BOUNDS ON THE NUMBER OF REAL SOLUTIONS TO POLYNOMIAL EQUATIONS

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Abstract. We use Gale duality for complete intersections and adapt the proof of the fewnomial bound for positive solutions to obtain the bound

$$\frac{e^4+3}{4}2^{\binom{k}{2}}n^k$$

for the number of non-zero real solutions to a system of n polynomials in n variables having $n+k+1$ monomials whose exponent vectors generate a subgroup of $\mathbb{Z}^n$ of odd index. This bound only exceeds the bound for positive solutions by the constant factor $(e^4+3)/(e^2+3)$ and it is asymptotically sharp for k fixed and n large.

Introduction

In [3], the sharp bound of $2n+1$ was obtained for the number of non-zero real solutions to a system of n polynomial equations in n variables having $n+2$ monomials whose exponents affinely span the lattice $\mathbb{Z}^n$. In [4], the sharp bound of $n+1$ was given for the positive solutions to such a system of equations. This last bound was generalized in [7], which showed that the number of positive solutions to a system of n polynomial equations in n variables having $n+k+1$ monomials was less than

$$\frac{e^2+3}{4}2^{\binom{k}{2}}n^k\,,$$

which is asymptotically sharp for k fixed and n large [5]. This dramatically improved Khovanskii's fewnomial bound [8] of $2^{\binom{n+k}{2}}(n+1)^{n+k}$.

We give a bound for all non-zero real solutions. Under the assumption that the exponent vectors $\mathcal{W}$ span a subgroup of $\mathbb{Z}^n$ of odd index, we show that the number of non-degenerate non-zero real solutions to a system of polynomials with support $\mathcal{W}$ is less than

$$\frac{e^4+3}{4}2^{\binom{k}{2}}n^k\ . \tag{1}$$

The novelty is that this bound exceeds the bound for solutions in the positive orthant by a fixed constant factor $(e^4+3)/(e^2+3)$, rather than by a factor of 2^n, which is the number of orthants. By the construction in [5], it is asymptotically sharp for k fixed and n large.

We follow the outline of [7]—we use Gale duality for real complete intersections [6] and then bound the number of solutions to the dual system of master functions. The

2000 *Mathematics Subject Classification.* 14M25, 14P25, 52C35.
Key words and phrases. sparse polynomial system, hyperplane arrangement, fewnomial.
Bates and Sottile supported by the Institute for Mathematics and its Applications.
Sottile supported by the NSF CAREER grant DMS-0538734.

key idea is that including solutions in all chambers in a complement of an arrangement of hyperplanes in $\mathbb{RP}^k$, rather than in just one chamber as in [7], does not increase our estimate on the number of solutions very much. This was discovered while implementing a numerical continuation algorithm for computing the positive solutions to a system of polynomials [1]. That algorithm was improved by this discovery to one which finds all real solutions. It does so without computing complex solutions and is based on [7] and the results of this paper. Its complexity depends on (1), and not on the number of complex solutions.

We state our main theorem in Section 1 and then use Gale duality to reduce it to a statement about systems of master functions, which we prove in Section 2.

1. Gale duality for systems of sparse polynomials

Let $\mathcal{W} = \{w_0 = 0, w_1, \ldots, w_{n+k}\} \subset \mathbb{Z}^n$ be a collection of $n+k+1$ integer vectors ($|\mathcal{W}| = n+k+1$), which correspond to monomials in variables $x_1, \ldots, x_n$. A (Laurent) polynomial f with *support $\mathcal{W}$* is a real linear combination of monomials with exponents from $\mathcal{W}$,

$$f(x_1, \ldots, x_n) \ = \ \sum_{i=0}^{n+k} c_i x^{w_i} \qquad \text{with } c_i \in \mathbb{R}\,. \tag{2}$$

A *system with support $\mathcal{W}$* is a system of polynomial equations

$$f_1(x_1, \ldots, x_n) \ = \ f_2(x_1, \ldots, x_n) \ = \ \cdots \ = \ f_n(x_1, \ldots, x_n) \ = \ 0\,, \tag{3}$$

where each polynomial f_i has support $\mathcal{W}$. Since multiplying every polynomial in (3) by a monomial x^α does not change the set of non-zero solutions but translates $\mathcal{W}$ by the vector α, we see that it was no loss of generality to assume that $0 \in \mathcal{W}$.

The system (3) has infinitely many solutions if $\mathcal{W}$ does not span $\mathbb{R}^n$. We say that $\mathcal{W}$ *spans $\mathbb{Z}^n$ mod 2* if the $\mathbb{Z}$-linear span of $\mathcal{W}$ is a subgroup of $\mathbb{Z}^n$ of odd index.

Theorem 1. *Suppose that $\mathcal{W}$ spans $\mathbb{Z}^n$ mod 2 and $|\mathcal{W}| = n+k+1$. Then there are fewer than* (1) *non-degenerate non-zero real solutions to a sparse system* (3) *with support $\mathcal{W}$.*

The importance of this bound for the number of real solutions is that it has a completely different character than Kouchnirenko's bound for the number of complex solutions.

Proposition 2 (Kouchnirenko [2])**.** *The number of non-degenerate solutions in $(\mathbb{C}^\times)^n$ to a system* (3) *with support $\mathcal{W}$ is no more than $n!\mathrm{vol}(\mathrm{conv}(\mathcal{W}))$.*

Here, $\mathrm{vol}(\mathrm{conv}(\mathcal{W}))$ is the Euclidean volume of the convex hull of $\mathcal{W}$.

Perturbing coefficients of the polynomials in (3) so that they define a complete intersection in $(\mathbb{C}^\times)^n$ can only increase the number of non-degenerate solutions. Thus it suffices to prove Theorem 1 under this assumption. Such a complete intersection is equivalent to a complete intersection of master functions in a hyperplane complement [6].

Let $\mathbb{R}^{n+k}$ have coordinates $z_1, \ldots, z_{n+k}$. A polynomial (2) with support $\mathcal{W}$ is the pull-back $\Phi^*_{\mathcal{W}}(\Lambda)$ of the degree 1 polynomial $\Lambda := c_0 + c_1 z_1 + \cdots + c_{n+k} z_{n+k}$ along the map

$$\Phi_{\mathcal{W}} \ : \ (\mathbb{R}^\times)^n \ni x \ \longmapsto \ (x^{w_i} \mid i = 1, \ldots, n+k) \in \mathbb{R}^{n+k}\,.$$

If we let $\Lambda_1, \ldots, \Lambda_n$ be the degree 1 polynomials which pull back to the polynomials in the system (3), then they cut out an affine subspace L of $\mathbb{R}^{n+k}$ of dimension k.

Let $\{p_i \mid i=1,\dots,n{+}k\}$ be degree 1 polynomials on $\mathbb{R}^k$ which induce an isomorphism between $\mathbb{R}^k$ and L,

$$\Psi_p \;:\; \mathbb{R}^k \ni y \;\longmapsto\; (p_1(y),\dots,p_{n+k}(y)) \in L \subset \mathbb{R}^{n+k}\,.$$

Let $\mathcal{A}\subset\mathbb{R}^k$ be the arrangement of hyperplanes defined by the vanishing of the $p_i(y)$. This is the pullback along Ψ_p of the coordinate hyperplanes of $\mathbb{R}^{n+k}$.

The image $\Phi_{\mathcal{W}}((\mathbb{R}^\times)^n)$ inside of the torus $(\mathbb{R}^\times)^{n+k}$ has equations

$$z^{\beta_1} \;=\; z^{\beta_2} \;=\; \cdots \;=\; z^{\beta_k} \;=\; 1\,,$$

where the *weights* $\{\beta_1,\dots,\beta_k\}$ form a basis for the $\mathbb{Z}$-submodule of $\mathbb{Z}^{n+k}$ of linear relations among the vectors $\mathcal{W}$. To these data, we associate a system of *master functions* on the complement $M_{\mathcal{A}}$ of the arrangement $\mathcal{A}$ of $\mathbb{R}^k$,

$$p(y)^{\beta_1} \;=\; p(y)^{\beta_2} \;=\; \cdots \;=\; p(y)^{\beta_k} \;=\; 1\,. \tag{4}$$

Here, if $\beta=(b_1,\dots,b_{n+k})$ then $p^\beta := p_1(y)^{b_1}\cdots p_{n+k}(y)^{b_{n+k}}$.

A basic result of [6] is that if $\mathcal{W}$ spans $\mathbb{Z}^n$ modulo 2 and either of the systems (3) or (4) defines a complete intersection, then the other defines a complete intersection and the maps $\Phi_{\mathcal{W}}$ and Ψ_p induce isomorphisms between the two solution sets, as analytic subschemes of $(\mathbb{R}^\times)^n$ and $M_{\mathcal{A}}$. Since we assumed that the system (3) is general, these hypotheses hold and the arrangement is *essential* in that the polynomials p_i span the space of all degree 1 polynomials on $\mathbb{R}^k$.

Theorem 3. *A system* (4) *of master functions in the complement of an essential arrangement of $n{+}k$ hyperplanes in $\mathbb{R}^k$ has at most* (1) *non-degenerate real solutions.*

We actually prove a bound for a more general system than (4), namely for

$$p(z)^{2\beta_1} \;=\; p(z)^{2\beta_2} \;=\; \cdots \;=\; p(z)^{2\beta_k} \;=\; 1\,.$$

We write this more general system as

$$|p(z)|^{\beta_1} \;=\; |p(z)|^{\beta_2} \;=\; \cdots \;=\; |p(z)|^{\beta_k} \;=\; 1\,. \tag{5}$$

In a system of this form we may have real number weights $\beta_i \in \mathbb{R}^{n+k}$. We give the strongest form of our theorem.

Theorem 4. *A system of the form* (5) *with real weights β_i in the complement of an essential arrangement of $n{+}k$ hyperplanes in $\mathbb{R}^k$ has at most* (1) *non-degenerate real solutions.*

2. Proof of Theorem 4

We follow [7] with minor, but important, modifications. Perturbing the polynomials $p_i(y)$ and the weights β_j will not decrease the number of non-degenerate real solutions in $M_{\mathcal{A}}$. This enables us to make the following assumptions.

The arrangement $\mathcal{A}^+ \subset \mathbb{RP}^k$, where we add the hyperplane at infinity, is general in that every j hyperplanes of $\mathcal{A}^+$ meet in a $(k{-}j)$ dimensional linear subspace, called a *codimension j face of $\mathcal{A}$*. If B is the matrix whose columns are the weights $\beta_1,\dots,\beta_k$, then the entries of B are rational numbers and no minor of B vanishes. This last technical condition as well as the freedom to further perturb the β_j and the p_i are necessary for the results in [7, Section 3] upon which we rely.

For functions $f_1, \ldots, f_j$ on $M_{\mathcal{A}}$, let $V(f_1, \ldots, f_j)$ be the subvariety they define. Suppose that $\beta_j = (b_{1,j}, \ldots, b_{n+k,j})$. For each $j = 1, \ldots, k$, define

$$\psi_j(y) \ := \ \sum_{i=1}^{n+k} b_{i,j} \log |p_i(y)| \,.$$

Then (5) is equivalent to $\psi_1(y) = \cdots = \psi_k(y) = 0$. Inductively define $\Gamma_k, \Gamma_{k-1}, \ldots, \Gamma_1$ by

$$\Gamma_j \ := \ \mathrm{Jac}(\psi_1, \ldots, \psi_j,\ \Gamma_{j+1}, \ldots, \Gamma_k)\,,$$

the Jacobian determinant of $\psi_1, \ldots, \psi_j, \Gamma_{j+1}, \ldots, \Gamma_k$. Set

$$C_j \ := \ V(\psi_1, \ldots, \psi_{j-1},\ \Gamma_{j+1}, \ldots, \Gamma_k)\,,$$

which is a curve in $M_{\mathcal{A}}$.

Let $\flat(C)$ be the number of unbounded components of a curve $C \subset M_{\mathcal{A}}$. We have the estimate from [7], which is a consequence of the Khovanskii-Rolle Theorem,

$$|V(\psi_1, \ldots, \psi_k)| \ \leq \ \flat(C_k) + \cdots + \flat(C_1) \ + \ |V(\Gamma_1, \ldots, \Gamma_k)|\,. \tag{6}$$

Here, $|S|$ is the cardinalty of the set S. We estimate these quantities.

Lemma 5.

(1) $|V(\Gamma_1, \ldots, \Gamma_k)| \leq 2^{\binom{k}{2}} n^k$.

(2) *C_j is a smooth curve and*

$$\flat(C_j) \ \leq \ \frac{1}{2} 2^{\binom{k-j}{2}} n^{k-j} \binom{n+k+1}{j} \cdot 2^j \ \leq \ \frac{1}{2} 2^{\binom{k}{2}} n^k \cdot \frac{2^{2j-1}}{j!}\,.$$

Proof of Theorem 4. By (6) and Lemma 5, we have

$$|V(\psi_1, \ldots, \psi_k)| \ \leq \ 2^{\binom{k}{2}} n^k \Big(1 + \frac{1}{4}\sum_{j=1}^{k} \frac{4^j}{j!}\Big) \ < \ 2^{\binom{k}{2}} n^k \cdot \frac{e^4+3}{4}\,. \qquad \square$$

Proof of Lemma 5. The bound (1) is from Lemma 3.4 of [7]. Statements analogous to (2) for $\widetilde{C}_j$, the restriction of C_j to a single chamber (connected component) of $M_{\mathcal{A}}$, were established in Lemma 3.4 and the proof of Lemma 3.5 in [7]:

$$\flat(\widetilde{C}_j) \ \leq \ \frac{1}{2} 2^{\binom{k-j}{2}} n^{k-j} \binom{n+k+1}{j} \ \leq \ \frac{1}{2} 2^{\binom{k}{2}} n^k \cdot \frac{2^{j-1}}{j!}\,. \tag{7}$$

The bound we claim for $\flat(C_j)$ has an extra factor of 2^j. *A priori* we would expect to multiply this bound (7) by the number of chambers of $M_{\mathcal{A}}$ to obtain a bound for $\flat(C_j)$, but the correct factor is only 2^j.

We work in $\mathbb{RP}^k$ and use the extended hyperplane arrangement $\mathcal{A}^+$, as we will need points in the closure of C_j in $\mathbb{RP}^k$. The first inequality in (7) for $\flat(\widetilde{C}_j)$ arises as each unbounded component of $\widetilde{C}_j$ meets $\mathcal{A}^+$ in two distinct points (this accounts for the factor $\frac{1}{2}$) which are points of codimension j faces where the polynomials

$$F_i(y) \ := \ \Gamma_{k-i}(y) \cdot \Bigl(\prod_{i=1}^{n+k} p_i(y)\Bigr)^{2^i}$$

for $i = 0, \ldots, k-j-1$ vanish. (By Lemma 3.4(1) of [7], F_i is a polynomial of degree $2^i n$.) The genericity of the weights and the linear polynomials $p_i(y)$ imply that these points will lie on faces of codimension j but not of higher codimension. The factor $2^{\binom{k-j}{2}} n^{k-j}$ is the Bézout number of the system $F_0 = \cdots = F_{k-j-1}$ on a given codimension j plane, and there are exactly $\binom{n+k+1}{j}$ codimension j faces of $\mathcal{A}^+$.

At each of these points, C_j will have one branch in each chamber of $M_{\mathcal{A}}$ incident on that point. Since the hyperplane arrangement $\mathcal{A}^+$ is general there will be exactly 2^j such chambers. □

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