

SIGN CONDITIONS FOR THE EXISTENCE OF AT LEAST ONE POSITIVE SOLUTION OF A SPARSE POLYNOMIAL SYSTEM

FRÉDÉRIC BIHAN, ALICIA DICKENSTEIN, AND MAGALÍ GIAROLI

ABSTRACT. We give sign conditions on the support and coefficients of a sparse system of d generalized polynomials in d variables that guarantee the existence of at least one positive real root, based on degree theory and Gale duality. In the case of integer exponents, we relate our sufficient conditions to algebraic conditions that emerged in the study of toric ideals.

1. INTRODUCTION

Deciding whether a real polynomial system has a positive solution is a basic question, that is decidable via effective elimination of quantifiers [1]. There are few results on lower bounds of the number of real or positive roots of polynomial systems (see e.g. [3, 18, 19, 22]). In this paper, we give sign conditions on the support and coefficients of a sparse system of d generalized polynomials (that is, polynomials with real exponents, for which the positive solutions are well defined) in d variables, that guarantee the existence of at least one positive real root, based on degree theory and Gale duality.

We fix an exponent set $\mathcal{A} = \{a_1, \dots, a_n\} \subset \mathbb{R}^d$ of cardinality n and for any given real matrix $C = (c_{ij}) \in \mathbb{R}^{d \times n}$ we consider the associated sparse generalized multivariate polynomial system in d variables $x = (x_1, \dots, x_d)$ with support \mathcal{A} :

$$(1.1) \quad f_i(x) = \sum_{j=1}^n c_{ij} x^{a_j} = 0, \quad i = 1, \dots, d.$$

We will be interested by the existence of the positive solutions of (1.1) in $\mathbb{R}_{>0}^d$. Denoting by $n_{\mathcal{A}}(C)$ the (possibly infinite) number of positive real solutions of the system (1.1), our main goal is to give sufficient conditions on the exponent set \mathcal{A} and the coefficient matrix C that ensure that $n_{\mathcal{A}}(C) > 0$. When $\mathcal{A} \subset \mathbb{Z}^d$ we will consider the existence of solutions in the real torus $(\mathbb{R}^*)^d$ of points in \mathbb{R}^d with nonzero coordinates, and we will relate our conditions to well-studied algebraic properties of lattice ideals associated with the configuration \mathcal{A} .

FB is partially supported by the grant ANR-18-CE40-0009 of Agence Nationale de Recherche, France. AD and MG are partially supported by UBACYT 20020170100048BA, CONICET PIP 11220150100473, and ANPCyT PICT 2016-0398, Argentina.

In applications, for example, in the context of chemical reaction networks, lower bounds of positive roots of polynomial systems guarantee the existence of (stoichiometrically compatible) positive steady states. In [13], sign conditions are used to decide if a family of polynomial systems associated with a given reaction network cannot admit more than one positive solution for any choice of the parameters and, in this case, conditions for the existence of one positive solution are given as a corollary of a result from [14], based on degree theory. Our point of view of searching for conditions on the exponent and the coefficient matrices of the system comes from this paper. As we do not assume injectivity (at most one root), we cannot use tools from these papers or the more recent article [15], as Hadamard’s theorem.

In [6] the authors use degree theory in the study of chemical reaction networks to describe parameters for which there is a single positive solution or for which there are more (this is called *multistationarity*). We apply some of these techniques in a Gale duality setting, more precisely, based on Theorem 3.1, which is version of a particular case of Theorem 2 in the Supplementary Information of [6].

We can use different convex sets to apply Theorem 3.1. The first one which comes in mind is the positive orthant, which is not bounded. Another natural idea is to consider the Newton polytope of the polynomials in the system, or some dilates of it. This is reasonable since it is completely determined by the *monomials* appearing in the system. In this paper, we use another convex polytope which seems natural since it is determined by *the coefficients* of the system. This polytope is obtained using the Gale duality trick for polynomial systems that was studied by Bihan and Sottile in [4], see also [5]. We can think of this polytope as a “shadow” of the positive orthant via Gale duality, which has the advantage that it can be chosen to be bounded.

The paper is organized as follows. In Section 2 we recall the notion of Gale duality and the basic duality of solutions (see Theorem 2.5), and we introduce useful notation as well as the necessary condition (2.3). In Section 3, we recall the basic concepts of degree theory and we present our main result Theorem 3.7, which gives conditions on the Gale duality side to guarantee the existence of positive solutions.

In the following sections we give sufficient conditions on the support and the matrix of coefficients that ensure that Theorem 3.7 can be applied. In Section 4, we consider the notion of mixed dominating matrices from [11] to get Theorem 4.6. In Section 5, we give geometric conditions on \mathcal{A} and C that guarantee that the hypotheses of Theorem 3.7 are satisfied. Based on results from [10], we obtain Theorem 5.8. In Section 6, we concentrate our study on integer configurations \mathcal{A} ; we relate the dominance conditions to algebraic conditions that emerged in the study of toric ideals, and we naturally extend in this case our study to the existence of solutions in the real torus $(\mathbb{R}^*)^d$.

2. GALE DUALITY FOR POSITIVE SOLUTIONS OF POLYNOMIAL SYSTEMS

We first present basic definitions and results on Gale duality. Given a matrix $M \in \mathbb{R}^{r \times s}$ of maximal rank r , a *Gale dual matrix* of M is any matrix $N \in \mathbb{R}^{s \times (s-r)}$ of maximal rank whose columns vectors are a basis of the kernel of M . Clearly a Gale dual matrix is not unique as it corresponds to a choice of a basis: it is unique up to right multiplication by an invertible $(s-r) \times (s-r)$ -matrix. We will also say that the s row vectors of N define a Gale dual configuration in \mathbb{R}^{s-r} to the configuration in \mathbb{R}^r defined by the s column vectors of M . We will introduce a Gale dual system (2.8) and polyhedra Δ_P (2.7), depending on the choice of a Gale dual matrix to the coefficient matrix C . We will then recall Theorem 2.5, which gives a fundamental link between the positive real roots of system (1.1) and the solutions in Δ_P of the Gale dual system (2.8).

2.1. Matrices and their Gale duals. Let $\mathcal{A} = \{a_1, \dots, a_n\}$ be a finite subset of \mathbb{R}^d of cardinality n and $C = (c_{ij}) \in \mathbb{R}^{d \times n}$. As we mentioned in the introduction, we are interested in the solvability of the associated sparse generalized multivariate polynomial system (1.1) in d variables $x = (x_1, \dots, x_d)$ with support \mathcal{A} and coefficient matrix C .

Note that if we multiply each equation of system (1.1) by a monomial (i.e, we translate the configuration \mathcal{A}), the number of positive real solutions does not change, and then $n_{\mathcal{A}}(C)$ is an affine invariant of the point configuration \mathcal{A} . It is then natural to consider the matrix $A \in \mathbb{R}^{(d+1) \times n}$ with columns $(1, a_1), (1, a_2), \dots, (1, a_n) \in \mathbb{R}^{d+1}$:

$$(2.1) \quad A = \begin{pmatrix} 1 & \dots & 1 \\ a_1 & \dots & a_n \end{pmatrix}.$$

We will refer to the matrix A as the corresponding matrix of the point configuration \mathcal{A} .

We will always assume that C is of maximal rank d and A is of maximal rank $d+1$. Then, we need to have $n \geq d+1$. If equality holds, it is easy to see that system (1.1) has a positive solution if and only if the necessary condition (2.3) holds. So we will suppose that $n \geq d+2$.

We denote by $k = n - d - 1$ the codimension of A (and of \mathcal{A}). Note that the codimension of C equals $k+1$. Let $B = (b_{ij}) \in \mathbb{R}^{n \times k}$ be a matrix which is Gale dual to A , and let $D = (d_{ij}) \in \mathbb{R}^{n \times (k+1)}$ be any a matrix which is Gale dual to C . We will number the *columns* of B from 1 to k and the *columns* of D from 0 to k and denote by $P_1, \dots, P_n \in \mathbb{R}^{k+1}$ the *row* vectors of D , that is, the Gale dual configuration to the columns of C .

2.2. A necessary condition. There is a basic necessary condition for $n_{\mathcal{A}}(C)$ to be positive. Denote by $C_1, \dots, C_n \in \mathbb{R}^d$ the *column* vectors of the coefficient matrix C and call

$$(2.2) \quad \mathcal{C}^\circ = \mathbb{R}_{>0}C_1 + \dots + \mathbb{R}_{>0}C_n,$$

the positive cone generated by them. Given a solution $x \in \mathbb{R}_{>0}^d$ of system (1.1), the vector $(x^{a_1}, \dots, x^{a_n})$ is positive and so the origin $\mathbf{0} \in \mathbb{R}^d$ belongs to \mathcal{C}° . Then, necessarily

$$(2.3) \quad \mathbf{0} \in \mathcal{C}^\circ.$$

It is a well-known result that Condition (2.3) holds if and only if the vectors P_1, \dots, P_n lie in an open halfspace through the origin. Note that Condition (2.3), together with the hypothesis that C is of maximal rank d , is equivalent to $\mathcal{C}^\circ = \mathbb{R}^d$.

2.3. Defining cones and polytopes in Gale dual space. We also define other cones that we will use. Denote by

$$(2.4) \quad \mathcal{C}_P = \mathbb{R}_{>0}P_1 + \dots + \mathbb{R}_{>0}P_n,$$

the positive cone generated by the rows of a Gale dual matrix D and let

$$(2.5) \quad \mathcal{C}_P^\vee = \{y \in \mathbb{R}^{k+1} : \langle P_i, y \rangle > 0, i = 1, \dots, n\},$$

be its dual open cone. Note that if C has maximal rank d and Condition (2.3) holds, the cone \mathcal{C}_P is strictly convex. Therefore, its dual open cone \mathcal{C}_P^\vee is a nonempty full dimensional open convex cone. We will also consider the closed cone

$$(2.6) \quad \bar{\mathcal{C}}_P = \mathbb{R}_{\geq 0}P_1 + \dots + \mathbb{R}_{\geq 0}P_n.$$

The following Lemma is straightforward.

Lemma 2.1. *Assume that C has maximal rank d and that $\mathbf{0} \in \mathcal{C}^\circ$. Then for any nonzero $u \in \bar{\mathcal{C}}_P$ and any $c \in \mathbb{R}_{>0}$, the polytope $\mathcal{C}_P^\vee \cap \{y \in \mathbb{R}^{k+1} : \langle u, y \rangle = c\}$ has dimension k . Moreover, this polytope is bounded if and only if $u \in \mathcal{C}_P$.*

Define

$$(2.7) \quad \Delta_P = \mathcal{C}_P^\vee \cap \{y \in \mathbb{R}^{k+1} : y_0 = 1\}.$$

Corollary 2.2. *Assume that C has maximal rank, $\mathbf{0} \in \mathcal{C}^\circ$ and let D be a Gale dual matrix of C . Then $(1, 0, \dots, 0) \in \mathcal{C}_P$ if and only if Δ_P has dimension k and is bounded.*

We next show that we can always find a Gale matrix D such that Δ_P is nonempty and bounded.

Lemma 2.3. *Assume that C has maximal rank. Then there is a Gale dual matrix D of C such that $(1, 0, \dots, 0) \in \mathcal{C}_P$.*

Proof. Start with any Gale dual matrix D of C and pick any vector $u \in \mathcal{C}_P$. Then there is an invertible matrix $R \in \mathbb{R}^{(k+1) \times (k+1)}$ such that $u \cdot R = (1, 0, \dots, 0)$, where u is written as a row vector. Consider the matrix $D' = DR$ and denote by P'_1, \dots, P'_n its row vectors. Then D' is Gale dual to C , and $(1, 0, \dots, 0) \in \mathcal{C}_{P'} = \mathbb{R}_{>0}P'_1 + \dots + \mathbb{R}_{>0}P'_n$. \square

To any choice of Gale dual matrices B and D of A and C respectively, we associate the following system with unknowns $y = (y_0, \dots, y_k)$:

$$(2.8) \quad \prod_{i=1}^n \langle P_i, y \rangle^{b_{ij}} = 1, \quad j = 1, \dots, k,$$

which is called a *Gale dual system* of (1.1). Denote $G_j(y) = \prod_{i=1}^n \langle P_i, y \rangle^{b_{ij}}$. Another choice D' of a Gale dual matrix for C corresponds to another choice y' of linear coordinates for \mathbb{R}^{k+1} : if $D' = DR$ with $R \in \mathbb{R}^{(k+1) \times (k+1)}$ invertible, then setting $y' = R^{-1}(y)$ we get $D'y' = Dy$, where y and y' are considered as column vectors. Another choice B' of a Gale dual matrix for A gives an equivalent Gale system $H_1 = \dots = H_k = 1$, where for each j there exist exponents (μ_1, \dots, μ_k) such that $H_j = G_1^{\mu_1} \dots G_k^{\mu_k}$.

Note that (2.8) is homogeneous of degree zero since the columns of B sum up to zero. For any cone $\mathcal{C} \in \mathbb{R}^n$ with apex the origin, its projectivization $\mathbb{P}\mathcal{C}$ is the quotient space \mathcal{C}/\sim under the equivalence relation \sim defined by: for all $y, y' \in \mathcal{C}$, we have $y \sim y'$ if and only if there exists $\alpha > 0$ such that $y = \alpha y'$.

We will often use the following observation.

Remark 2.4. *If $(1, 0, \dots, 0) \in \bar{\mathcal{C}}_P$, then \mathcal{C}_P^ν is contained in the open half-space defined by $y_0 > 0$ and thus the map $(y_0, y_1, \dots, y_k) \mapsto (1, y_1/y_0, \dots, y_k/y_0)$ induces a bijection between $\mathbb{P}\mathcal{C}_P^\nu$ and Δ_P .*

2.4. The equivalence of solutions. Here is a slight variation of Theorem 2.2 in [4].

Theorem 2.5. *There is a bijection between the positive solutions of the initial system (1.1) and the solutions of the Gale dual system (2.8) in $\mathbb{P}\mathcal{C}_P^\nu$, which induces a bijection between the positive solutions of (1.1) and the solutions of (2.8) in Δ_P when $(1, 0, \dots, 0) \in \bar{\mathcal{C}}_P$.*

Proof. If $x \in \mathbb{R}_{>0}^d$ is a solution of the system (1.1), then $(x^{a_1}, \dots, x^{a_n})$ belongs to $\ker(C) \cap \mathbb{R}_{>0}^n$. Thus, there exists $y \in \mathbb{R}^{k+1}$ (which is unique since D has maximal rank) such that $x^{a_i} = \langle P_i, y \rangle$ for $i = 1, \dots, n$. Then, $y \in \mathcal{C}_P^\nu$ and y is a solution of the Gale dual system (2.8). If furthermore $(1, 0, \dots, 0) \in \bar{\mathcal{C}}_P$, then dividing by y_0 if necessary, a solution $y \in \mathcal{C}_P^\nu$ of (2.8) gives a solution of the same system in Δ_P because it is homogeneous of degree zero. We showed in Remark 2.4 that the previous map is bijective by giving explicitly its inverse map.

Now, let $y \in \mathcal{C}_P^\nu$ be a solution of (2.8). Let (e_1, \dots, e_d) be the canonical basis of \mathbb{R}^d . Since A has maximal rank, there exists $\alpha_j = (\alpha_{1j}, \dots, \alpha_{nj}) \in \mathbb{R}^n$, for $j = 1, \dots, d$, such that $e_j = \sum_{i=1}^n \alpha_{ij} a_i$. To any column vector $z \in \mathbb{R}^{k+1}$, we associate the vector $D \cdot z$ with coordinates $\langle P_i, z \rangle$, $i = 1, \dots, n$. Consider now the map

$$\begin{aligned} \varphi: \mathbb{R}^{k+1} &\rightarrow \mathbb{R}^d \\ z &\mapsto ((D \cdot z)^{\alpha_1}, \dots, (D \cdot z)^{\alpha_d}), \end{aligned}$$

where $(D \cdot z)^{\alpha_j} = \prod_{i=1}^n \langle P_i, z \rangle^{\alpha_{ij}}$. Let $x = \varphi(y)$. Then, $x^{a_i} = \langle P_i, y \rangle$ for $i = 1, \dots, n$, which gives $(x^{a_1}, \dots, x^{a_n}) \in \ker(C)$. Moreover, since $y \in \mathcal{C}_P^\nu$, we have that $x \in \mathbb{R}_{>0}^d$, and then x is a positive solution of system (1.1). \square

Remark 2.6. *Theorem 2.2 in [4] is a particular case of Theorem 2.5 taking a Gale dual matrix D with the identity matrix I_{k+1} at the top (in which case the condition that $(1, 0, \dots, 0) \in \bar{\mathcal{C}}_P$ is trivially satisfied).*

3. EXISTENCE OF POSITIVE SOLUTIONS VIA GALE DUALITY AND DEGREE THEORY

In this section, we present Theorem 3.7, which gives conditions on the Gale dual matrices B and D that guarantee the existence of at least one positive solution of the system (1.1). As we mentioned in the Introduction, our results are based on degree theory. Assume from now on that the matrix C is *uniform*, that is, that no maximal minor of C vanish, and the necessary condition (2.3) is satisfied.

Given an open set $U \subset \mathbb{R}^k$, a function $h \in \mathcal{C}^0(\bar{U}, \mathbb{R}^k)$ and $y \in \mathbb{R}^k \setminus h(\partial U)$, the symbol $\deg(h, U, y)$ denotes the Brouwer degree (which belongs to \mathbb{Z}) of h with respect to (U, y) . A main result in degree theory is that if $\deg(h, U, y) \neq 0$, then there exists at least one $x \in U$ such that $y = h(x)$. For background and the main properties about Brouwer degree, we refer to Section 2 in the Supplementary Information of [6] and Section 14.2 in [21].

We present the version of the Brouwer's theorem that we will use. This version is a particular case of Theorem 2 in the Supplementary Information of [6] (here we take W empty), and also appears in the proof of Lemma 2 of [7]. Recall that a vector $v \in \mathbb{R}^k$ *points inwards* a $U \subset \mathbb{R}^k$ at a boundary point $x \in \partial U$, if for small $\varepsilon > 0$ it holds that $x + \varepsilon v \in \bar{U}$.

Theorem 3.1 ([6, 7]). *Let $h : \mathbb{R}^k \rightarrow \mathbb{R}^k$ be a \mathcal{C}^1 -function. Let U be an open, nonempty, bounded and convex subset of \mathbb{R}^k such that*

- i) $h(x) \neq 0$ for any $x \in \partial U$.*
- ii) for every $x \in \partial U$, the vector $h(x)$ points inwards U at x .*

Then,

$$\deg(h, U, 0) = (-1)^k.$$

In particular, there exists a point x in U such that $h(x) = 0$. Moreover, assuming the zeros are nondegenerate, if there exists a zero $x^ \in U$ where the sign of the Jacobian at x^* is $(-1)^{k+1}$, then there are at least three zeros and always an odd number.*

Define the sign of any real number r by $\text{sign}(r) = +1, -1, 0$ according as $r > 0$, $r < 0$ or $r = 0$ respectively. The sign of any vector $r = (r_1, \dots, r_k) \in \mathbb{R}^k$ is then defined by $\text{sign}(r) = (\text{sign}(r_1), \dots, \text{sign}(r_k))$.

In view of Theorem 2.5, we look for the solutions of (2.8) in Δ_P . Plugging $y_0 = 1$ in (2.8) and clearing the denominators, we get a generalized

polynomial system in Δ_P on variables $y = (y_1, \dots, y_k)$:

$$(3.1) \quad g_j(y) = 0, \quad j = 1, \dots, k, \quad g_j(y) = \prod_{b_{ij} > 0} p_i(y)^{b_{ij}} - \prod_{b_{ij} < 0} p_i(y)^{-b_{ij}},$$

where

$$(3.2) \quad p_i(y) = \langle P_i, (1, y) \rangle.$$

We denote by g the Gale map:

$$(3.3) \quad g = (g_1, \dots, g_k): \mathbb{R}^k \rightarrow \mathbb{R}^k.$$

Definition 3.2. Given $C \in \mathbb{R}^{d \times n}$ uniform, we denote by $I_C \subset \{1, \dots, n\}$ the set of indexes corresponding to the minimal set of generators $\{P_i, i \in I_C\}$ of the polyhedral cone \mathcal{C}_P .

Note that the set I_C is unique since C is uniform and satisfies Condition (2.3), which implies that P_1, \dots, P_n lie in an open halfspace through the origin. The facets of \mathcal{C}_P° are supported on the orthogonal hyperplanes P_i^\perp for $i \in I_C$. Note that for any $i \in I_C$ the vector P_i is an inward normal vector of \mathcal{C}_P° at any point in the relative interior of the facet supported on P_i^\perp . It follows that the facets of the polytope Δ_P are supported on the hyperplanes $p_i(y) = 0$ for $i \in I_C$, and that (d_{i1}, \dots, d_{ik}) is an inward normal vector of Δ_P at any point in the relative interior of the facet supported on $p_i(y) = 0$. Note also that I_C depends on C and is independent of the choice of a Gale dual matrix D . In fact, it can be characterized by the following property: for any (z_1, \dots, z_n) in the kernel of C , we have $z_i > 0$ for $i = 1, \dots, n$ if and only if $z_i > 0$ for all $i \in I_C$.

Definition 3.3. For any $i \in I_C$ denote by F_i the facet of Δ_P supported on $p_i(y) = 0$, and set

$$F_L = \bigcap_{i \in L} F_i, \quad \text{for any } L \subset I_C.$$

Here by a face of Δ_P we mean a face of the closure of Δ_P . We denote by F_L° the relative interior of F_L . We set

$$\mathcal{F}(\Delta_P) = \{L \subset I_C : F_L \text{ is a face of } \Delta_P\}.$$

We want to compute the number of zeros of the Gale map g in (3.3) inside Δ_P . The sign of g along the boundary of Δ_P can sometimes be determined as follows.

Lemma 3.4. Let $A \in \mathbb{R}^{(d+1) \times n}$ as in (2.1), $C \in \mathbb{R}^{d \times n}$, and $B \in \mathbb{R}^{n \times k}$ and $D \in \mathbb{R}^{n \times (k+1)}$ Gale dual matrices of A and C respectively. Let $g = (g_1, \dots, g_k)$ the Gale map as in (3.3). Let $j \in \{1, \dots, k\}$.

- (1) Let F_i be any facet of Δ_P and let $x \in F_i^\circ$. If $b_{ij} \neq 0$, then $\text{sign}(g_j(x)) = -\text{sign}(b_{ij})$.
- (2) Let $L \in \mathcal{F}(\Delta_P)$ and $x \in F_L$. Assume that $\{b_{\ell j} : \ell \in L\} \neq \{0\}$.
 - (i) If there exists $\ell_0, \ell_1 \in L$ such that $b_{\ell_0 j} \cdot b_{\ell_1 j} < 0$, then $g_j(x) = 0$.
 - (ii) If $b_{\ell j} \geq 0$ for all $\ell \in L$ then $\text{sign}(g_j(x)) = -1$, and if $b_{\ell j} \leq 0$ for all $\ell \in L$ then $\text{sign}(g_j(x)) = +1$.

Corollary 3.5. *Let $A \in \mathbb{R}^{(d+1) \times n}$ as in (2.1), $C \in \mathbb{R}^{d \times n}$, and $B \in \mathbb{R}^{n \times k}$ and $D \in \mathbb{R}^{n \times (k+1)}$ Gale dual matrices of A and C respectively. Let g be the Gale map (3.3) associated to B and D . If $g(x) = 0$ and $x \in F_L$ (so $L \in \mathcal{F}(\Delta_P)$), then for $j = 1, \dots, k$, either $\{b_{\ell j} : \ell \in L\} = \{0\}$, or $\{b_{\ell j} : \ell \in L\}$ contains a (strictly) positive and a (strictly) negative element.*

In particular, if g vanishes in the relative interior of a facet F_ℓ then the ℓ -th row of B contains only zero entries.

Definition 3.6. *We say that a matrix M is weakly mixed if any column of M either has only zero entries, or contains a positive and a negative element.*

Otherwise said, a matrix M is not weakly mixed if and only if it has a non zero column whose entries are all either nonpositive, or nonnegative.

Given $B \in \mathbb{R}^{n \times k}$ and $L \subset \{1, \dots, n\}$, we denote by $B_L \in \mathbb{R}^{|L| \times k}$ the submatrix of B given by the rows with indexes in L . We now present the main result of this section.

Theorem 3.7. *Let $\mathcal{A} = \{a_1, \dots, a_n\} \subset \mathbb{R}^d$ and $C \in \mathbb{R}^{d \times n}$ uniform. Let $A \in \mathbb{R}^{(d+1) \times n}$ as in (2.1), and $B \in \mathbb{R}^{n \times k}$ and $D \in \mathbb{R}^{n \times (k+1)}$ Gale dual matrices of A and C respectively. Assume that $\mathbf{0} \in C^\circ$ and that Δ_P is a full dimensional bounded polytope.*

Assume furthermore that the following conditions hold:

- (1) *For any $L \in \mathcal{F}(\Delta_P)$ the submatrix $B_L \in \mathbb{R}^{|L| \times k}$ is not weakly mixed.*
- (2) *For any $i \in I_C$ the following holds:*
 - $b_{ij} \cdot d_{ij} \geq 0$ for $j = 1, \dots, k$,
 - there exists $j \in \{1, \dots, k\}$ such that $b_{ij} \cdot d_{ij} > 0$,
 - for all $j \in \{1, \dots, k\}$, if $b_{ij} = 0$ then $d_{ij} = 0$.

Then $n_{\mathcal{A}}(C) > 0$.

Proof. Since Δ_P is full dimensional and bounded, $(1, 0, \dots, 0) \in C_P$. By Theorem 2.5 it is sufficient to show that the Gale system (3.1) has at least one solution in Δ_P . First note that a vector $v \in \mathbb{R}^k$ points inwards Δ_P at a point y contained in the relative interior of a facet F_i ($i \in I_C$) if and only if $\langle (d_{i1}, \dots, d_{ik}), v \rangle \geq 0$. More generally $v \in \mathbb{R}^k$ points inwards Δ_P at a point y in the relative interior of a face F_L ($L \in \mathcal{F}(\Delta_P)$) if and only if $\langle (d_{\ell 1}, \dots, d_{\ell k}), v \rangle \geq 0$ for any $\ell \in L$, by a classical result of convex geometry. The assumption (1) ensures that g does not vanish at $\partial\Delta_P$, by Corollary 3.5. Condition (2) ensures that $-g$ points inwards Δ_P at each point x in the relative interior of any facet F_i . Then $-g$ also point inwards Δ_P at any point x in the relative interior of a face F_L . The result follows now from Theorem 3.1, taking $U = \Delta_P$ and $h = -g$. \square

Example 3.8. *Consider the codimension one case $k = 1$ (which is treated carefully in [2]). Then $B \in \mathbb{R}^{(d+2) \times 1}$ is a column matrix and its entries are the coefficients $\lambda_1, \dots, \lambda_{d+2}$ of a nontrivial affine relation on \mathcal{A} . Assume*

that A is uniform (equivalently, assume that \mathcal{A} is a circuit¹). Then, B has no zero entry. Assume moreover that C is uniform and that $\mathbf{0} \in C^\circ$. Then, there exists a Gale dual matrix D such that Δ_P is a bounded interval of \mathbb{R} . Moreover, there exists a vector $\delta \in \mathbb{R}^2$ such that $\langle P_i, \delta \rangle > 0$ for $i = 1, \dots, d+2$, where $P_1, \dots, P_{d+2} \in \mathbb{R}^2$ are the row vectors of D . Let $\alpha : \{1, \dots, d+2\} \rightarrow \{1, \dots, d+2\}$ be the bijection such that all determinants $\det(P_{\alpha_i}, P_{\alpha_{i+1}})$ for $i = 1, \dots, d+1$ are positive. Then, by Theorem 2.9 in [2], we have $n_{\mathcal{A}}(C) \leq \text{signvar}(\lambda_{\alpha_1}, \lambda_{\alpha_2}, \dots, \lambda_{\alpha_{d+2}})$ and moreover the difference is an even integer number (see Proposition 2.12 in [2]). The endpoints of the interval Δ_P are the roots of the two extremal polynomials p_{α_1} and $p_{\alpha_{d+2}}$, equivalently, $I_C = \{\alpha_1, \alpha_{d+2}\}$. Now the Gale polynomial $g = g_1 : \mathbb{R} \rightarrow \mathbb{R}$ points inwards Δ_P at its vertices if and only if $\lambda_{\alpha_1} \cdot \lambda_{\alpha_{d+2}} < 0$, which is equivalent to $\text{signvar}(\lambda_{\alpha_1}, \lambda_{\alpha_{d+2}}) = 1$. But, $\text{signvar}(\lambda_{\alpha_1}, \lambda_{\alpha_2}, \dots, \lambda_{\alpha_{d+2}})$ and $\text{signvar}(\lambda_{\alpha_1}, \lambda_{\alpha_{d+2}})$ have the same parity. Thus $g : \mathbb{R} \rightarrow \mathbb{R}$ points inwards Δ_P at its vertices if and only if $n_{\mathcal{A}}(C)$ is odd by Proposition 2.12 in [2]. Therefore, in the circuit case the sufficient condition to have $n_{\mathcal{A}}(C) > 0$ which is given by Theorem 3.7 is equivalent to $n_{\mathcal{A}}(C)$ being odd. Now, for any integer $d \geq 2$, it is not difficult to get examples of circuits $\mathcal{A} \subset \mathbb{R}^d$ and matrices C such that $n_{\mathcal{A}}(C)$ is odd and is different from 1. This shows that our sufficient condition does not imply $n_{\mathcal{A}}(C) = 1$ in general, and thus is not equivalent to the condition given in [13] ensuring that $n_{\mathcal{A}}(C) = 1$.

We now present an example with $k = d = 2$ to illustrate Theorem 3.7.

Example 3.9. Let $\mathcal{A} \subset \mathbb{Z}^2$ be the set of points $a_1 = (0, 4)$, $a_2 = (5, 4)$, $a_3 = (2, 8)$, $a_4 = (3, 0)$ and $a_5 = (3, 5)$. Consider the matrix of coefficients

$$C = \begin{pmatrix} -1 & -1 & 1 & 1 & 0 \\ -(3c+8) & -c & 2c+8 & 0 & 2 \end{pmatrix},$$

where $c \in \mathbb{R}$ is a parameter. The polynomial system of two polynomial equations and two variables x, y :

$$\begin{aligned} -y^4 - x^5y^4 + x^2y^8 + x^3 &= 0, \\ -(3c+8)y^4 - cx^5y^4 + (2c+8)x^2y^8 + 2x^3y^5 &= 0, \end{aligned}$$

has support \mathcal{A} and coefficient matrix C . Let A as in (2.1). Choose the following Gale dual matrices of A and C :

$$B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \\ 0 & 1 \\ -4 & -4 \end{pmatrix} \quad D = \left(\begin{array}{c|cc} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \\ c & -4 & -4 \end{array} \right)$$

Then $p_1(y) = 1 + y_1$, $p_2(y) = 1 + y_1 + 2y_2$, $p_3(y) = 1 + 2y_1 + y_2$, $p_4(y) = 1 + y_2$ and $p_5(y) = c - 4y_1 - 4y_2$. If $c > 0$, the convex polytope Δ_P is nonempty,

¹A point configuration \mathcal{A} of $d+2$ points is a *circuit* if any subset of $d+1$ points of \mathcal{A} is affinely independent.

bounded and it has five facets supported on the lines $p_i = 0$ for $i = 1, \dots, 5$, see Figure 1. Moreover, if $c > 0$, then the assumptions of Theorem 3.7 are satisfied and thus $n_{\mathcal{A}}(C) > 0$.

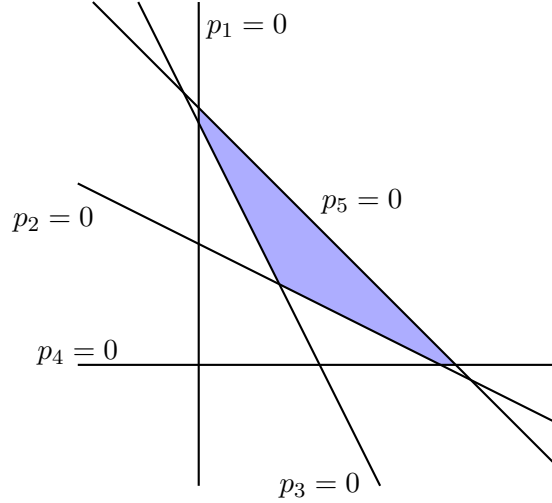


FIGURE 1. The polytope Δ_P of Example 3.9, with $c > 0$.

We use *Singular* [8], a free software of computer algebra system, to check what happens when we vary the value of $c > 0$.

```
LIB "signcond.lib";
ring r=0, (c,x,y,t), dp;
poly f1=-y^4-x^5*y^4+x^2*y^8+x^3;
poly f2=-(3*c+8)*y^4-c*x^5*y^4+(2*c+8)*x^2*y^8+2*x^3*y^5;
ideal i=f1,f2, diff(f1,x)*diff(f2,y)-diff(f1,y)*diff(f2,x),x*y*t-1;
ideal j=std(i);
ideal k=eliminate(j, x*y*t);
k;
k[1]=48c12+1280c11+12288c10+49152c9+65467c8-2560c7-24576c6-
98304c5-131078c4+1280c3+12288c2+49152c+65563
```

The roots of this last polynomial in c correspond to systems with a degenerate solution, and we can check that the only positive root of f_1, f_2 and their jacobian is 1. We check, again using *Singular* [8], with the library “*signcond.lib*” (implemented by E. Tobis, based on the algorithms described in [1]) that if we take for example $c = \frac{1}{2}$ ($c < 1$), the system has 3 positive solutions, and if we take $c = \frac{8}{7}$ ($c > 1$), the system has only 1 positive solution. We use the command `firstoct`, that computes the number of roots of a system in the first octant, that is, the positive roots.

```
LIB "signcond.lib";
ring r=0, (x,y), dp;
poly f1=-y^4-x^5*y^4+x^2*y^8+x^3;
poly f2=-(3*(1/2)+8)*y^4-(1/2)*x^5*y^4+(2*(1/2)+8)*x^2*y^8+2*x^3*y^5;
```

```

poly f3=-(3*(8/7)+8)*y^4-(8/7)*x^5*y^4+(2*(8/7)+8)*x^2*y^8+2*x^3*y^5;
ideal i2 = f1,f2;
ideal j2 = std(i2);
firstoct(j2);
3
ideal i3 = f1,f3;
ideal j3 = std(i3);
firstoct(j3);
1

```

For $c = \frac{1}{2}$, the condition in [13] to ensure exactly one positive solution is trivially not satisfied (as expected since the system has 3 positive solutions).

Observe that this procedure is symbolic and thus certified, as opposed to numerical algorithms. This computation of the number of positive solutions with the command `firstoct` works for moderately sized polynomial systems with coefficients in \mathbb{Q} or an algebraic extension of it. Our results are particularly useful to study families of polynomials.

4. DOMINATING MATRICES

In this section, we present some conditions on A and C that guarantee that the hypotheses of Theorem 3.7 are satisfied. Our main result is Theorem 4.6.

We first present conditions that guarantee that a matrix A admits a choice of a Gale dual matrix B , which satisfies condition (1) of Theorem 3.7, for any uniform matrix of coefficients C satisfying (2.3) (which means that it does not depend on I_C). When $A \in \mathbb{Z}^{(d+1) \times n}$ we will relate these conditions with complete intersection lattice ideals in Section 6.

We recall some definitions from [11], with the difference that we replace rows by columns and allow matrices with real entries.

Definition 4.1. *A vector is said to be mixed if contains a strictly positive and a strictly negative coordinate. More generally, a real matrix is called mixed if every column contains a strictly positive and a strictly negative entry. A real matrix is called dominating if it contains no square mixed submatrix. An empty matrix is mixed and also dominating.*

Observe that since a matrix A as in (2.1) has a row of ones, the columns of any Gale dual matrix B add up to zero, and thus B is always mixed. Also note that a mixed matrix is weakly mixed (see Definition 3.6), but the converse is not true in general as a weakly mixed matrix can also contain a column with only zero entries.

Lemma 4.2. *Assume that $A \in \mathbb{R}^{(d+1) \times n}$ is a uniform matrix. If $B \in \mathbb{R}^{n \times k}$ is a Gale dual matrix of A which is dominating, then condition (1) of Theorem 3.7 is satisfied for all $C \in \mathbb{R}^{d \times n}$ uniform satisfying $\mathbf{0} \in C^\circ$.*

Proof. Let $C \in \mathbb{R}^{d \times n}$ uniform, and take any Gale dual matrix $D \in \mathbb{R}^{n \times (k+1)}$ of C such that Δ_P is nonempty and bounded (which exists due to Lemma 2.3 and Corollary 2.2).

If $L = \{\ell\} \in \mathcal{F}(\Delta_P)$, then B_L weakly mixed means that it has only zeros, which forces the matrix A minus the ℓ -th column to have rank $< d + 1$. Consider $L \in \mathcal{F}(\Delta_P)$ such that $|L| \geq 2$. Note that $|L| \leq k$ since C is uniform. If B_L is weakly mixed then at least $k - |L| + 1$ columns of B_L contain only zero entries, for otherwise we would get a square submatrix of size $|L| \times |L|$ containing a positive and a negative coefficient in each column. But if $k - |L| + 1$ columns of B_L contain only zeros, then we get $k - |L| + 1$ linearly independent vectors in the kernel of the matrix $A_{\setminus L}$ obtained from A by removing the columns indexed by L . This forces the matrix $(A_{\setminus L})$ to have rank less than $d + 1$, which is a contradiction, since A is uniform. \square

The following results will be useful. The following propositions are only stated for matrices with integer matrices in [11], but clearly the proofs given in that paper also work for real matrices.

Proposition 4.3 ([11], Corollary 2.7 and 2.8). *If a real matrix is mixed dominating, then any nonzero linear combination of its columns is a mixed vector. In particular, its columns are linearly independent.*

Proposition 4.4 ([10], Proposition 4.1). *The left kernel of any mixed dominating real matrix contains a positive vector.*

We will also need the following Lemma.

Lemma 4.5. *Assume that $C \in \mathbb{R}^{d \times n}$ has maximal rank d and that $\mathbf{0} \in \mathcal{C}^\circ$. Let $\tilde{D} \in \mathbb{R}^{n \times k}$ be any matrix of maximal rank k such that $C\tilde{D} = 0$. Assume that*

$$(4.1) \quad \mathbf{0} \in \mathbb{R}_{>0}\tilde{P}_1 + \cdots + \mathbb{R}_{>0}\tilde{P}_n,$$

where $\tilde{P}_1, \dots, \tilde{P}_n$ are the row vectors of \tilde{D} . Then, there exists a positive vector D_0 in the kernel of C which does not belong to the linear span of the column vectors of \tilde{D} , and the matrix $D \in \mathbb{R}^{n \times (k+1)}$ obtained from \tilde{D} by adding D_0 as a first column vector is Gale dual to C and satisfies $(1, 0, \dots, 0) \in \mathcal{C}_P$.

Proof. By (4.1) there exists a positive vector in the left kernel of \tilde{D} , in other words, a row vector λ with positive coordinates² such that $\lambda \cdot \tilde{D} = (0, \dots, 0)$. Since $\mathbf{0} \in \mathcal{C}^\circ$ we have $\ker(C) \cap \mathbb{R}_{>0}^n \neq \emptyset$. Then, as $\ker(C)$ has dimension $k+1$ and \tilde{D} has rank k , there exists a vector $D_0 \in \ker(C) \cap \mathbb{R}_{>0}^n$ which does not belong to the linear span of the column vectors of \tilde{D} . The matrix $D \in \mathbb{R}^{n \times k}$ obtained from \tilde{D} by adding D_0 as a first column vector is Gale dual to C . Moreover, we have $\lambda \cdot D = (\lambda \cdot D_0, 0, \dots, 0)$ and thus $(1, 0, \dots, 0) \in \mathcal{C}_P$ since $\lambda \cdot D_0 > 0$ (here λ is a row vector, D_0 is a column vector so that $\lambda \cdot D_0$ is a real number, which is positive since λ and D_0 are positive vectors). \square

If $S \subset \mathbb{R}^n$ is a subspace, we denote $\text{sign}(S) = \{\text{sign}(v) : v \in S\}$. Recall that we denote the column vectors of a matrix B by B_1, \dots, B_k .

²In fact, it is sufficient that λ is a nonzero vector with only nonnegative coordinates.

Theorem 4.6. *Let $\mathcal{A} = \{a_1, \dots, a_n\} \subset \mathbb{R}^d$. Assume $A \in \mathbb{R}^{(d+1) \times n}$ as in (2.1), and $C \in \mathbb{R}^{d \times n}$ are uniform matrices. Suppose there exist a dominating Gale dual matrix $B \in \mathbb{R}^{n \times k}$ of A . Assume $\mathbf{0} \in C^\circ$ and $\text{sign}(B_j) \in \text{sign}(\ker(C))$ for each $j = 1, \dots, k$. Then, $n_{\mathcal{A}}(C) > 0$.*

Proof. As B is dominating and A, C are uniform, condition (1) of Theorem 3.7 is satisfied by Lemma 4.2. As $\text{sign}(B_j) \in \text{sign}(\ker(C))$ for $j = 1, \dots, k$, there exist vectors D_1, \dots, D_k in $\ker(C)$ such that $\text{sign}(D_j) = \text{sign}(B_j)$ for each $j = 1, \dots, k$. Consider the matrix \tilde{D} with column vectors D_1, \dots, D_k . Since B is mixed dominating (it is mixed since A contains a row of ones) the matrix \tilde{D} is mixed dominating and furthermore \tilde{D} has rank k by Proposition 4.3. Moreover, by Proposition 4.4, there is a positive vector in the left kernel of \tilde{D} . Then, condition (4.1) is satisfied, and thus by Lemma 4.5 and Corollary 2.2, there is a positive vector D_0 such that the matrix D with column vectors D_0, \dots, D_k is Gale dual to C and the associated polytope Δ_P is nonempty and bounded. By construction, condition (2) of Theorem 3.7 is also satisfied, and thus $n_{\mathcal{A}}(C) > 0$. \square

Recall that the support of a vector $v \in \mathbb{R}^n$ is defined as the set of its nonzero coordinates, and we denote it by $\text{supp}(v)$. Given a subspace $S \subset \mathbb{R}^n$, a circuit of S is a nonzero element $s \in S$ with minimal support (with respect to inclusion). Given a vector v , a circuit $s = (s_1, \dots, s_n)$ is said to be conformal to $v = (v_1, \dots, v_n)$ if for any index i in $\text{supp}(s)$, $\text{sign}(s_i) = \text{sign}(v_i)$. The next lemma shows that if A admits a Gale dual mixed dominating matrix, then there exist a choice of Gale mixed dominating matrix of A whose columns are circuits of $\ker(A)$. Note that all the circuits of $\ker(A)$ can be described in terms of vectors of maximal minors of A , and so they only depend on the *associated oriented matroid* of A .

Lemma 4.7. *Assume $A \in \mathbb{R}^{(d+1) \times n}$ as in (2.1). Suppose there exist a dominating Gale dual matrix $B \in \mathbb{R}^{n \times k}$ of A . Then, there exists a dominating Gale dual matrix $B' \in \mathbb{R}^{n \times k}$ of A such that every column of B' is a circuit of $\ker(A)$.*

Proof. It is a known result that every vector in $\ker(A)$ can be written as a nonnegative sum of circuits conformal to it (see [17]). In particular, for every vector in $\ker(A)$, there exists a circuit conformal to it. For each column B_i of B , $i = 1, \dots, k$, take a circuit B'_i of $\ker(A)$ such that B'_i is conformal to B_i . Now, we take B' the matrix with columns B'_1, \dots, B'_k . Every column of B' is a circuit of $\ker(A)$, B' is mixed since A has a row of ones, and is dominating because B'_i is conformal to B_i for each $i = 1, \dots, k$ and the matrix B is dominating. Since B' is mixed dominating, the columns of B' are linearly independent by Proposition 4.3, and then B' is a Gale dual matrix of A . \square

5. GEOMETRIC CONDITIONS ON A AND C

The main result of this section is Theorem 5.8, where we give geometric conditions on A and C that guarantee that the hypotheses of Theorem 3.7 are satisfied.

A characterization of matrices A admitting a mixed dominating Gale dual matrix B can be found in [10]. Recall that our definition of mixed dominating matrix differs from the one in [10] by replacing rows by columns. Here we present this result with our notation. We denote the convex hull of a point configuration \mathcal{A} by $\text{chull}(\mathcal{A})$. Recall also that we assume $n \geq d + 2$, so that \mathcal{A} cannot be the set of vertices of a d -dimensional simplex.

Theorem 5.1 ([10], Theorem 5.6). *Let $\mathcal{A} = \{a_1, \dots, a_n\} \subset \mathbb{R}^d$, with $n \geq d + 2$ and $A \in \mathbb{R}^{(d+1) \times n}$ as in (2.1). Then A admits a mixed dominating Gale dual matrix B if and only if \mathcal{A} can be written as a disjoint union $\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2$ such that*

- (1) *the polytopes $\text{chull}(\mathcal{A}_1)$ and $\text{chull}(\mathcal{A}_2)$ intersect in exactly one point,*
- (2) *the corresponding matrices A_1 and A_2 as in (2.1) of \mathcal{A}_1 and \mathcal{A}_2 respectively, admit mixed dominating Gale dual matrices, and*
- (3) *$\dim \text{chull}(\mathcal{A}) = \dim \text{chull}(\mathcal{A}_1) + \dim \text{chull}(\mathcal{A}_2)$.*

Moreover, we have:

Lemma 5.2 ([10], Corollary 5.7). *If A admits a mixed dominating Gale dual matrix B then $\text{chull}(\mathcal{A})$ has at most $2d$ vertices.*

In particular, by Lemma 4.2, we have:

Corollary 5.3. *Let $\mathcal{A} = \{a_1, \dots, a_n\} \subset \mathbb{R}^d$. Assume that A as in (2.1) is uniform and that $\mathcal{A} \subset \mathbb{R}^d$ can be decomposed as a disjoint union $\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2$ such that conditions (1), (2) and (3) of Theorem 5.1 hold. Then, there exists a Gale dual matrix $B \in \mathbb{Z}^{n \times k}$ of A such that condition (1) of Theorem 3.7 is satisfied.*

The following observation says that if we have a point configuration $\mathcal{A}_v \subset \mathbb{R}^d$ such that the corresponding matrix A_v admits a Gale dual mixed dominating matrix, then, for any other point configuration $\mathcal{A} \subset \mathbb{R}^d$ that contains \mathcal{A}_v and their convex hulls $\text{chull}(\mathcal{A}), \text{chull}(\mathcal{A}_v)$ coincide (that is, \mathcal{A} can be obtained from \mathcal{A}_v adding points inside the convex hull), the corresponding matrix A also admits a Gale dual mixed dominating matrix.

Lemma 5.4. *Let $\mathcal{A} = \{a_1, \dots, a_n\}$, $\mathcal{A}_v \subset \mathbb{R}^d$ be two point configurations such that $\mathcal{A}_v \subset \mathcal{A}$. Assume that the corresponding matrix $A \in \mathbb{R}^{(d+1) \times n}$ is uniform and that the following conditions hold:*

- (1) *$\text{chull}(\mathcal{A}) = \text{chull}(\mathcal{A}_v)$*
- (2) *The corresponding matrix $A_v \in \mathbb{R}^{(d+1) \times |\mathcal{A}_v|}$ has a Gale dual matrix B_v which is dominating.*

Then, there exists a mixed dominating Gale dual matrix $B \in \mathbb{Z}^{n \times k}$ of A and thus condition (1) of Theorem 3.7 is satisfied.

Note that Lemma 5.4 follows from applying several times Theorem 5.1 (taking one point from \mathcal{A}_v as \mathcal{A}_2), but we present a constructive proof.

Proof of Lemma 5.4. Without loss of generality, we may assume that $\mathcal{A}_v = \{a_1, \dots, a_s\}$, with $s \geq d$. For $i = s + 1, \dots, n$, there exists a subset \mathcal{A}_i of \mathcal{A}_v such that \mathcal{A}_i is the set of vertices of a d -simplex and a_i is contained in the interior of $\text{chull}(\mathcal{A}_i)$. Then there exists an affine relation on $\mathcal{A}_i \cup \{a_i\}$ where the coefficient of a_i is equal to one and the coefficients of the points of \mathcal{A}_i are all negative. Using the affine relations on \mathcal{A}_v given by the column vectors of B_v , we get k linearly independent vectors in the kernel of A which are the column vectors of a upper triangular block matrix B Gale dual to A of the following form:

$$B = \begin{pmatrix} B_v & R \\ 0 & I_{n-s} \end{pmatrix},$$

where R has only nonpositive entries (and at least two negative entries in each column) and I_{n-s} is the identity matrix of size $n - s$. Clearly, if B_v is dominating then B is dominating and thus by Lemma 4.2 the first item of Theorem 3.7 is satisfied. \square

We have the following corollary.

Corollary 5.5. *Let $\mathcal{A} = \{a_1, \dots, a_n\}$, $\mathcal{A}_v \subset \mathbb{R}^d$ be two point configurations such that $\mathcal{A}_v \subset \mathcal{A}$ and $\text{chull}(\mathcal{A}) = \text{chull}(\mathcal{A}_v)$. Assume that the corresponding matrix A is uniform and that \mathcal{A}_v is either the set of vertices of d -simplex, or a circuit in \mathbb{R}^d . Then, there exists a Gale dual matrix $B \in \mathbb{Z}^{n \times k}$ of A such that condition (1) of Theorem 3.7 is satisfied.*

Consider \mathcal{A} and the point configuration $\mathcal{C} = \{C_1, \dots, C_n\}$ given by the columns of the coefficient matrix C . Consider the $(d + 1) \times n$ -matrix

$$\bar{C} = \begin{pmatrix} 1 & \cdots & 1 \\ & & C \end{pmatrix}$$

and assume that A and \bar{C} are uniform. Given a subset $J \subset \{1, \dots, n\}$, we denote $\mathcal{A}_J = \{a_j : j \in J\}$.

Definition 5.6. *Given a subset $I \subset \{1, \dots, n\}$, we say that A and C are I -compatible if the following conditions hold:*

- (1) *The corresponding matrices A_I and \bar{C}_I admit Gale dual matrices which are mixed dominating and have the same sign pattern,*
- (2) *$\text{chull}(\mathcal{A}_I) = \text{chull}(\mathcal{A})$ and $\text{chull}(\mathcal{C}_I) = \text{chull}(\mathcal{C})$,*
- (3) *For each $j \notin I$, there exist $J \subset I$, with $|J| = d + 1$, such that $a_j \in \text{chull}(\mathcal{A}_J)$ and $C_j \in \text{chull}(\mathcal{C}_J)$.*

The condition that A and C are I -compatible can be translated in terms of signs of maximal minors of A and \bar{C} . Also note that the configurations \mathcal{A} and \mathcal{C} may have different matroids. The following Example 5.7 shows two I -compatible configurations with different matroids.

Example 5.7. We show in Figure 5 an example of two point configurations, $\mathcal{A} = \{a_1, \dots, a_6\}$ and $\mathcal{C} = \{C_1, \dots, C_6\}$ with $d = 2$ and $k = 3$, which are I -compatible, for $I = \{1, 2, 3, 4\}$. In this case $a_5 \in \text{ch}(\mathcal{A}_{I_5})$, $C_5 \in \text{ch}(\mathcal{C}_{I_5})$ for $I_5 = \{1, 3, 4\}$ and $a_6 \in \text{ch}(\mathcal{A}_{I_6})$, $C_6 \in \text{ch}(\mathcal{C}_{I_6})$ for $I_6 = \{1, 2, 3\}$.

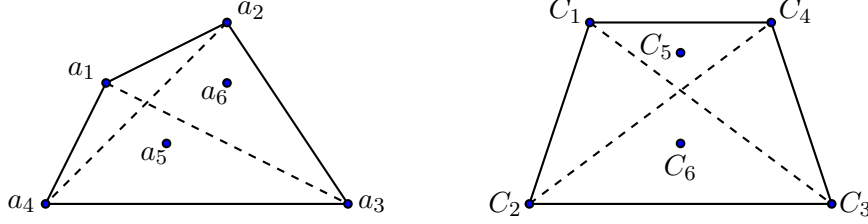


FIGURE 2. A and C are I -compatible for $I = \{1, 2, 3, 4\}$.

We have the following result:

Theorem 5.8. Assume that A , C and \bar{C} are uniform. Suppose $\mathbf{0} \in \mathcal{C}^\circ$, and there exists $I \subset \{1, \dots, n\}$ such that A and C are I -compatible. Then, $n_{\mathcal{A}}(C) > 0$.

Proof. Let B_I be a Gale dual matrix of A_I as in Condition (1) of Definition 5.6. As $a_j \in \text{chull}(\mathcal{A}_I)$ for each $j \notin I$, we can use Lemma 5.4. We construct a dominating matrix B , using the matrix B_I and using for each a_j , $j \notin I$, the affine relation given by the circuit $a_j \cup \mathcal{A}_J$, with J as in Condition (3) of Definition 5.6, to obtain a vector in the kernel of A as in the proof of Lemma 5.4. Conditions (1) and (3) of the definition of being I -compatible mean that there exist k vectors in the kernel of \bar{C} with the same sign patterns as the columns of the constructed B , and these k vectors are linearly independent because they form a mixed dominating matrix (Proposition 4.3). We have that $\ker(\bar{C}) \subset \ker(C)$, and $\text{sign}(B_1), \dots, \text{sign}(B_k) \in \text{sign}(\ker(C))$. We can apply Theorem 4.6 and then $n_{\mathcal{A}}(C) > 0$. \square

Remark 5.9. When $|I| = d+2$, condition (1) in Definition 5.6 can be translated in terms of signatures of circuits. Given a circuit $\mathcal{U} = \{u_1, \dots, u_{d+2}\} \subset \mathbb{R}^d$, and a nonzero affine relation $\lambda \in \mathbb{R}^{d+2}$ among the u_i , we call $\Lambda_+ = \{i \in \{1, \dots, d+2\} : \lambda_i > 0\}$ and $\Lambda_- = \{i \in \{1, \dots, d+2\} : \lambda_i < 0\}$. The pair (Λ_+, Λ_-) is usually called a signature of \mathcal{U} . As \mathcal{U} is a circuit, the pairs (Λ_+, Λ_-) and (Λ_-, Λ_+) are the two possible signatures. Then, we consider the (unordered) signature partition $\mathcal{S}(\mathcal{U}) = \{\Lambda_+, \Lambda_-\}$. Given a subset $I \subset \{1, \dots, n\}$, with $|I| = d+2$, and A and C uniform, condition (1) in Definition 5.6 is equivalent to the following condition:

$$(1') \mathcal{S}(\mathcal{A}_I) = \mathcal{S}(\mathcal{C}_I)$$

In this case, condition (3) in Definition 5.6 implies that $\{a_j\} \cup \mathcal{A}_J$ and $\{C_j\} \cup \mathcal{C}_J$ have the same signature partition, which is $(d+1, 1)$.

5.1. **The case $k = 2$.** The point configurations such that the corresponding matrix admits a Gale dual which is dominating are limited. So, if we are not in this case, checking condition (1) of Theorem 3.7 involves knowing the incidences of the facets of the polytope Δ_P . However, we now show that in case A has codimension $k = 2$, there always exists a choice of Gale dual matrix B such that we can conclude that $n_{\mathcal{A}}(C) > 0$ with the help of Theorem 3.7 without checking Condition (1) as it becomes a consequence of the other conditions.

Lemma 5.10. *Assume that A and C are uniform matrices and $k = 2$. Suppose that $\mathbf{0} \in C^\circ$. Then there exists a matrix B Gale dual to A such that for any matrix D Gale dual to C for which Δ_P is nonempty, bounded and the condition (2) of Theorem 3.7 is satisfied, condition (1) of Theorem 3.7 is satisfied, and thus $n_{\mathcal{A}}(C) > 0$.*

Proof. Let B be any Gale dual matrix of A with row vectors b_1, \dots, b_n . Choose any $i_2 \in I_C$. Then there exist $i_1 \in I_C$ such that the cone $\mathbb{R}_{>0}b_{i_1} + \mathbb{R}_{<0}b_{i_2}$ does not contain vectors b_i with $i \in I_C$. Note that the latter cone has dimension two since A is uniform (which implies that B is uniform as well). There exists a matrix R of rank two such that $B_{\{i_1, i_2\}} \cdot R = I_2$ (if we assume that A, B have integer entries, then there exists an integer matrix R of rank two such that $B_{\{i_1, i_2\}} \cdot R = a \cdot I_2$ where $a = |\det(B_{\{i, j\}})|$). Consider the matrix $B' = B \cdot R$, with row vectors b'_1, \dots, b'_n . Then B' is a Gale dual matrix to A such that $b'_{i_1} = (1, 0)$, $b'_{i_2} = (0, 1)$ and the open quadrant $\mathbb{R}_{>0} \times \mathbb{R}_{<0}$ does not contain any vector b'_i with $i \in I_C$. Note also that if $i \in I_C$ and $i \neq i_1, i_2$ then both coordinates of b'_i are nonzero for otherwise this would give a vanishing maximal minor of B' . In particular, we get $b'_i \neq 0$, and thus b'_i is not weakly mixed, for all $i \in I_C$. Suppose now that there are two distinct vectors b'_i and b'_j with $i, j \in I_C$ such that the submatrix $B_{\{i, j\}}$ is weakly mixed. Then these row vectors lie in opposite quadrants of \mathbb{R}^2 and these quadrants should be $\mathbb{R}_{>0}^2$ and $\mathbb{R}_{<0}^2$. But then the cone $\mathbb{R}_{>0}b'_i + \mathbb{R}_{>0}b'_j$ contains either $b'_{i_1} = (1, 0)$ or $b'_{i_2} = (0, 1)$, and thus $\{i, j\} \notin F_L$. \square

Given a vector $v \in \mathbb{R}^n$ and $I \subset \{1, \dots, n\}$ we denote by $v_I \in \mathbb{R}^{|I|}$ the vector obtained from v after removing the coordinates with indexes that do not belong to I . Given a set $S \subset \mathbb{R}^n$, we denote $S_I = \{v_I : v \in S\}$.

Consider the four open quadrants of \mathbb{R}^2 numbered from 1 to 4, where the signs of the two coordinates are $(+, +)$, $(-, +)$, $(-, -)$, and $(+, -)$ for the first, second, third and fourth quadrant respectively. In case that there exists a Gale dual matrix B with rows in each of the quadrants, we have the following result.

Lemma 5.11. *Given $A \in \mathbb{R}^{(d+1) \times (d+3)}$ uniform, let $B \in \mathbb{R}^{(d+3) \times 2}$ be a Gale dual matrix of A . Suppose there exists rows of B , b_{i_j} , with $1 \leq j \leq 4$, such that b_{i_j} lies in the j -th open quadrant of \mathbb{R}^2 . Let $C \in \mathbb{R}^{d \times n}$ uniform. Suppose that $\mathbf{0} \in C^\circ$. Assume moreover that given a Gale dual matrix of C , the row*

vectors P_{i_1}, \dots, P_{i_4} define normals to facets of the closure of the cone \mathcal{C}_P^\vee in (2.5). If $\text{sign}((B_j)_{I_C}) \in \text{sign}((\ker(C))_{I_C})$ for $j = 1, 2$, then $n_{\mathcal{A}}(C) > 0$.

Note that the condition that P_{i_1}, \dots, P_{i_4} define normals to facets of the associated cone \mathcal{C}_P^\vee is independent of the choice of Gale dual matrix of C .

Proof. As $\text{sign}((B_j)_{I_C}) \in \text{sign}((\ker(C))_{I_C})$ for $j = 1, 2$, there are vectors $D_1, D_2 \in \ker(C)$ such that $\text{sign}((D_j)_{I_C}) = \text{sign}((B_j)_{I_C})$ for each $j = 1, 2$. We can assume that D_1 and D_2 are linearly independent. If not, the zero coordinates of D_1 and D_2 (which are at most two, since C is uniform) have to be the same. That is, $(D_1)_j = 0$ if and only if $(D_2)_j = 0$ (otherwise, they cannot be linearly dependent). Suppose that $(D_1)_j = (D_2)_j = 0$ for certain j . If $j \in I_C$, then $(B_1)_j = (B_2)_j = 0$, but since A is uniform, B_1 and B_2 have at most one zero coordinate, and then, B_1 and B_2 are scalar multiples of each other, a contradiction. Then if $(D_1)_j = (D_2)_j = 0$, $j \notin I_C$. We take a vector v in $\ker(C)$ such that D_1 and v are linearly independent. Then we can take $D'_2 = D_2 + \lambda v$, with λ small enough such that $\text{sign}((D'_2)_{I_C}) = \text{sign}((B_2)_{I_C})$.

So, we can suppose that D_1 and D_2 are linearly independent. Consider the matrix \tilde{D} with column vectors D_1 and D_2 . We have that $\mathbf{0}$ belongs to the open cone generated by the rows of \tilde{D} , because the i_j -th row of \tilde{D} belongs to the j -th open quadrant, then Condition 4.1 of Lemma 4.5 is satisfied. As $\mathbf{0} \in \mathcal{C}^\circ$, by Lemma 4.5 and Corollary 2.1, there exists a positive vector D_0 such that the matrix obtained from \tilde{D} by adding D_0 as a first column vector is Gale dual to C and the associated polytope Δ_P is nonempty and bounded. Also note that Δ_P has a facet for each row vector i_j of \tilde{D} , each one in the j -quadrant of \mathbb{R}^2 , for $j = 1, \dots, 4$. Then, if we have a 2×2 mixed submatrix of B , it does not correspond to a submatrix B_L , with $L \in \mathcal{F}(\Delta_P)$ (and any row of \tilde{D} corresponding to $i \in I_C$ is not equal to zero). Then, all the conditions of Theorem 3.7 are satisfied and $n_{\mathcal{A}}(C) > 0$. □

6. ALGEBRAIC CONDITIONS AND REAL SOLUTIONS OF INTEGER CONFIGURATIONS

In this section we will consider integer configurations \mathcal{A} and thus, integer matrices A . Interestingly, in Corollary 6.2 we will relate Lemma 4.2 with known algebraic results in the study of toric ideals [20, Ch.4]. Indeed, we summarize in § 6.1 some known algebraic results that show the existence of a mixed dominating Gale dual matrix is equivalent to the fact that there is a full dimensional sublattice of the integer kernel $\ker_{\mathbb{Z}}(A)$ whose associated lattice ideal (6.1) is a complete intersection. This means that it can be generated by as many polynomials as the codimension of its zero set. In the opposite spectrum, an ideal is not Cohen-Macaulay when its homological behavior is complicated (see for instance [9]). When $k = 2$, we also consider lattice ideals which are not Cohen-Macaulay. Proposition 6.4 shows how to deal with this bad algebraic case. Also, in § 6.2 we naturally extend the

search for positive solutions to the search for real solutions with nonzero coordinates.

6.1. Algebraic conditions. A polynomial ideal is called *binomial* if it can be generated with polynomials with at most two terms. A subgroup $\mathcal{L} \subset \mathbb{Z}^n$ is called a *lattice*. We associate to a lattice \mathcal{L} the following binomial ideal:

$$(6.1) \quad I_{\mathcal{L}} = \langle x^{u^+} - x^{u^-} : u \in \mathcal{L} \rangle \subset \mathbb{R}[x_1, \dots, x_n],$$

where $u = u^+ - u^-$ is the decomposition in positive and negative components. For example, if $u = (1, -2, 1, 0) \in \mathbb{Z}^4$, then $x^{u^+} - x^{u^-} = x_1 x_3 - x_2^2$.

Given a configuration $\mathcal{A} = \{a_1, \dots, a_n\} \subset \mathbb{Z}^d$ of integral points, and the associated matrix $A \in \mathbb{Z}^{(d+1) \times n}$, let $B \in \mathbb{Z}^{n \times k}$ a Gale dual matrix of A , and denote by B_1, \dots, B_k the columns vectors of B . Note that $\{B_1, \dots, B_k\}$ is a \mathbb{Q} -basis of $\ker_{\mathbb{Z}}(A)$, but it is not necessarily a \mathbb{Z} -basis unless the greatest common divisor of the maximal minors of B is equal to 1. When this is the case, we will say that B is a \mathbb{Z} -Gale dual of A . We associate to any choice of Gale dual B of A the following lattice:

$$\mathcal{L}_B = \mathbb{Z}B = \mathbb{Z}B_1 \oplus \dots \oplus \mathbb{Z}B_k \subset \mathbb{Z}^n,$$

and its corresponding lattice ideal $I_{\mathcal{L}_B}$. In particular, when $\mathcal{L}_B = \ker_{\mathbb{Z}}(A)$, then the lattice ideal $I_{\mathcal{L}_B}$ is the so called *toric ideal* I_A . We have the following known result from [11]. See also Theorem 2.1 of [12], where the notation is similar to the notation of this paper.

Theorem 6.1 ([11], Theorem 2.9). *The lattice ideal $I_{\mathcal{L}_B}$ is a complete intersection if and only if $\mathcal{L}_B = \mathcal{L}_{B'}$ for some dominating matrix $B' \in \mathbb{Z}^{n \times k}$. In this case, $I_{\mathcal{L}_B} = \langle x^{u^+} - x^{u^-} : u \text{ is a column of } B' \rangle$.*

The following result is a direct consequence of Lemma 4.2 and Theorem 6.1.

Corollary 6.2. *If $A \in \mathbb{Z}^{(d+1) \times n}$ and $C \in \mathbb{R}^{d \times n}$ are uniform matrices and $B \in \mathbb{Z}^{n \times k}$ is a Gale dual matrix of A such that the lattice ideal $I_{\mathcal{L}_B}$ is a complete intersection, then there exists a Gale dual matrix $B' \in \mathbb{Z}^{n \times k}$ of A which satisfies the condition (1) of Theorem 3.7.*

Given A , let $B \in \mathbb{Z}^{n \times k}$ a Gale dual matrix of A , and consider the lattice $\mathcal{L}_B = \mathbb{Z}B$. The set of rows of B , $\{b_1, \dots, b_n\} \subset \mathbb{Z}^k$ is called a Gale diagram of \mathcal{L}_B . Any other \mathbb{Z} -basis for \mathcal{L}_B yields a Gale diagram, which means that Gale diagrams are unique up to an invertible matrix with integer coefficients.

The following proposition from [16] relates Gale diagrams with algebraic properties of the lattice ideal \mathcal{L}_B when $k = 2$:

Proposition 6.3 ([16], Proposition 4.1). *Given $A \in \mathbb{Z}^{(d+1) \times (d+3)}$, let $B \in \mathbb{Z}^{n \times 2}$ be a Gale dual matrix of A . The lattice ideal $I_{\mathcal{L}_B}$ is not Cohen-Macaulay if and only if it has a Gale diagram which intersects all the four open quadrants of \mathbb{R}^2 .*

The following result follows from Proposition 6.3 and Lemma 5.11.

Proposition 6.4. *Given $A \in \mathbb{Z}^{(d+1) \times (d+3)}$ uniform, let $B \in \mathbb{Z}^{n \times 2}$ be a Gale dual matrix of A . Suppose that the lattice ideal $I_{\mathcal{L}_B}$ is not Cohen-Macaulay and let B' be any other Gale Dual matrix of A such that the columns B'_1, B'_2 of B' form a \mathbb{Z} -basis of \mathcal{L}_B and such that the corresponding Gale diagram $\{b'_1, \dots, b'_n\}$ intersects all the four open quadrants of \mathbb{R}^2 . Let b'_{i_j} , with $1 \leq j \leq 4$, be rows of B' each lying in the interior of a different open quadrant in \mathbb{R}^2 . Let $C \in \mathbb{R}^{d \times n}$ uniform satisfying $\mathbf{0} \in C^\circ$. Assume moreover that given a Gale dual matrix of C , the row vectors P_{i_1}, \dots, P_{i_4} define normals to facets of the closure of the cone C_P^\vee in (2.5).*

Then, if $\text{sign}((B'_j)_{I_C}) \in \text{sign}((\ker(C))_{I_C})$ for $j = 1, 2$, then $n_{\mathcal{A}}(C) > 0$.

6.2. Real solutions. When A has integer entries, (1.1) is a system of Laurent polynomials with real coefficients, which are defined over the real torus $(\mathbb{R}^*)^d$. In this subsection, we are interested on the existence of real solutions of (1.1) with nonzero coordinates for integer matrices A of exponents. Our main result is Theorem 6.10. We will only consider matrices B which are \mathbb{Z} -Gale dual to A , whose columns generate $\ker_{\mathbb{Z}}(A)$ over \mathbb{Z} .

Given any $s = (s_1, \dots, s_d) \in \mathbb{Z}^d$, denote by \mathbb{R}_s^d the orthant

$$\mathbb{R}_s^d = \{x \in \mathbb{R}^d : (-1)^{s_i} x_i > 0, i = 1, \dots, d\}.$$

In particular, $\mathbb{R}_s^d = \mathbb{R}_{>0}^d$ if $s \in 2\mathbb{Z}^d$. Let $x \in (\mathbb{R}^*)^d$ be a solution of (1.1). Then $x \in \mathbb{R}_s^d$ for some $s \in \mathbb{Z}^d$ (which is unique up to adding a vector in $2\mathbb{Z}^d$). Setting $z_i = (-1)^{s_i} x_i$, we get that $z = (z_1, \dots, z_d)$ is a positive solution of the system with exponent matrix A and coefficient matrix C_s defined by $(C_s)_{ij} = (-1)^{\langle s, a_j \rangle} c_{ij}$. Moreover, if D is a matrix Gale dual to C , then the matrix D_s defined by $(D_s)_{ij} = (-1)^{\langle s, a_i \rangle} d_{ij}$ is Gale dual to C_s . Denote by $P_{i,s}$ the i -th row vector of D_s . Thus $P_{i,0} = P_i$ (i -th row of D) and $P_{i,s} = (-1)^{\langle s, a_i \rangle} P_i$, $i = 1, \dots, n$. Denote by $C_{P_s}^\vee$ the positive cone generated by $P_{i,s}$ for $i = 1, \dots, n$.

Let \mathcal{M}_P denote the complement in \mathbb{R}^{k+1} of the hyperplane arrangement given by the hyperplanes $\{y \in \mathbb{R}^{k+1} : \langle P_i, y \rangle = 0\}$, $i = 1, \dots, n$. For any $\varepsilon \in \mathbb{Z}^n$ denote by C_ε^\vee the connected component of \mathcal{M}_P defined by

$$C_\varepsilon^\vee = \{y \in \mathbb{R}^{k+1} : (-1)^{\varepsilon_i} \langle P_i, y \rangle > 0, i = 1, \dots, n\}.$$

Note that $C_0^\vee = C_P^\vee$.

Write A' for the matrix with column vectors a_1, \dots, a_n (A' is obtained by removing the first row of A). It is convenient to introduce the map $\psi : \mathbb{Z}^d \rightarrow \mathbb{Z}^{1 \times n}$ defined by $\psi(s) = s \cdot A'$ (here we see $s \in \mathbb{Z}^d$ as a row vector, i.e, as an element of $\mathbb{Z}^{1 \times d}$). Then, for any integer vector $b \in \ker(A)$ we have:

$$\prod_{i=1}^n \langle P_{i,s}, y \rangle^{b_i} = (-1)^{\langle s, A'b \rangle} \prod_{i=1}^n \langle P_i, y \rangle^{b_i} = \prod_{i=1}^n \langle P_i, y \rangle^{b_i}.$$

Thus, applying Theorem 2.5 to the system with coefficient matrix C_s and exponent matrix A , we obtain that the real solutions of (1.1) contained in the orthant \mathbb{R}_s^d are in bijection with the solutions of (2.8) in the quotient

$\mathbb{P}\mathcal{C}_{\psi(s)}^\nu$ of the open cone $\mathcal{C}_{\psi(s)}^\nu$ by the equivalence relation \sim ($y \sim y'$ if and only if there exists $\alpha > 0$ such that $y = \alpha y'$), defined in Section 2.

We have proved the following result:

Proposition 6.5. *For any $s \in \mathbb{Z}^d$, there is a bijection between the real solutions of (1.1) contained in \mathbb{R}_s^d and the solutions of (2.8) in $\mathbb{P}\mathcal{C}_{\psi(s)}^\nu$, which induces a bijection between the solutions of (1.1) in \mathbb{R}_s^d and the solutions of (2.8) in $\Delta_{P_s} = \mathcal{C}_{\psi(s)}^\nu \cap \{y_0 = 1\}$ when $(1, 0, \dots, 0)$ lies in the closure of the cone \mathcal{C}_{P_s} .*

If M is any matrix or vector with integer entries, we denote by $[M]_2$ the matrix or vector with coefficients in the field $\mathbb{Z}/2\mathbb{Z}$ obtained by taking the image of each entry by the quotient map $\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$. Note that the following relation between the ranks holds: $\text{rk}([A]_2) = \text{rk}([A']_2)$ if $[(1, 1, \dots, 1)]_2$ belongs to the row span of $[A']_2$ and $\text{rk}([A]_2) = \text{rk}([A']_2) + 1$ otherwise. The following result is straightforward.

Lemma 6.6. *For any $s, s' \in \mathbb{Z}^d$, we have $\mathcal{C}_{\psi(s)}^\nu = \mathcal{C}_{\psi(s')}^\nu$ if and only if $[s' - s]_2$ belongs to the left kernel of $[A']_2$. For each $s \in \mathbb{Z}^d$, there are $2^{d - \text{rk}([A']_2)}$ distinct orthants \mathbb{R}_s^d such that $\mathcal{C}_{\psi(s)}^\nu = \mathcal{C}_{\psi(s')}^\nu$. Finally, the image of \mathbb{Z}^d via the map $s \mapsto \mathcal{C}_{\psi(s)}^\nu$ consists of $2^{\text{rk}([A']_2)}$ connected components of \mathcal{M}_P .*

Recall that since A contains a row of ones, each polynomial in (2.8) is homogeneous of degree 0, which implies the following fact.

Lemma 6.7. *For any $\varepsilon \in \mathbb{Z}^n$, the map $y \mapsto -y$ induces a bijection between the solutions of (2.8) in $\mathcal{C}_\varepsilon^\nu$ and the solutions of (2.8) in $\mathcal{C}_{\varepsilon + (1, 1, \dots, 1)}^\nu$.*

Let \mathcal{A} be a configuration of n integer points in \mathbb{Z}^d . Consider the associated matrix $A \in \mathbb{Z}^{(d+1) \times n}$ as in (2.1) and $A' \in \mathbb{Z}^{d \times n}$ the matrix obtained by removing the first row of A as before. Choose a \mathbb{Z} -Gale dual matrix B of A and consider the Gale dual system (2.8) it defines for a given Gale dual matrix D of a matrix $C \in \mathbb{R}^{d \times n}$ of full rank.

Proposition 6.8. *With the previous notations, the following holds:*

- (1) *Assume $\text{rk}([A]_2) = \text{rk}([A']_2)$ and let $\varepsilon \in \mathbb{Z}^n$. If (2.8) has a solution in $\mathcal{C}_\varepsilon^\nu$ then there exists $s \in \mathbb{Z}^d$ such that $[\varepsilon]_2 = [\psi(s)]_2$.*
- (2) *Assume now $\text{rk}([A]_2) > \text{rk}([A']_2)$ and let $\varepsilon \in \mathbb{Z}^n$. If (2.8) has a solution in $\mathcal{C}_\varepsilon^\nu$ then either there exists $s \in \mathbb{Z}^d$ such that $[\varepsilon]_2 = [\psi(s)]_2$, or there exists $s \in \mathbb{Z}^d$ such that $[\varepsilon]_2 + [(1, 1, \dots, 1)]_2 = [\psi(s)]_2$. Moreover, there do not exist $s, s' \in \mathbb{Z}^d$ such that $[\psi(s')]_2 = [(1, 1, \dots, 1)]_2 + [\psi(s)]_2$, so that only one of the two previous cases occurs.*

Proof. Let y be a solution of (2.8) in $\mathcal{C}_\varepsilon^\nu$ and let b be any element of $\ker(A) \cap \mathbb{Z}^n$. Writing b as an integer linear combination of the column vectors of B and using (2.8), we get $\prod_{i=1}^n \langle P_i, y \rangle^{b_i} = 1$. Then, using $y \in$

$\mathcal{C}_\varepsilon^\nu$ we obtain that $\sum_{i=1}^n \varepsilon_i b_i$ is an even integer number. The fact that the column vectors of B form a basis of $\ker(A) \cap \mathbb{Z}^n$ implies that the column vectors of $[B]_2$ form a basis of $\ker([A]_2)$ (in other words $[B]_2$ is Gale dual to $[A]_2$). Then, $\sum_{i=1}^n \varepsilon_i b_i \in 2\mathbb{Z}$ for any $b \in \ker(A) \cap \mathbb{Z}^n$ is equivalent to the fact that $[\varepsilon]_2$ belongs to the left kernel of $[B]$. This left kernel is the image of the map $\mathbb{Z}^{d+1} \rightarrow \mathbb{Z}^{1 \times n}$ sending (s_0, s_1, \dots, s_d) to $[(s_0, s_1, \dots, s_d)]_2 \cdot [A]_2 = [s_0(1, 1, \dots, 1)]_2 + [\psi(s)]_2$, where $s = (s_1, \dots, s_d)$. The image of this map coincides with the image of the map $s \mapsto [\psi(s)]_2$ precisely when $\text{rk}([A]_2) = \text{rk}([A']_2)$, which proves item 1). To finish it remains to see that if $\text{rk}([A]_2) > \text{rk}([A']_2)$ there do not exist distinct $s, s' \in \mathbb{Z}^{1 \times d}$ such that $[\psi(s')]_2 = [(1, 1, \dots, 1) + \psi(s)]_2$ for otherwise $[(1, \dots, 1)]_2$ would belong to the row span of $[A']_2$. \square

Example 6.9. *If $a_1, \dots, a_n \in 2\mathbb{Z}^d$, then $\text{rk}([A']_2) = 0$ and $\text{rk}([A]_2) = 1$. Moreover, the number of real solutions of (1.1) is 2^d times its number of positive solutions, the latter number being equal to the number of solutions of (2.8) in $\mathcal{C}_0^\nu = \mathcal{C}_P^\nu$ by Theorem 2.5.*

As a direct consequence of Proposition 6.5, Proposition 6.8, and Lemma 6.6, we get the following result.

Theorem 6.10. *There exists a solution of (1.1) in $(\mathbb{R}^*)^d$ if and only if there exists a solution of (2.8) in the complement \mathcal{M}_P of the hyperplane arrangement defined by P_1, \dots, P_n . Moreover,*

- (1) *For any $s \in \mathbb{Z}^d$ there is a bijection between the solutions of (1.1) in \mathbb{R}_s^d and the solutions of (2.8) in $\mathbb{P}\mathcal{C}_\varepsilon^\nu$, with $[\varepsilon]_2 = [\psi(s)]_2$.*
- (2) *There are at most $2^{\text{rk}[A]_2}$ connected components $\mathcal{C}_\varepsilon^\nu$ of \mathcal{M}_P where (2.8) has a solution.*

Given A and C and a choice of Gale dual matrices B, D , we saw in the proof of Theorem 3.7, that under the hypotheses of the theorem, it follows from Theorem 2.5 that $n_{\mathcal{A}}(C) > 0$ is indeed equivalent to the existence of a solution to (3.1) in Δ_P . In the previous sections, we have given different sufficient conditions on D and B such that system (3.1) has at least one solution in Δ_P . When A has integer entries it is then enough to check if these sufficient conditions are satisfied by B and any matrix D_ε obtained by multiplying the i -th row of D by $(-1)^{\varepsilon_i}$ for some $\varepsilon \in \mathbb{Z}^n$. In this case, (1.1) has at least one solution in $(\mathbb{R}^*)^d$ by Theorem 6.10.

REFERENCES

- [1] S. Basu, R. Pollack and M.-F. Roy. *Algorithms in Real Algebraic Geometry*, volume 10 of *Algorithms and Computation in Mathematics*. Springer-Verlag Berlin Heidelberg, 2nd. ed., 2016.
- [2] F. Bihan and A. Dickenstein. *Descartes' Rule of Signs for Polynomial Systems supported on Circuits*, *Int. Math. Res. Notices*, Vol. 2017 (22), 6867-6893, 2016.
- [3] F. Bihan, F. Santos and P.-J. Spaenlehauer. *A polyhedral method for sparse systems with many positive solutions*, *SIAM J. Appl. Algebra Geometry* 2(4), 620-645, 2018.

- [4] F. Bihan and F. Sottile. *New feynomial upper bounds from Gale dual polynomial systems*. Mosc. Math. J., Volume 7, Number 3, 2007.
- [5] F. Bihan and F. Sottile. *Gale duality for complete intersection*, Ann. Inst. Fourier 58, no. 3, 877–891, 2008.
- [6] C. Conradi, E. Feliu, M. Mincheva and C. Wiuf, *Identifying parameter regions for multistationarity*, PLoS Comput. Biol., 13(10), e1005751, 2017.
- [7] P. De Leenheer, D. Angeli and E. D. Sontag. *Monotone chemical reaction networks*. J. Math. Chem., 41(3), 295–314, 2007.
- [8] W. Decker, G.-M. Greuel, G. Pfister and H. Schönemann, SINGULAR 4-1-1 — A computer algebra system for polynomial computations. <http://www.singular.uni-kl.de>, 2018.
- [9] D. Eisenbud. *Commutative Algebra: with a View Toward Algebraic Geometry*. Graduate Texts in Mathematics 150, Springer, 1999.
- [10] K. G. Fischer, W. Morris and J. Shapiro *Mixed dominating matrices*. Lineal Algebra Appl., 270(1-3), 191–214, 1998.
- [11] K. G. Fischer and J. Shapiro *Mixed matrices and binomial ideals*. J. Pure Appl. Algebra, 113(1), 39–54, 1996.
- [12] L. F. Matusевич and A. Sobieska. *Counterexamples for CohenMacaulayness of lattice ideals*, Commun. Algebra, 2019.
- [13] S. Müller, E. Feliu, G. Regensburger, C. Conradi, A. Shiu and A. Dickenstein. *Sign conditions for injectivity of generalized polynomial maps with appl. to chemical reaction networks and real algebraic geometry*, Found. Comput. Math. 16(1), 69–97, 2016
- [14] S. Müller and G. Regensburger. *Generalized mass action systems: Complex balancing equilibria and sign vectors of the stoichiometric and kinetic-order subspaces*, SIAM J. Appl. Math. 72(6), 1926–1947, 2012.
- [15] S. Müller, J. Hofbauer, and G. Regensburger. *On the bijectivity of families of exponential/generalized polynomial maps* to appear: SIAM J. Appl. Algebra Geometry , available at: arXiv:1804.01851, 2018.
- [16] I. Peeva and B. Sturmfels. *Syzygies of codimension 2 lattice ideals*. Math. Z., 229(1), 163–194, 1998.
- [17] R. T. Rockafellar. *The elementary vectors of a subspace of \mathbb{R}^n* . Combinatorial Mathematics and Its Applications. In Proc. of the Chapel Hill Conf., 104–127, University of North Carolina Press, 1969.
- [18] E. Soprunova and F. Sottile. *Lower bounds for real solutions to sparse polynomial systems*. Adv. Math., 204(1), 116–151, 2006.
- [19] F. Sottile. *Real solutions to equations from geometry*, University Lecture Series, vol. 57, American Mathematical Society, Providence, RI, 2011.
- [20] B. Sturmfels. *Gröbner bases and convex polytopes*, University Lecture Series Vol. 8, American Mathematical Society, 1996.
- [21] G. Teschl. *Topics in real and functional analysis*. Available at <http://www.mat.univie.ac.at/~gerald/ftp/book-fa/>, 2004.
- [22] J. Wang. *On systems of polynomials with at least one positive real zero*. Preprint, available at: arXiv:1809.10825, 2018.

LABORATOIRE DE MATHÉMATIQUES, UNIVERSITÉ SAVOIE MONT BLANC, 73376 LE
BOURGET-DU-LAC CEDEX, FRANCE

E-mail address: `Frederic.Bihan@univ-savoie.fr`

URL: `http://www.lama.univ-savoie.fr/~bihan/`

DTO. DE MATEMÁTICA, FCEN, UNIVERSIDAD DE BUENOS AIRES, AND IMAS (UBA-
CONICET), CIUDAD UNIVERSITARIA, PAB. I, C1428EGA BUENOS AIRES, ARGENTINA

E-mail address: `alidick@dm.uba.ar`

URL: `http://mate.dm.uba.ar/~alidick`

DTO. DE MATEMÁTICA, FCEN, UNIVERSIDAD DE BUENOS AIRES, AND IMAS (UBA-
CONICET), CIUDAD UNIVERSITARIA, PAB. I, C1428EGA BUENOS AIRES, ARGENTINA

E-mail address: `mgiaroli@dm.uba.ar`