-\dim\sigma}$ under the action of $\mathbb{T}_N(\mathbb{R})$, and the Zariski closure of $\mathcal{O}_\sigma(\mathbb{R})$ is the union of all $\mathcal{O}_\tau(\mathbb{R})$ such that σ is a face of τ.

As pointed out in [13], this construction works for any sub-semigroup of $\mathbb{C} = \mathbb{C}^* \cup \{0\}$. In particular, considering the semigroup $\mathbb{R}_+ = \mathbb{R}^*_+ \cup \{0\}$ instead of $\mathbb{R} = \mathbb{R}^* \cup \{0\}$, one obtains the *positive part* of a toric variety X_Δ. We will denote the positive part of X_Δ by X^+_Δ. The positive part of X_Δ is a semialgebraic subset of the real part of X_Δ.

For any toric variety X_Δ, we have $X^+_\Delta \subset X_\Delta(\mathbb{R}) \subset X_\Delta(\mathbb{C})$ due to the semigroup inclusions $\mathbb{R}_+ \subset \mathbb{R} \subset \mathbb{C}$. Moreover, the absolute value map $z \to |z|$ gives rise to a retraction $\mathbb{R}_+ \subset \mathbb{C} \to \mathbb{R}_+$ which restricts to a retraction $\mathbb{R}_+ \subset \mathbb{R} \to \mathbb{R}_+$. The absolute value map can be extended in order to obtain the following retractions.

$$\begin{array}{ccccc} X^+_\Delta & \subset & X_\Delta(\mathbb{C}) & \to & X^+_\Delta \\ \| & & \cup & & \| \\ X^+_\Delta & \subset & X_\Delta(\mathbb{R}) & \to & X^+_\Delta \end{array}$$

For any lattice N, define the *compact torus* T_N by

$$T_N = \mathrm{Hom}_{\mathbb{Z}}(M, S^1) \subset \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = \mathbb{T}_N(\mathbb{C}),$$

where S^1 is the unit circle in $\mathbb{C}$. Note that if N has rank n then $T_N \simeq (S^1)^n$. The set of 2-torsion points of T_N will be denoted by $T_N[2]$. We have

$$T_N[2] = \mathrm{Hom}_{\mathbb{Z}}(M, S^0) \subset \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{R}^*) = \mathbb{T}_N(\mathbb{R}),$$

where $S^0 = \{\pm 1\}$ is the set of 2-torsion points of S^1. If N has rank n, then $T_N[2] \simeq \{\pm 1\}^n$. The isomorphism $\mathbb{C}^* \simeq S^1 \times \mathbb{R}^*_+$ given by the map $z \mapsto (z/|z|, |z|)$ produces the identification

$$\mathbb{T}_N(\mathbb{C}) = \mathrm{Hom}_{\mathbb{Z}}(M, S^1) \times \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{R}^*_+) = T_N \times \mathbb{T}_N^+.$$

Then, using the isomorphism $\mathbb{R}^*_+ \to \mathbb{R}$ given by the logarithm, we obtain

$$\mathbb{T}_N^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{R}^*_+) = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{R}) = N_{\mathbb{R}},$$

hence

$$\mathbb{T}_N(\mathbb{C}) = T_N \times N_{\mathbb{R}}.$$

Similarly, we have that

$$\mathbb{T}_N(\mathbb{R}) = T_N[2] \times \mathbb{T}_N^+ = T_N[2] \times N_{\mathbb{R}}.$$

Applying this to the lattice $N(\sigma)$ corresponding to a cone $\sigma \in \Delta$, we obtain

$$\begin{aligned}
\mathcal{O}_\sigma^+ &= N(\sigma)_{\mathbb{R}} \simeq \mathbb{R}^{n-\dim(\sigma)},\\
\mathcal{O}_\sigma(\mathbb{C}) &= T_{N(\sigma)} \times N(\sigma)_{\mathbb{R}},\\
\mathcal{O}_\sigma(\mathbb{R}) &= T_{N(\sigma)}[2] \times N(\sigma)_{\mathbb{R}}.
\end{aligned}$$

From this discussion we get the following result, which is well-known for the complex case (see [13, Section 4.1, p. 79]).

Proposition 3.1. *The retraction $r : X_\Delta(\mathbb{C}) \to X_\Delta^+$ given by the absolute value map identifies X_Δ^+ with the quotient space of $X_\Delta(\mathbb{C})$ by the action of the compact torus T_N, and it also identifies X_Δ^+ with the quotient space of $X_\Delta(\mathbb{R})$ by the action of $T_N[2]$.*

The fibers of the quotient maps

$$X_\Delta(\mathbb{C}) \to X_\Delta^+, \quad X_\Delta(\mathbb{R}) \to X_\Delta^+,$$

over a point $p \in X_\Delta^+$ are $T_{N(\sigma)}$ and $T_{N(\sigma)}[2]$, respectively, where $\sigma \in \Delta$ is the unique cone such that $p \in \mathcal{O}_\sigma^+ \simeq \mathbb{R}^{n-\dim(\sigma)}$. Since the exponential map gives the obvious identification

$$T_N[2] = \frac{((1/2)N)}{N} \subset N_{\mathbb{R}}/\overline{N} = T_N,$$

where $(1/2)N = \{u \in N_{\mathbb{R}} \mid 2u \in N\}$, Proposition 3.1 can be restated as follows, *cf.* [14, Theorem 11.5.4], [7, Proposition 4.1.1].

Proposition 3.2. *The toric variety $X_\Delta(\mathbb{C})$ is homeomorphic to the quotient space $X_\Delta^+ \times T_N/\sim$, where the equivalence relation on $X_\Delta^+ \times T_N$ is given by $(p, t) \sim (p', t')$ if and only if $p = p'$ and $t - t' \in T_{N_\sigma} = (N_\sigma)_{\mathbb{R}}/N_\sigma$ for the unique cone $\sigma \in \Delta$ such that $p \in \mathcal{O}_\sigma^+$.*

The real part $X_\Delta(\mathbb{R})$ is homeomorphic to the quotient space $X_\Delta^+ \times T_N[2]/\sim$, where the equivalence relation on $X_\Delta^+ \times T_N[2]$ is given by $(p, t) \sim (p', t')$ if and only if $p = p'$ and $t - t' \in T_{N_\sigma}[2] = (1/2)N_\sigma/N_\sigma$ for the unique cone $\sigma \in \Delta$ such that $p \in \mathcal{O}_\sigma^+$.

4. Betti numbers of nonsingular complete toric varieties

The fact that any nonsingular complete toric variety $X = X_\Delta$ is an M-variety can be deduced from known results by the following arguments.

1. The Jurkiewicz–Danilov theorem [5, Proposition 10.4] implies that the cycle map from the Chow groups $\mathrm{Ch}(X;\,R)$ to $H_*(X;\,R)$ is an isomorphism for arbitrary coefficients R. Since the Chow groups are generated by closures of $\mathbb{T}_N$-orbits, which are conjugation-invariant subvarieties, it follows from standard results in equivariant cohomology (*cf.* [1, Remark 1.2.4 (2)]) that X is maximal. Moreover, results of Krasnov [17] and Borel–Haefliger [3, Section 5.15] imply that there exists a degree-halving isomorphism of algebras.
2. Using virtual Poincare polynomials (for homology with $\mathbb{Z}/2$ coefficients), it is easy to show that the Betti numbers of $X(\mathbb{C})$ are the entries of the combinatorial h-vector of Δ, *cf.* [13, Sections 4.5 and 5.6] for the case of rational coefficients. One can imitate this proof for $X(\mathbb{R})$, using the virtual Poincaré polynomial for real algebraic varieties defined by McCrory and Parusiński [19]. Here one starts with the virtual Poincaré polynomial $\tilde{P}_{\mathbb{R}^*}(t) = t-1$ instead of $\tilde{P}_{\mathbb{C}^*}(t) = t^2-1$. This implies the relations between the individual Betti numbers mentioned in the introduction:
$$b_{2k}(X_\Delta(\mathbb{C})) = b_k(X_\Delta(\mathbb{R})) = h_k(\Delta), \quad b_{2k+1}(X_\Delta(\mathbb{C})) = 0. \tag{3}$$
3. Nonsingular projective toric varieties are manifolds with a Hamiltonian torus action. Since all fixed points for the action on $X(\mathbb{C})$ are contained in $X(\mathbb{R})$, a result of Duistermaat [9, Theorem 3.1] implies that the Betti sum for $X(\mathbb{R})$ is the number of these fixed points. Because the same is true for $X(\mathbb{C})$, this shows that every nonsingular projective toric variety is an M-variety. Extending Duistermaat's methods, Biss, Guillemin and Holm also showed that there exists a degree-halving isomorphism of algebras $H^{2*}(X(\mathbb{C})) \to H^*(X(\mathbb{R}))$ [2, Corollary 5.8].
4. Relation (3) between the individual Betti numbers of the complex and real points of a nonsingular projective toric variety is also a special case of a result of Davis and Januszkiewicz [6, Theorem 3.1]. Both Duistermaat's and Davis–Januszkiewicz's arguments are Morse-theoretic.
5. Another proof in the projective case is by "shelling," as in [13, section 5.2]. (This is closely related to the proof of Davis and Januszkiewicz.) If X is nonsingular and projective, there is an ordering of the top-dimensional cones of Δ that defines a filtration
$$\emptyset \subset Z_m(\mathbb{C}) \subset \cdots \subset Z_1(\mathbb{C}) = X(\mathbb{C})$$
by closed subvarieties $Z_i(\mathbb{C})$ with algebraic cells $Y_i(\mathbb{C}) = Z_i(\mathbb{C}) \setminus Z_{i+1}(\mathbb{C}) \simeq \mathbb{C}^{k_i}$. The corresponding filtration $Z_i(\mathbb{R})$ of $X(\mathbb{R})$ has real algebraic cells $Y_i(\mathbb{R}) \simeq \mathbb{R}^{k_i}$. Since the closure of the cell $Y_i(\mathbb{R})$ is a real algebraic variety, which is a cycle mod 2 [3], it follows that $b_{2k}(X(\mathbb{C})) = b_k(X(\mathbb{R}))$ and $b_{2k+1}(X(\mathbb{C})) = 0$ for all k.

5. The complex toric homology spectral sequence

Following Totaro [22], Fischli [10], and Jordan [16], we describe a spectral sequence associated to the filtration of $X = X_\Delta(\mathbb{C})$ by the orbits of the torus action. Let Δ^p be the set of cones of codimension p of the rational fan Δ in $N_\mathbb{R} \simeq \mathbb{R}^n$. Recall that for each $\sigma \in \Delta^p$ the orbit $\mathcal{O}_\sigma$ has dimension p, and the Zariski closure $\overline{\mathcal{O}}_\sigma$ is the union of all orbits $\mathcal{O}_\tau$ such that σ is a face of τ. Taking

$$X_p := \bigcup_{\sigma \in \Delta^p} \overline{\mathcal{O}}_\sigma$$

we get a filtration (in fact, a stratification)

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X_\Delta(\mathbb{C}) \tag{4}$$

with each open stratum $X_p^\circ := X_p \setminus X_{p-1}$ equal to the disjoint union of the p-dimensional orbits:

$$X_p^\circ = \bigcup_{\sigma \in \Delta^p} \mathcal{O}_\sigma. \tag{5}$$

Hence,

$$H_i\left(X_p^\circ\right) = \bigoplus_{\sigma \in \Delta^p} H_i(\mathcal{O}_\sigma). \tag{6}$$

The filtration (4) gives rise to a spectral sequence

$$E^1_{p,q} = H_{p+q}\left(X_p^\circ\right) \Rightarrow H_{p+q}(X) \tag{7}$$

converging to the homology (with closed supports) of X (*cf.* [18, p. 327]). The differentials

$$\mathrm{d}^1_{p,q} \colon H_{p+q}\left(X_p^\circ\right) \to H_{p+q-1}\left(X_{p-1}^\circ\right)$$

at the E^1 level coincide with the connecting homomorphisms in the long exact sequence

$$\begin{aligned} \cdots \to H_{p+q}\left(X_{p-1}^\circ\right) &\to H_{p+q}\left(X_p \setminus X_{p-2}\right) \\ \to H_{p+q}\left(X_p^\circ\right) &\to H_{p+q-1}\left(X_{p-1}^\circ\right) \to \cdots \end{aligned}$$

for the pair $X_p^\circ \subset X_p \setminus X_{p-2} = X_p^\circ \cup X_{p-1}^\circ$. It follows from (6) and from the functoriality of the connecting homomorphism that we may describe each differential $\mathrm{d}^1_{p,q}$ in block matrix form $\left(\mathrm{d}_{q,\sigma,\tau}\right)_{\sigma \in \Delta^p,\, \tau \in \Delta^{p-1}}$ with

$$\mathrm{d}_{q,\sigma,\tau} \colon H_{p+q}(\mathcal{O}_\sigma) \to H_{p+q-1}(\mathcal{O}_\tau)$$

being the connecting homomorphism for the pair $\mathcal{O}_\tau \subset \mathcal{O}_\sigma \cup \mathcal{O}_\tau$. The latter is zero unless $\mathcal{O}_\tau$ is in the boundary of $\mathcal{O}_\sigma$, that is, unless σ is a face of τ.

Remark 5.1. The spectral sequence (7) is isomorphic with the Leray spectral sequence of the retraction $X_\Delta(\mathbb{C}) \to X_\Delta^+$. In particular, if X_Δ is projective, (7) is isomorphic with the Leray spectral sequence of the moment map.

The differential $\mathrm{d}^1_{p,q}$ permits a simple description in terms of the fan Δ, which we are going to derive now. Recall that for each cone σ of codimension p we have

$$\mathcal{O}_\sigma \simeq T_{N(\sigma)} \times \mathbb{R}^p.$$

Since

$$H_k(\mathbb{R}^p) = \begin{cases} \mathbb{Z}/2 & \text{if } k = p, \\ 0 & \text{otherwise,} \end{cases}$$

and since $\mathbb{Z}/2$ has a unique generator, the Künneth formula gives for any q a canonical isomorphism

$$H_{p+q}(\mathcal{O}_\sigma) = H_q\left(T_{N(\sigma)}\right) \tag{8}$$

in homology with closed supports. Hence each component $\mathrm{d}_{q,\sigma,\tau}$ of the differential d^1 is canonically identified with a map

$$\mathrm{d}'_{q,\sigma,\tau} \colon H_q\left(T_{N(\sigma)}\right) \to H_q\left(T_{N(\tau)}\right).$$

(If the characteristic of the coefficients was different from 2, we would have to choose orientations, and signs would appear as in simplicial homology.)

If σ is a face of τ, then the lattice $N_\sigma \subset N$ of the isotropy group of $\mathbb{T}_N$ acting on $\mathcal{O}_\sigma$ is contained in the corresponding lattice N_τ. Hence we get a natural surjection $N(\sigma) = N/N_\sigma \to N/N_\tau = N(\tau)$, which gives natural split surjections $\mathbb{T}_{N(\sigma)} \to \mathbb{T}_{N(\tau)}$ and $T_{N(\sigma)} \to T_{N(\tau)}$, both of which we will denote by $\pi_{\sigma,\tau}$.

Proposition 5.1 ([10, Theorem 2.1], [16, Section 2.3]). *If σ is a facet of τ, then the homomorphism*

$$\mathrm{d}'_{q,\sigma,\tau} \colon H_q\left(T_{N(\sigma)}\right) \to H_q(T_{N(\tau)})$$

coincides with the homomorphism induced by the split surjection

$$\pi_{\sigma,\tau} \colon T_{N(\sigma)} \to T_{N(\tau)}.$$

Proof. See for example [16, Section 2.3], where it is derived from Proposition 3.1. Alternatively, it follows easily from the fact that the pair $\mathcal{O}_\tau \subset \mathcal{O}_\sigma \cup \mathcal{O}_\tau$ is isomorphic with the pair $(\mathbb{C}^*)^{p-1} \times \{0\} \subset (\mathbb{C}^*)^{p-1} \times \mathbb{C}$ for $\sigma \in \Delta^p$. To see this, consider the star of σ, that is, the set of cones $\tau \in \Delta$ having σ as face. Taking the image of each such τ in $N(\sigma)$ gives a new fan, whose associated toric variety is $\overline{\mathcal{O}_\sigma}$, *cf.* [13, pp. 52–54]. Hence, the pair $(\mathcal{O}_\tau, \mathcal{O}_\sigma)$ is isomorphic with the pair given by ray and the origin in $N(\sigma)$.

Totaro [22] noticed that for rational coefficients this spectral sequence degenerates at the E^2 level for any toric variety (*cf.* [16, Proposition 2.4.5]). In [12], the second author proved that over an arbitrary coefficient ring R this happens for toric varieties which are R-homology manifolds, in particular in the nonsingular case. Moreover, he conjectured that degeneration occurs for arbitrary toric varieties, as with rational coefficients.

6. Homology of compact tori and their two-torsion points

In order to construct a spectral sequence for $X(\mathbb{R})$ which compares well with the spectral sequence introduced for $X(\mathbb{C})$, we need to relate the homology of an n-dimensional compact torus T and its 2-torsion points $T[2]$.

The homology of any compact topological group G is a graded algebra by the *Pontryagin product* $H_*(G)\otimes H_*(G) \to H_*(G)$, which is constructed in the obvious way from the Eilenberg–Zilber map $H_*(G)\otimes H_*(G) \to H_*(G\times G)$ and the multiplication $G\times G \to G$. (The restriction to compact groups is caused by our choice of homology with closed supports.) The unit in $H_*(G)$ is the homology class $[1]$ of the identity element $1 \in G$. Since T and $T[2]$ are commutative, the Pontryagin product is (anti)commutative in these cases. Note that $H_*(T[2]) = H_0(T[2])$ is nothing but the group algebra of the finite group $T[2]$ with coefficients in $\mathbb{Z}/2$.

$H_*(S^1)$ is an exterior algebra on the fundamental class of S^1. Similarly, the homology of $S^1[2] = S^0 = \{1, g\}$ is an exterior algebra on the fundamental class $[1]+[g]$ because

$$([1]+[g])^2 = [1]^2 + [g]^2 = 2\,[1] = 0.$$

Hence, $H_*(S^1)$ and $H_0(S^0)$ are isomorphic as ungraded algebras. Note that in both cases the generators are unique, so that the isomorphism is canonical.

Choosing a decomposition of T into circles

$$T \simeq \left(S^1\right)^n, \tag{9}$$

the Künneth formula gives isomorphisms

$$H_*(T) \simeq H_*\left(S^1\right)^{\otimes n}, \tag{10}$$

$$H_0(T[2]) \simeq H_0\left(S^0\right)^{\otimes n}. \tag{11}$$

Since (9) is a decomposition of topological groups, (10) and (11) are isomorphisms of algebras. This gives an isomorphism

$$H_*(T) \simeq H_0(T[2]) \quad \text{as ungraded algebras.} \tag{12}$$

The following example shows that this isomorphism does depend on the decomposition of T as a product of copies of S^1 and that it is not functorial, except for homomorphisms that are compatible with the chosen product decompositions.

Example 6.1. Let $T = S^1 \times S^1$, and let $D \subset T$ be the diagonal torus. Then

$$[D] = \left[S^1 \times 1\right] + \left[1 \times S^1\right] \in H_*(T),$$

whereas

$$[D[2]] = \left[S^0 \times 1\right] + \left[1 \times S^0\right] + \left[S^0 \times S^0\right] \in H_0(T[2]).$$

By functoriality, the same relations hold for any two circles representing different elements in $H_*(T)$ and a circle representing their sum.

In order to get a better comparison of $H_*(T)$ and $H_0(T[2])$ we will construct a natural filtration on $H_0(T[2])$. Let $\mathcal{I} = \mathcal{I}(T[2])$ be the kernel of the augmentation $H_0(T[2]) \to \mathbb{Z}/2$ induced by the group homomorphism from T to a point. It is an ideal in $H_0(T[2])$, and there is a canonical direct sum decomposition of vector spaces

$$H_0(T[2]) \simeq \langle [1] \rangle \oplus \mathcal{I}, \tag{13}$$

functorial with respect to homomorphisms of tori.

Lemma 6.1. *For $k \geq 1$, the k-th power $\mathcal{I}^k$ of $\mathcal{I}$ is additively generated by the fundamental classes of rank k subtori of $T[2]$. In particular, $\mathcal{I}^k = 0$ if k is greater than the rank n of $T[2]$.*

Proof. An element $a \in H_0(T[2])$ is a formal linear combination of points of $T[2]$. We have $a \in \mathcal{I}$ if and only if there is an even number of such points. Denoting by A the set of these points, we can rewrite a as

$$a = \sum_{g \in A} [g] = \sum_{1 \neq g \in A} ([1] + [g]) ,$$

which is a sum of fundamental classes of rank 1 subtori.

Suppose the claim is true for k. Then $\mathcal{I}^{k+1}$ is additively generated by products $[H] * ([1] + [g])$ with $g \in T$ and H a rank k subtorus. If g is contained in H, this equals $[H] + [H] = 0$. Otherwise we get the fundamental class of a subtorus of rank $k + 1$.

We now look at graded quotient associated to the filtration

$$H_0(T[2]) = \mathcal{I}^0 \supset \mathcal{I}^1 \supset \cdots \supset \mathcal{I}^n \supset \mathcal{I}^{n+1} = 0.$$

Writing $V = \mathrm{Hom}(S^1, T) \otimes \mathbb{Z}/2$, the assignment $\chi \mapsto \chi_*([S^1])$ gives a canonical isomorphism of vector spaces $V = H_1(T)$, which extends to an isomorphism of graded algebras

$$\bigwedge\nolimits^* V = H_*(T), \tag{14}$$

natural with respect to homomorphisms of tori.

The following result is crucial for the present paper in that it will finally enable us to compare the homology of complex toric varieties with their real parts.

Proposition 6.1. *There is a natural isomorphism of graded algebras*

$$\mathrm{Gr}^*_{\mathcal{I}} H_0(T[2]) = H_*(T).$$

Proof. The assignment $\chi \mapsto \chi_*([S^0])$ gives a map (not a homomorphism) $V \to \mathcal{I} \subset H_0(T[2])$, which induces a map

$$V \simeq \mathcal{I}/\mathcal{I}^2.$$

We claim that the latter is a homomorphism, i.e., compatible with sums: Take two elements from V. We may assume that they are distinct and non-zero. Then Example 6.1 shows that the image of their sum and the sum of their images in $\mathcal{I}$ differ by an element from $\mathcal{I}^2$.

The square of any element $a \in \mathcal{I}$ is zero because a is a sum of terms of the form $[1] + [g]$. Therefore, we get a homomorphism of graded algebras

$$\bigwedge^* V \to \operatorname{Gr}^*_{\mathcal{I}} H_0(T[2]). \tag{15}$$

Since $\mathcal{I}^k$ is additively generated by the fundamental classes of rank k subgroups, the map (15) is surjective, hence an isomorphism.

The desired map is the composition of (14) and (15). It is clear from the definitions that both isomorphisms do not depend on any choice and that they are functorial in T.

Remark 6.1. Together with the isomorphism (12), Proposition 6.1 implies that $H_0(T[2])$ and $\operatorname{Gr}^*_{\mathcal{I}} H_0(T[2])$ are isomorphic as algebras. Again the isomorphism is not natural in general, but it is so with respect to homomorphisms that are compatible with the chosen product decompositions. In particular, one can choose isomorphisms compatible with a single injection or a single split projection.

7. The real toric homology spectral sequence

As for $X(\mathbb{C})$, one can consider for the real toric variety $X(\mathbb{R})$ the filtration by $\mathbb{T}_N(\mathbb{R})$-orbit dimension. This gives the real analogue

$$E^1_{p,q} = H_{p+q}\left(X^\circ_p(\mathbb{R})\right) \Rightarrow H_{p+q}(X(\mathbb{R})) \tag{16}$$

of the spectral sequence (7). As in the complex case, the spectral sequence (16) is isomorphic to the Leray spectral sequence of $X_\Delta(\mathbb{R}) \to X^+_\Delta$.

Because $\mathbb{T}_{N(\sigma)} = T_{N(\sigma)}[2] \times \mathbb{R}^p \simeq (\mathbb{R}^*)^p$ is a disjoint union of p-cells for $\sigma \in \Delta^p$, the filtration on $X(\mathbb{R})$ is actually cellular, and $E^1_{p,q} = 0$ for $q \neq 0$, so our spectral sequence degenerates at the E^2 level for trivial reasons. The nontrivial row $E^1_{*,0}$ is just the cellular chain complex associated to the filtration. We will denote this chain complex by $C_*(\Delta)$.

Similar to the identification

$$E^1_{p,q} = \bigoplus_{\sigma \in \Delta^p} H_q\left(T_{N(\sigma)}\right) \tag{17}$$

in the complex case (*cf.* Section 5), we can identify the E^1 term of the spectral sequence at hand as

$$E^1_{p,0} = C_p(\Delta) = \bigoplus_{\sigma \in \Delta^p} H_p\left(T_{N(\sigma)}\right) = \bigoplus_{\sigma \in \Delta^p} H_0\left(T_{N(\sigma)}[2]\right). \tag{18}$$

The following real analogue of Proposition 5.1 is obvious.

Lemma 7.1. *Under the above identification, the component* $H_0\left(T_{N(\sigma)}[2]\right) \to H_0\left(T_{N(\tau)}[2]\right)$ *of the differential is induced by the surjection* $T_{N(\sigma)}[2] \to T_{N(\tau)}[2]$ *for* σ *a facet of* τ*, and zero otherwise.*

Using the filtration on 2-tori introduced in the previous section, we introduce a filtration on $C_*(\Delta)$. It is this new filtration and not the one discussed above which will lead to a generalization of Proposition 6.1 from tori to toric varieties. For convenience, the new filtration is increasing instead of decreasing; its pieces are

$$F_p(C_*(\Delta)) = \bigoplus_{\sigma\in\Delta} \mathcal{I}^{-p}\left(T_{N(\sigma)}[2]\right) \quad \text{for } p \le 0.$$

Then $F_{-n-1}(C_*(\Delta)) = 0$ and $F_0(C_*(\Delta)) = C_*(\Delta)$. We write $G^k_{p,q}$ for the terms of the resulting spectral sequence, which is concentrated in the second quadrant. For $n = 3$, the possibly non-zero terms are located as follows:

$*$				$6 = q$
	$*$			
	$*$	$*$		
		$*$	$*$	
		$*$	$*$	
			$*$	
			$*$	0
$p = -3$			0	

As a direct consequence of Proposition 6.1, we obtain the following result.

Proposition 7.1. *The term* $G^0_{p,q}$ *of the spectral sequence for* $X(\mathbb{R})$ *is isomorphic to the term* $E^1_{p+q,-p}$ *of the spectral sequence for* $X(\mathbb{C})$*, and this is compatible with the differentials. Hence also* $G^1_{p,q} = E^2_{p+q,-p}$*. In particular, the* G^1 *and* E^2 *levels have the same total dimension.*

Proof. By construction, $G^0_{p,q} = \bigoplus_{\sigma\in\Delta^{p+q}} \mathrm{Gr}^{-p}_{\mathcal{I}}\, H_0(T_{N(\sigma)}[2])$. It now follows from Proposition 6.1 and the discussion in section 5 that $G^0_{p,q} = E^1_{p+q,-p}$ for all p, q. The fact that for each p, q differential $\mathrm{d}^0_{pq} : G^0_{p,q} \to G^0_{p,q-1}$ corresponds to the differential $\mathrm{d}^1_{pq} : E^1_{p+q,-p} \to E^1_{p+q-1,-p}$ is easy to check from Lemma 7.1 and Proposition 5.1.

Remark 7.1. The isomorphism $G^1_{p,q} \simeq E^2_{p+q,-p}$ is not compatible with the differentials. Indeed, the differential with source $G^1_{p,q}$ has target $G^1_{p-1,q} \simeq E^2_{p+q-1,-p+1}$, whereas the one with source $E^2_{p+q,-p}$ has target $E^2_{p+q-2,-p+1}$.

Remark 7.2. The decomposition (13) induces a direct sum decomposition of $C_*(\Delta)$. One summand is the complex $G^0_{0,*} = E^1_{*,0}$, the other one the direct sum of the augmentation ideals $\mathcal{I}\left(T_{N(\sigma)}\right)$. Therefore, all differentials starting at the rightmost column $G^k_{0,*}$ vanish for $k \ge 1$.

8. Comparing the spectral sequences

We can now sharpen the conjecture stated in the introduction.

Conjecture 8.1. The spectral sequence $G^k_{p,q}$ for $H_*(X(\mathbb{R}))$ degenerates at the G^1 level.

Notice that this implies that the spectral sequence $E^k_{p,q}$ converging to $H_*(X(\mathbb{C}))$ degenerates at the E^2 level: the comparison between G^1 and E^2 (Proposition 7.1) implies that otherwise the Betti sum for $X(\mathbb{R})$ would exceed that for $X(\mathbb{C})$, in contradiction to the Smith–Thom inequality (1). In particular, the conjecture implies that X is an M-variety.

Numerous examples indicate that degeneration is at least not a rare phenomenon. We have checked many examples using the Maple package Torhom by the second author [11]. So far this has not led to an example of a toric variety for which the spectral sequence $G^k_{p,q}$ does not degenerate. Here is one example:

Example 8.1. Let Δ be the normal fan (*cf.* [13, section 5.6]) of the five-dimensional cyclic polytope with vertices $(k^1, \dots, k^5)$ for $k = 0, 1, \dots, 6$. The dimensions of the E^2 terms and the G^1 terms are:

$E^2_{p,q}$

$q \backslash p$	0	1	2	3	4	5
5						1
4					11	4
3				13	27	6
2			6	17	21	4
1		1	1	4	5	1
0	1	0	0	0	0	0

$G^1_{p,q}$

$q \backslash p$	−5	−4	−3	−2	−1	0
10	1					
9		4				
8		11	6			
7			27	4		
6			13	21	1	
5				17	5	0
4				6	4	0
3					1	0
2					1	0
1						0
0						1

Clearly there is plenty of room for higher differentials in both cases. Yet both spectral sequences degenerate, as Torhom's direct computation of $H_*(X(\mathbb{R}))$ shows.

We now prove Theorem 1.1 by verifying Conjecture 8.1 in each case.

Proof (of the first part). This is essentially an adaption of a result of Brion's. In [4, Section 1.2], Brion considered the double complex

$$C_{p,q}(\Delta) = \bigoplus_{\sigma \in \Delta^p} \bigwedge^q N_{\mathbb{R}}(\sigma) \tag{19}$$

with a horizontal differential, which is induced by the canonical projections $N_{\mathbb{R}}(\tau) \to N_{\mathbb{R}}(\sigma)$ for τ a face of σ. (In fact, Brion considered the dual complex, but that does not matter here.) Note that in case of a rational fan, this is the E^1 term of the homology spectral sequence for the toric variety X_Δ as discussed in section 5, but with real coefficients. Brion proved that the homology of the complex (19), which is the E^2 term in our context, is concentrated on the diagonals $0 \le p - q \le k + 1$ if all cones of codimension at most k are simplicial. For rational fans, Brion's argument generalises directly to coefficients in an arbitrary field K instead of $\mathbb{R}$ if one replaces "simplicial cones" by "K-regular cones". Here a rational cone σ in $N_{\mathbb{R}}$ is called K-regular if the images in $N \otimes_{\mathbb{Z}} K$ of the minimal generators of the extremal rays of σ can be extended to a basis of $N \otimes_{\mathbb{Z}} K$.

Since we assume X_Δ to have at most isolated singularities, all cones in Δ which are not full-dimensional are K-regular for any K, in particular for $K = \mathbb{Z}/2$. It follows immediately from the shape of the E^2 term that non-trivial higher differentials are impossible for G^1.

Proof (of the second part). There can be no higher differentials starting at $G^1_{0,*}$ by Remark 7.2. Hence the only higher differential can be $\mathrm{d}^1\colon G^1_{-1,4} \to G^1_{-2,4}$ for three-dimensional varieties. Clearly, the source of this differential is non-trivial only if the previous differential $\mathrm{d}^0\colon G^0_{-1,4} \to G^0_{-1,3}$ is not injective. By Proposition 7.1 this is the case only if the differential $\mathrm{d}^1\colon E^1_{3,1} \to E^1_{2,1}$ is not injective, in other words, if the map

$$H_1(T_N) \to \bigoplus_{\tau \in \Delta^2} H_1(T_{N(\tau)})$$

is not injective. Since the kernel of each map $H_1(T_N) \to H_1\left(T_{N(\tau)}\right)$ is generated by the image of the minimal representative of τ in $V = N/2N$, this implies that all $\tau \in \Delta^2$ have the same image v in V. As a consequence, all two-tori

$$T_{N(\sigma)}[2] = \frac{((1/2)N)/v}{(N/v)} =: \tilde{T}$$

of rank 2 are actually identical. By Remark 6.1, one can therefore choose isomorphisms $H_0(T_N[2]) \simeq \mathrm{Gr}^*_{\mathcal{I}} H_0(T_N[2])$ and $H_0(\tilde{T}) \simeq \mathrm{Gr}^*_{\mathcal{I}} H_0(\tilde{T})$ which intertwine the differential

$$\mathrm{d}^0\colon \bigoplus_{p+q=3} G^0_{p,q} = \mathrm{Gr}^*_{\mathcal{I}} H_0(T_N[2]) \to \bigoplus_{\sigma \in \Delta^2} \mathrm{Gr}^*_{\mathcal{I}} H_0\left(T_{N(\sigma[2])}\right) = \bigoplus_{p+q=2} G^0_{p,q}$$

and the underlying differential

$$\mathrm{d}\colon C_3(\Delta) = H_0(T_N[2]) \to \bigoplus_{\sigma \in \Delta^2} H_0\left(T_{N(\sigma[2])}\right) = C_2(\Delta).$$

Hence the kernels of these differentials have the same dimension, and all higher differentials $\mathrm{d}^k_{p,q}$ must vanish for $p + q = 3$ and $k \ge 1$.

9. Explicit calculations in dimension 2

Jordan [16, Theorem 3.4.2] computed the integral homology with closed supports of an arbitrary two-dimensional toric variety. Together with our result, this leads to the following classification in the case of complete toric surfaces.

Proposition 9.1. *Let X be the complete toric surface associated to a fan Δ with respect to a lattice N of rank 2. Let s be the number of one-dimensional cones of Δ.*

(1) If at least two primitive generators of one-dimensional cones of Δ have different images in the quotient lattice $N/2N$, then

$$\begin{aligned} &b_0(X(\mathbb{R})) = 1, \quad b_1(X(\mathbb{R})) = s-2, \quad b_2(X(\mathbb{R})) = 1, \\ &b_0(X(\mathbb{C})) = 1, \quad b_1(X(\mathbb{C})) = 0, \quad b_2(X(\mathbb{C})) = s-2, \\ &b_3(X(\mathbb{C})) = 0, \quad b_4(X(\mathbb{C})) = 1. \end{aligned}$$

(2) If the primitive generators of one-dimensional cones of Δ all have the same image in $N/2N$, then

$$\begin{aligned} &b_0(X(\mathbb{R})) = 1, \quad b_1(X(\mathbb{R})) = s-1, \quad b_2(X(\mathbb{R})) = 2, \\ &b_0(X(\mathbb{C})) = 1, \quad b_1(X(\mathbb{C})) = 0, \quad b_2(X(\mathbb{C})) = s-1, \\ &b_3(X(\mathbb{C})) = 1, \quad b_4(X(\mathbb{C})) = 1. \end{aligned}$$

Proof. Recall that every complete two-dimensional fan Δ is the normal fan of some polytope P. The dimensions of the E^1 level of the spectral sequence for $X(\mathbb{C})$ are as follows, where the $q = 0$ row is the cellular chain complex of such a P:

	0		$2 = p$
$q = 2$			1
		s	2
0	s	s	1

The $q = 0$ row of E^2 is the homology of P. As in the proof of Theorem 1.1, we know that the differential $E^1_{2,1} \to E^1_{1,1}$ is either injective or has a one-dimensional kernel. The latter occurs if and only if the minimal generators of the one-dimensional cones in Δ have the same image in $N/2N$. Hence, the E^2 term is one of the following:

(1)

	0		$2 = p$
$q = 2$			1
		$s-2$	0
0	1	0	0

(2)

	0		$2 = p$
$q = 2$			1
		$s-1$	1
0	1	0	0

From this we can read off the Betti numbers of $X(\mathbb{C})$. To get the Betti numbers of $X(\mathbb{R})$, we simply have to sum up the dimensions in each column.

If X is a nonsingular complete toric surface, then the primitive generators of two consecutive (with respect to either of the two cyclic orders) one-dimensional cones have different images in $N/2N$, so that case (1) applies. It is instructive to use Proposition 3.2 to find generators for the homology groups of $X(\mathbb{R})$.

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