Lecture Notes on Navier-Stokes-Fourier system

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Abstract

We investigate three issues in the theory of the Navier-Stokes-Fourier system describing the motion of a compressible, viscous and heat conducting fluid: weak compactness of the family of weak solutions (that is the main ingredient in the proof of the existence of weak solutions), relative entropy inequality and weak-strong uniqueness principle.

1 Introduction

These Lecture Notes are devoted to some aspects of the theory of the Navier-Stokes-Fourier system. We shall discuss 1) existence of weak solutions, 2) existence of suitable weak solutions and relative entropies, 3) weak strong uniqueness property in the class of weak solutions. For physical reasons, we shall limit ourselves to the three dimensional physical space, and for the sake of simplicity, to the flows in bounded domains with no-slip boundary conditions.

1.1 Nonexhausting bibliographic remarks

1.1.1 Weak solutions

There are several ways to define weak solutions for the complete Navier-Stokes-Fourier system. Here, we shall mention three of them: the convenience of each definition depends on the mathematical assumptions that one imposes on the constitutive laws for pressure (internal energy) on one hand, and on the transport coefficients on the other hand. We shall shortly introduce three of them: First one and the third one are continuations of the theories based on the so called effective viscous flux identity started by P.L. Lions [32] and the second one, due to Bresch, Desjardins [2] can be considered as a continuation of theories based on new a priory estimates in the line started by Kazhikov [28].

The first approach due to Feireisl [14] based on a weak formulation of the continuity, momentum and internal energy equations is convenient for the pressure laws

\[ p(\rho, \vartheta) = p_c(\rho) + \vartheta p_\vartheta(\rho), \]
where
\[ p_c(\varrho) \approx \varrho^\gamma, \quad p_\vartheta(\varrho) \approx \varrho^\Gamma, \quad \Gamma \leq \gamma/3. \]

Here, \( \vartheta \) denotes the temperature, \( \varrho \) the density and \( \gamma \) is the adiabatic coefficient of the fluid. The heat conductivity in this approach has to be temperature dependent (with a convenient polynomial growth), and the viscosity coefficients have to be constant.

The second approach due to Bresch, Desjardins [2], [3] (see also Mellet, Vasseur [36]) is convenient in the case when the shear viscosity \( \mu \) and the bulk viscosity \( \eta \) depend on the density and satisfy the differential identity
\[ (\eta - \frac{2}{3} \mu)'(\varrho) = 2\varrho \mu'(\varrho) - 2\mu(\varrho), \]
and the pressure satisfies
\[ p(\varrho, \vartheta) = \varrho \vartheta + p_c(\varrho), \]
where \( p_c(\varrho) \) is singular at \( \varrho \to 0 \) and behaves as \( \varrho^\gamma \) at \( \varrho \to \infty \). The main ingredient in the prof in this situation is the fact that the particular relation between viscosities stated above makes possible to establish a new mathematical entropy identity, that provides estimates for the gradient of density. This estimate implies compactness of the sequence of approximating densities.

The third approach is due to [17]. It is based on the weak formulation of continuity and momentum equations and on the formulation of the conservation of energy in terms of the specific entropy that involves explicitly the second law of thermodynamics with entropy production rate being a non negative measure. This approach is applicable for the pressure laws \( p(\varrho, \vartheta) \) exhibiting the coercivity of type \( \varrho^{\gamma} \) and \( \vartheta^4 \) for large densities and temperatures. The viscosity coefficients are temperature dependent and have to behave as \( 1 + \vartheta^\alpha \) (\( 1 \geq \alpha > 2/5 \)), and the heat conductivity has to behave as \( 1 + \vartheta^3 \). This setting includes at least one physically reasonable case of a monoatomic gas in the situation when the radiation is not neglected and is given by the Stephan-Boltzman law.

The above formulation is sufficiently weak to allow existence of variational solutions for large data. On the other hand it is sufficiently robust to yield most of low Mach and low Reynolds number limits of mathematical fluid mechanics, and - more surprisingly - it obeys the weak-strong uniqueness principle.

In this Lecture Notes, we shall concentrate on the third approach. We shall discuss within this concept three issues: 1) existence of weak solutions (or, more precisely, in order to concentrate on the essence of the problem, the compactness of the family of weak solutions); 2) Relative entropy inequality for the complete system; 3) Weak strong uniqueness property in the class of weak solutions. This text does not contain any new results: it is a compilation of our recent works [17, Chapter3] and [18].

1.1.2 Lion’s approach and Feireisl’s approach

The concept of weak solutions in fluid dynamics was introduced in 1934 by Leray [30] in the context of incompressible newtonian fluids. It has been extended more than 60 years later to the Newtonian compressible fluids in barotropic regime (meaning that \( p = p(\varrho) \approx \varrho^\gamma \) by Lions [32].

The Lions’ theory relies on two crucial observations:

1) A discovery of a certain weak continuity property of the quantity
\[ p(\varrho) + \left( \frac{4}{3} \mu + \eta \right) \text{div}\mathbf{u} \]
called effective viscous flux. This part is essential for the existence proof; it employs certain cancelation properties that are available due to the structure of the equations, that are mathematically expressed through a commutator involving density, momentum and the Riesz operator.
2) Theory of renormalized solutions to the transport equation that P.L. Lions introduced together with DiPerna in [7]. In the context of compressible Navier-Stokes equations, the DiPerna-Lions transport theory applies to the continuity equation. The theory asserts among others that the limiting density is a renormalized solution to the continuity equation provided it is squared integrable. This hypothesis is satisfied only provided $\gamma \geq \frac{9}{5}$. The condition on the squared integrability of the density is the principal obstacle to the improvement of the Lions result.

Notice that some indications on the particular importance of the effective viscous flux have been known at about the same time to several authors and used in different problems dealing with small data (see Hoff [26], Padula [38]) and that the suggestion to use the continuity equations to evaluate the oscillations in the sequence of approximating densities has been formulated and performed in the one dimensional case by D. Serre [44].

All physical reasonable adiabatic coefficients $\gamma$ belong to the interval $(1, \frac{5}{3}]$, the value $\gamma = \frac{5}{3}$ being reserved for the monoatomic gas. This is the reason why it is interesting and important to relax the condition on the adiabatic coefficient in the Lions theory. This has been done by Feireisl et al. in [19] (see also previous observations of Feireisl in [13]. The new additional aspects of this extension are the following:

1) The authors have introduced an oscillations defect measure to evaluate the oscillations in the sequence of approximating densities, and proved that it is bounded provided $\gamma > \frac{3}{2}$.

2) The boundedness of the oscillations defect measure is a criterion that replaces the condition of the squared integrability of the density in the DiPerna-Lions transport theory. Consequently if any term of the sequence of approximating densities satisfies the renormalized continuity equation, and if the oscillations defect measure of this sequence is bounded, then the weak limit of the sequence is again a renormalized solution of the continuity equation.

1.1.3 Weak solutions for the complete Navier-Stokes-Fourier system

The existence theory for the complete Navier-Stokes-Fourier system (with temperature dependent viscosities) employs both Lions’ and Feireisl’s techniques. In addition, it presents the following difficulties:

1) In order to reduce the investigation to a situation similar to the barotropic case, one has to prove first the strong convergence of the temperature sequence. This point involves the treatment of the entropy production rate as a Radon measure and a convenient use of the compensated compactness, namely of the Div-curl lemma in combination with the theory of parametrized Young measures.

2) Even after the strong convergence of temperature is known, the weak continuity of the effective viscous flux is not an obvious issue. It requires to use another cancelation property that mathematically expresses through another commutator including shear viscosity, symmetric velocity gradient and the Riesz operator.

3) Once the weak continuity property of the effective viscous flux is known, the proof follows the lines of Lion’s and Feireisl’s approaches: a) one proves first the boundedness of the oscillation defect measure for the sequence of densities; b) the boundedness of oscillations defect measure implies that the limiting density is a renormalized solution to the continuity equation; c) the renormalized continuity equation is used to show that the oscillations in the density sequence do not increase in time. This means the strong convergence of density.
1.1.4 Relative entropies and weak strong uniqueness

Weak solutions are not known to be uniquely determined (cf. e.g. exposition of Fefferman [12] dealing with three dimensional incompressible Navier-Stokes equations) and may exhibit rather pathological properties, see e. g. Hoff and Serre [27]. So far, the best property that one may expect in the direction of an uniqueness result, is the weak-strong uniqueness, meaning that any weak solution coincides with the strong solution emanating from the same initial data, as long as the latter exists. The weak-strong uniqueness principle is known for the incompressible Navier-Stokes equations since the works of Prodi[42], Serrin [45], 1959, 1962. About 50 years later, the weak-strong uniqueness problem has been revisited by Desjardins [6] and Germain [22]) for the compressible Navier-Stokes equations. They obtained only some conditional results. Finally, the unconditional weak strong uniqueness principle has been proved in [16] (see also related paper [20]).

Only very recently the weak strong uniqueness property has been proved in [18] for weak solutions of the complete Navier-Stokes-Fourier system in the entropy formulation introduced in [17].

In all cases cited above, the weak strong uniqueness principle has been achieved by the method of relative entropies. Relative entropy is a functional whose role is to measure the distance between a weak solution of the investigated equations and any arbitrary smooth function exhibiting some characteristic properties of solutions of the investigated equations, as e.g. the sign or the boundary conditions. This functional must satisfy a convenient differential inequality (called relative entropy inequality) that is then used to evaluate the evolution of the relative entropy along the time line. There is no general algorithm to construct this functional and this fact makes the construction of a convenient relative entropy functional for any given specific problem quite difficult. The method of relative entropies has been used in various context by different authors, see Dafermeos [5] and later on P.L. Lions [32], Saint-Raymond [43], Grenier [23], Masmoudi [33], Ukai [48], Wang and Jiang [49], among others. The notion of dissipative solutions introduced in Lions [31] for the incompressible Euler equations is very much related to the concept of relative entropies.

1.2 Navier-Stokes-Fourier system

Navier-Stokes-Fourier system describes the motion of a compressible, viscous and heat conducting fluid. For simplicity, we suppose that the fluid fills a fixed bounded domain Ω and we shall investigate its evolution through an arbitrary large time interval (0, T). The motion will be described by means of three basic state variables: the mass density \( ϱ = ϱ(t, x) \), the velocity field \( u = u(t, x) \), and the absolute temperature \( ϑ = ϑ(t, x) \), where \( t \in (0, T) \) is time variable and \( x \in Ω \subset R^3 \) is the space variable in the Eulerian coordinate system. We shall investigate the time evolution of these quantities. It is described by the following system of partial differential equations.

\[
\frac{∂}{∂t} ϱ + \text{div}_x(ϱu) = 0, \tag{1.1}
\]

\[
\frac{∂}{∂t}(ϱu) + \text{div}_x(ϱu ⊗ u) + \nabla_x p(ϱ, ϑ) = \text{div}_x S(ϑ, \nabla_x u), \tag{1.2}
\]

\[
\frac{∂}{∂t}(s(ϱ, ϑ)) + \text{div}_x(s(ϱ, ϑ)u) + \text{div}_x \left( q(ϑ, \nabla_x ϑ) \right) = σ, \tag{1.3}
\]

where \( p = p(ϱ, ϑ) \) is the pressure, \( s = s(ϱ, ϑ) \) is the (specific) entropy, \( σ \) is the entropy production rate \( S \) is the viscous stress tensor and \( q \) is the heat flux. The first equation expresses the mathematical formulation of the balance of mass, the second one, the conservation of momentum, and the third one, the conservation of energy in terms of the specific entropy.
We suppose that the viscous stress $\mathbb{S}$ is described by Newton’s law
\begin{equation}
\mathbb{S}(\vartheta, \nabla_x u) = \mu(\vartheta) T(\nabla_x u) + \eta(\vartheta) \text{div}_x u I, \quad T(\nabla_x u) = \nabla_x u + (\nabla_x u)^T - \frac{2}{3} \text{div}_x u I, \tag{1.4}
\end{equation}
while $q$ is the heat flux satisfying Fourier’s law
\begin{equation}
q = -\kappa(\vartheta) \nabla_x \vartheta. \tag{1.5}
\end{equation}
The entropy production rate, if all quantities are smooth, satisfies
\begin{equation}
\sigma = \frac{1}{\vartheta} \left( \mathbb{S}(\vartheta, \nabla_x u) : \nabla_x u - \frac{q(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right). \tag{1.6}
\end{equation}
The system of equations (1.1 - 1.3), with the constitutive relations (1.4), (1.5) is called \textit{Navier-Stokes-Fourier system}. Equations (1.1 - 1.6) are supplemented with initial conditions
\begin{equation}
\varrho(0, \cdot) = \varrho_0, \quad \varrho u(0, \cdot) = \varrho_0 u_0, \quad \varrho s(\varrho, \vartheta)(0, \cdot) = \varrho_0 s(\varrho_0, \vartheta_0), \quad \varrho_0 \geq 0, \quad \vartheta_0 > 0, \tag{1.7}
\end{equation}
and \textit{no-slip} boundary conditions for velocity and zero heat transfer conditions through the boundary
\begin{equation}
u|_{\partial \Omega} = 0, \quad q \cdot n|_{\partial \Omega} = 0, \tag{1.8}
\end{equation}
where $n$ denotes the external normal to the boundary $\partial \Omega$ of $\Omega$.

For simplicity, we have neglected external forces and heat sources. We however consider large motions, since the initial data may be arbitrary large.

In [17], we have introduced a concept of \textit{weak solution} to the Navier-Stokes-Fourier system (1.1 - 1.8). This concept postulates, in agreement with the Second law of thermodynamics, that the entropy production rate $\sigma$ is a non-negative measure,
\begin{equation}
\sigma \geq \frac{1}{\vartheta} \left( \mathbb{S}(\vartheta, \nabla_x u) : \nabla_x u - \frac{q(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right). \tag{1.9}
\end{equation}

With this postulate, equation (1.3) becomes inequality. In order to compensate the loss of information, we may postulate, that the total energy of the system in the volume $\Omega$ is conserved, namely
\begin{equation}
\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \varrho |u|^2 + \varrho e(\varrho, \vartheta) \right) \, dx = 0. \tag{1.10}
\end{equation}

We introduce the quantity $H_{\vartheta} = \varrho e - \varrho s$ (that is reminiscent of the \textit{Helmholtz free energy}), where $e = e(\varrho, \vartheta)$ denotes the specific energy of the gas. Adding equations (1.3) and (1.10), we obtain
\begin{equation}
\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \varrho |u|^2 + H_{\vartheta}(\varrho, \vartheta) - \partial_{\varrho} H_{\vartheta}(\varrho, \vartheta)(\varrho - \varrho_0) - H_{\vartheta}(\varrho_0, \vartheta_0) \right) \, dx \tag{1.11}
\end{equation}
\begin{equation}
+ \int_{\Omega} \vartheta \left( \mathbb{S}(\vartheta, \nabla_x u) : \nabla_x u - \frac{q(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \, dx \, dt \leq 0,
\end{equation}
where we have taken into account inequality (1.9)
On the other hand, if \((\varrho, \vartheta, u)\) \(\varrho > 0, \vartheta > 0\) is a trio of smooth functions satisfying (1.1–1.8) one may derive, at least formally, the so called relative entropy identity,

\[
\int_\Omega \left( \frac{1}{2} \varrho |u - U|^2 + \mathcal{E}(\varrho, \vartheta | r, \Theta) \right) (\tau, \cdot) \, dx \tag{1.12}
\]

\[
+ \int_0^\tau \int_\Omega \varrho(\partial_t U + u \cdot \nabla_x U) \cdot (U - u) \, dx \, dt,
\]

\[
+ \int_0^\tau \int_\Omega \varrho(U - u) \cdot \frac{\nabla_x p(r, \Theta)}{r} \, dx \, dt,
\]

\[
+ \int_0^\tau \int_\Omega \left( p(r, \Theta) - p(\varrho, \vartheta) \right) \text{div}_x U \, dx \, dt
\]

\[
+ \int_0^\tau \int_\Omega \varrho \left( s(r, \Theta) - s(\varrho, \vartheta) \right) \left( \partial_t \Theta + u \cdot \nabla_x \Theta \right) \, dx \, dt,
\]

\[
+ \int_0^\tau \int_\Omega \left( 1 - \frac{\varrho}{r} \right) \left( \partial_t p(r, \Theta) + U \cdot \nabla_x p(r, \Theta) \right) \, dx \, dt,
\]

where we have denoted

\[
\mathcal{E}(\varrho, \vartheta | r, \Theta) = H_\Theta(\varrho, \vartheta) - \frac{\partial_e H_\Theta(\varrho, \vartheta) (\varrho - r) - H_\Theta(r, \Theta)}{\varrho}.
\]

In (1.12), \((r, \Theta)\) is a couple of positive sufficiently smooth functions on \([0, T] \times \Omega\) and \(U\) is a sufficiently smooth vector field with compact support in \([0, T] \times \Omega\).

Conformably to (1.9), for a weak solution \((\varrho, \vartheta, u)\), the identity (1.12) has to be replaced by an inequality with the inequality sign \(\leq\). This inequality is usually called the relative entropy inequality.

Notice that the dissipation inequality (1.11) is a particular case of the relative entropy inequality, where \(r = \varrho, \Theta = \vartheta, U = 0\).

### 1.3 Constitutive relations

We assume that the thermodynamic functions \(p, e, \text{ and } s\) are interrelated through Gibbs’ equation

\[
\partial_t s(\varrho, \vartheta) = \partial_e s(\varrho, \vartheta) - \frac{p(\varrho, \vartheta)}{\varrho^2} d\varrho. \tag{1.13}
\]

We suppose that the fluid verifies the thermodynamic stability conditions,

\[
\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \quad \frac{\partial e(\varrho, \vartheta)}{\partial \varrho} > 0 \quad \text{for all } \varrho, \vartheta > 0. \tag{1.14}
\]
We easily verify by using (1.13), that
\[
\frac{\partial H_\vartheta}{\partial \vartheta} (\varrho, \vartheta) = \varrho \vartheta - \vartheta \vartheta \frac{\partial e}{\partial \vartheta} (\varrho, \vartheta) \quad \text{and} \quad \frac{\partial^2 H_\vartheta}{\partial \varrho^2} (\varrho, \vartheta) = \frac{1}{\varrho} \frac{\partial p}{\varrho \partial \varrho} (\varrho, \vartheta).
\] (1.15)

Thus, the thermodynamic stability in terms of the function $H_\vartheta$, can be reformulated as follows:
\[
\varrho \mapsto H_\vartheta (\varrho, \vartheta) \text{ is strictly convex,}
\] (1.16)
while
\[
\vartheta \mapsto H_\vartheta (\varrho, \vartheta) \text{ attains its global minimum at } \vartheta = \vartheta.
\] (1.17)

The above relations reflect stability of the equilibrium solutions to the Navier-Stokes-Fourier system (see Bechtel, Rooney, and Forest [1]) and play a crucial role in the analysis of the Navier-Stokes-Fourier system.

Taking into account the effects of radiation expressed through the Stefan-Boltzman law, we shall assume that the pressure $p = p(\varrho, \vartheta)$ can be written in the form, in agreement with the above considerations
\[
p(\varrho, \vartheta) = \vartheta^{\gamma/(\gamma-1)} P \left( \frac{\varrho}{\vartheta^{1/(\gamma-1)}} \right) + \frac{a}{3} \vartheta^4, \quad a > 0, \ \gamma > 3/2,
\] (1.18)
where
\[
P \in C^1[0, \infty), \ P(0) = 0, \ P'(Z) > 0 \text{ for all } Z > 0.
\] (1.19)

In agreement with Gibbs’ relation (1.13), the (specific) internal energy can be taken as
\[
e(\varrho, \vartheta) = \frac{1}{\gamma - 1} \frac{\varrho^{\gamma/(\gamma-1)}}{\vartheta} P \left( \frac{\varrho}{\vartheta^{1/(\gamma-1)}} \right) + \frac{a}{3} \vartheta^4.
\] (1.20)

Furthermore, by virtue of the second inequality in thermodynamic stability hypothesis (1.14), we have
\[
0 < \frac{\gamma P(Z) - P'(Z) Z}{Z} < c \text{ for all } Z > 0.
\] (1.21)

Relation (1.21) implies that the function $Z \mapsto P(Z)/Z^\gamma$ is decreasing, and we suppose that
\[
\lim_{Z \to \infty} \frac{P(Z)}{Z^\gamma} = P_\infty > 0.
\] (1.22)

Finally, the formula for (specific) entropy reads
\[
s(\varrho, \vartheta) = S \left( \frac{\varrho}{\vartheta^{1/(\gamma-1)}} \right) + \frac{4a}{3} \vartheta^3.
\] (1.23)

where, in accordance with Third law of thermodynamics,
\[
S'(Z) = - \frac{1}{\gamma - 1} \frac{\gamma P(Z) - P'(Z) Z}{Z^2} < 0, \ \lim_{Z \to \infty} S(Z) = 0.
\] (1.24)

From the point of view of statistical mechanics, the above hypotheses are physically reasonable at least in two cases: if $\gamma = 5/3$ they modelize the monoatomic gas, if $\gamma = 4/3$ they modelize the so called...
We say that a trio $(\rho, \vartheta, u)$ is a weak solution to the Navier-Stokes-Fourier system (1.1) if:

(i) the density and the absolute temperature satisfy $\rho(t, x) \geq 0$, $\vartheta(t, x) > 0$ for a.a. $(t, x) \in (0, T) \times \Omega$, $\rho \in L^\infty(0, T; L^{2/(\gamma+1)}(\Omega; \mathbb{R}^3))$, $\rho u \in L^\infty(0, T; L^1(\Omega))$, $\rho u^2 \in L^\infty(0, T; L^1(\Omega))$, $\vartheta \in L^4(\Omega) \cap L^2(0, T; W^{1,2}(\Omega))$, and $u \in L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^3))$;

(ii) $\rho \in C^{\text{weak}}([0, T]; L^1(\Omega))$ and equation (1.1) is replaced by a family of integral identities

$$\int_\Omega \rho \varphi \, dx \bigg|_0^\tau = \int_0^\tau \int_\Omega \left( \rho \partial_t \varphi + \rho u \cdot \nabla x \varphi \right) \, dx \, dt$$

for all $\tau \in [0, T]$ and for any $\varphi \in C^1([0, T] \times \overline{\Omega})$;

(iii) $\rho u \in C^{\text{weak}}([0, T]; L^{2/(\gamma+1)}(\Omega; \mathbb{R}^3))$ and momentum equation (1.2) is satisfied in the sense of distributions, specifically,

$$\int_\Omega \rho u \cdot \varphi \, dx \bigg|_0^\tau = \int_0^\tau \int_\Omega \left( \rho u \cdot \partial_t \varphi + \rho \otimes u : \nabla x \varphi + p(\rho, \vartheta) \nabla x \varphi - S(\vartheta, \nabla x u) : \nabla x \varphi \right) \, dx \, dt$$

for all $\tau \in [0, T]$ and for any $\varphi \in C^1([0, T] \times \Omega; \mathbb{R}^3)$;

(iv) the entropy balance (1.3), (1.9) is replaced by a family of integral inequalities

$$-\int_\Omega \rho s(\rho, \vartheta) \varphi \, dx \bigg|_0^\tau + \int_0^\tau \int_\Omega \varphi \left( S(\vartheta, \nabla x u) : \nabla x \varphi - \frac{q(\rho, \vartheta \vartheta) \cdot \nabla x \vartheta}{\vartheta} \right) \, dx \, dt$$

for all $\tau \in [0, T]$ and for any $\varphi \in C^1([0, T] \times \Omega; \mathbb{R}^3)$.

2 Definitions and main results

2.1 Weak and suitable weak solutions to the Navier-Stokes-Fourier system

Here and heareafter, we shall suppose that the initial data satisfy the following assumptions.

$$0 < \varrho \leq \varrho_0 \in L^\infty(\Omega), \quad \varrho_0 u_0 \in L^2((\gamma+1)(\Omega; \mathbb{R}^3),$$

$$0 < \vartheta_0 < \vartheta, \quad \rho s(\rho_0, \vartheta_0) \in L^1(\Omega),$$

$$\int_\Omega \left( \frac{1}{2} \varrho_0 |u_0|^2 + H_\vartheta(\varrho_0, \vartheta_0) - \partial_\vartheta H_\vartheta(\vartheta, \vartheta)(\varrho_0 - \vartheta) - H_\vartheta(\vartheta, \vartheta) \right) \, dx < \infty.$$
\[
\begin{align*}
\leq -\int_0^T \int_\Omega \left( \frac{1}{2} \partial_t \varphi + \rho \partial_\varphi \right) \partial_t \varphi - \rho \partial_\varphi \right) \partial_\varphi + \frac{\partial (\varphi, \nabla_\varphi)}{\partial_\varphi} \cdot \nabla_\varphi \varphi \right) dx dt & \\
\text{for a.a. } \tau \in (0, T) \text{ and for any } \varphi \in C^1([0, T] \times \Omega), \varphi \geq 0; \\
(v) \text{ the conservation of total energy in the volume } \Omega \text{ is verified} \\
\int_\Omega \left( \frac{1}{2} |\varphi_0| \right) \partial_\varphi \right) \partial_t \varphi - \rho \partial_\varphi \right) \partial_\varphi + \frac{\partial (\varphi, \nabla_\varphi)}{\partial_\varphi} \cdot \nabla_\varphi \varphi \right) dx & \\
\text{for a.a. } \tau \in (0, T). \\
\end{align*}
\]

Remark
Any weak solution satisfies the so called dissipation inequality:
\[
\int_\Omega \left( \frac{1}{2} |\varphi_0| \right) \partial_\varphi \right) \partial_t \varphi - \rho \partial_\varphi \right) \partial_\varphi + \frac{\partial (\varphi, \nabla_\varphi)}{\partial_\varphi} \cdot \nabla_\varphi \varphi \right) dx \\
+ \int_0^\tau \int_\Omega \left( \nabla_\varphi \cdot (\varphi_0 \nabla_\varphi) - \frac{\partial (\varphi, \nabla_\varphi)}{\partial_\varphi} \cdot \nabla_\varphi \varphi \right) dx dt \leq \\
\int_\Omega \left( \frac{1}{2} |\varphi_0| \right) \partial_\varphi \right) \partial_t \varphi - \rho \partial_\varphi \right) \partial_\varphi + \frac{\partial (\varphi, \nabla_\varphi)}{\partial_\varphi} \cdot \nabla_\varphi \varphi \right) dx \\
\text{for a.a. } \tau \in (0, T). \quad \text{Indeed, the dissipation inequality (2.8) is obtained from the sum of identity (2.7) and the entropy balance (2.6) multiplied by } \varphi. \\
\]

Definition 2
We say that the triplet \((\varphi, \vartheta, \mathbf{u})\) is a renormalized weak solution to the Navier-Stokes-Fourier system (1.1 - 1.8) if it is a very weak solution, if \(b(\varphi) \in C^\text{weak}([0, T], L^1(\Omega))\) and if the couple \((\varphi, \mathbf{u})\) satisfies the continuity equation in the renormalized sense,
\[
\int_\Omega b(\varphi) \varphi dx \bigg|_0^\tau = \int_0^\tau \int_\Omega b(\varphi) \left( \partial_t \varphi + \mathbf{u} \cdot \nabla_\varphi \right) dx dt + \int_0^\tau \int_\Omega \left( b'(\varphi) - b(\varphi) \right) \text{div} \mathbf{u} \varphi dx dt \\
\text{for any } \tau \in [0, T], \text{ any } \\
b \in C^2(0, \infty) \cap C^1(0, \infty), \ b' \in L^\infty(0, 1), \ b'/z^{5/6}, \ z^b'/z^{5/2} \in L^\infty(1, \infty) \quad \text{and} \\
\varphi \in C^2([0, T] \times \Omega). \\
\]

Definition 3
We say that the triplet \((\varphi, \vartheta, \mathbf{u})\) is a suitable weak solution to the Navier-Stokes-Fourier system (1.1 - 1.8) if it is a weak solution and if it satisfies the relative entropy inequality
\[
\int_\Omega \left( \frac{1}{2} |\varphi - \mathbf{U}|^2 + \mathcal{E} (\varphi, \vartheta | r, \Theta) \right) dx \\
\text{for any } \tau \in [0, T], \text{ any } \\
b \in C^2(0, \infty) \cap C^1(0, \infty), \ b' \in L^\infty(0, 1), \ b'/z^{5/6}, \ z^b'/z^{5/2} \in L^\infty(1, \infty) \quad \text{and} \\
\varphi \in C^2([0, T] \times \Omega). \\
\]
\[ \begin{align*}
+ \int_0^\tau \int_\Omega \frac{S(\vartheta, \nabla_x u)}{\vartheta} : \nabla_x u \, dx \, dt &= \int_0^\tau \int_\Omega \frac{q(\vartheta, \nabla_x \vartheta)}{\vartheta} : \nabla_x \vartheta \, dx \, dt \\
\leq \int_\Omega \left( \frac{1}{2} \varrho_0 |u_0 - U(0, \cdot)|^2 + E(\varrho_0, \vartheta_0|0, \cdot), \Theta(0, \cdot) \right) \, dx \\
+ \int_0^\tau \int_\Omega S(\vartheta, \nabla_x u) : \nabla_x U \, dx \, dt &= \int_0^\tau \int_\Omega q(\vartheta, \nabla_x \vartheta) : \nabla_x \Theta \, dx \, dt \\
+ \int_0^\tau \int_\Omega \varrho \left( \frac{1}{2} \varrho_0 |u_0 - U(0, \cdot)|^2 + E(\varrho_0, \vartheta_0|0, \cdot), \Theta(0, \cdot) \right) \, dx \\
+ \int_0^\tau \int_\Omega \varrho (U - u) \cdot \nabla_x p(r, \Theta) \, dx \, dt \\
+ \int_0^\tau \int_\Omega \left( p(r, \Theta) - p(\varrho, \vartheta) \right) \div_x U \, dx \, dt \\
+ \int_0^\tau \int_\Omega \varrho \left( s(r, \Theta) - s(\varrho, \vartheta) \right) \left( \partial_t \Theta + u \cdot \nabla_x \Theta \right) \, dx \, dt \\
+ \int_0^\tau \int_\Omega \left( 1 - \frac{\varrho}{r} \right) \left( \partial_t p(r, \Theta) + U \cdot \nabla_x p(r, \Theta) \right) \, dx \, dt
\end{align*} \]

for a.a. \( \tau \in (0, T) \) with

\[ (r, \Theta, U) \in C^1([0, T] \times \Omega; R^5), \quad r > 0, \quad \Theta > 0, \quad U|_{\partial \Omega} = 0, \quad \text{(2.11)} \]

where

\[ E(\varrho, \vartheta | r, \Theta) = H_{\varrho}(\varrho, \vartheta) - \partial_\varrho H_{\varrho}(r, \Theta)(\varrho - r) - H_{\varrho}(r, \Theta), \]

\[ H_{\varrho}(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \Theta \varrho s(\varrho, \vartheta). \]

\[ \text{(2.12)} \]

2.2 Main results

In this section, we shall formulate four theorems. First theorem asserts existence of weak solutions.

**Theorem 2.1** Let \( \Omega \subset R^3 \) be a bounded Lipschitz domain. Suppose that the thermodynamic functions \( p, e, s \) satisfy hypotheses (1.18 - 1.24), and that the transport coefficients \( \mu, \eta, \kappa \) obey (1.25), (1.26). Finally assume that the initial data (1.7) verify (2.1–2.3). Then the complete Navier-Stokes-Fourier system (1.1–1.8) admits at least one renormalized weak solution.

The proof of this result can be found in [17, Chapter 3] for domains with regularity \( C^{2,\nu}, \nu \in (0, 1) \). The necessary modifications to accommodate the Lipschitz domains are explained in Poul [40]. In this Lecture Notes, we shall prove solely the weak compactness property of the (hypothetical) set of weak solutions. This property is usually considered to be the main ingredient in the existence proof. Note however, that the compressible models are very much "approximation sensitive". Therefore, the way from the weak compactness to the real existence can still be very long and should not be underestimated.
Theorem 2.2 Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Suppose that the thermodynamic functions $p$, $e$, $s$ satisfy hypotheses (1.18 - 1.24), and that the transport coefficients $\mu$, $\eta$, and $\kappa$ obey (1.25), (1.26). Finally assume that the initial data (1.7) verify (2.1–2.3). Let $(\varrho_n, \vartheta_n, u_n)$ be a sequence of weak solutions to the complete Navier-Stokes-Fourier system (1.1–1.8). Then there exists a subsequence (denoted again $(\varrho_n, \vartheta_n, u_n)$) such that

$$
\varrho_n \rightharpoonup^* \varrho \text{ in } L^\infty(0, T; L^\gamma(\Omega)),
$$

$$
\vartheta_n \rightharpoonup \vartheta \text{ in } L^2(0, T; W^{1,2}(\Omega)),
$$

$$
u_n \rightharpoonup u \text{ in } L^2(0, T; W^{1,2}_0(\Omega; \mathbb{R}^3)),
$$

and the trio $(\varrho, \vartheta, u)$ is again a weak solution of the complete Navier-Stokes-Fourier system (1.1–1.8).

This compactness result will be proved in Section 3.

The second result to be discussed in this Lecture Notes claims that any weak solution satisfies the relative entropy inequality.

Theorem 2.3 Let all assumptions of Theorem 2.1 be satisfied and let the trio $(\varrho, \vartheta, u)$ be a weak solution to the complete Navier-Stokes-Fourier system (1.1–1.8).

Then $(\varrho, \vartheta, u)$ is a suitable weak solution. In particular, it satisfies the relative entropy inequality (2.10) with test functions $(r, \Theta, U)$ belonging to the class (2.11).

Theorem 2.3 has been proved in [18]. We shall reproduce this proof in Section 4.

The third result to be dealt with in this Lecture Notes is the weak strong uniqueness principle. It asserts that any weak solution coincides with the strong solution emanating from the same initial data as long as the latter exists.

Theorem 2.4 Let assumptions of Theorem 2.1 be satisfied, where, moreover, the function $P$ introduced in (1.19) is twice continuously differentiable on $(0, \infty)$. Let $(\varrho, \vartheta, u)$ be a weak solution of the Navier-Stokes-Fourier system (1.1–1.8). Assume that $(\tilde{\varrho}, \tilde{\vartheta}, \tilde{u})$ is a classical (strong) solution to the Navier-Stokes-Fourier system (1.1–1.8) in $(0, T) \times \Omega$ that satisfies dissipation inequality (2.8) and that belongs to the class

$$
0 < \tilde{\varrho} \leq \tilde{\vartheta} \leq \tilde{\varrho} < \infty, \quad 0 < \varrho \leq \vartheta \leq \varrho < \infty, \quad \tilde{u} \in L^\infty(0, T; L^\infty(\Omega, \mathbb{R}^3)),
$$

$$
\partial_t \tilde{\varrho}, \partial_t \tilde{\vartheta}, \partial_t \tilde{u}, \nabla_x \tilde{\vartheta}, \nabla^m_x \tilde{\varrho}, \nabla^m_x \tilde{u} \in L^\infty(0, T; L^\infty(\Omega)), \quad m = 1, 2
$$

emanating from the same initial data. Then

$$(\varrho, \vartheta, u) = (\tilde{\varrho}, \tilde{\vartheta}, \tilde{u}) \text{ in } [0, T] \times \overline{\Omega}.$$ 

The weak strong uniqueness property in the class of weak solutions has been proved in [18]. This proof will be reproduced in Section 5.

Remark(Relaxing regularity of test functions in Theorem 2.3.)

Using a density argument we can extend the class of test functions $r$, $\Theta$, $U$ appearing in the relative entropy inequality (2.10), (2.11).

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• Sufficient conditions for the left hand side of the relative entropy inequality to be well defined are, for example,
\[ 0 < \varrho \leq r \leq \bar{\varrho} < \infty, \quad 0 < \vartheta \leq \Theta \leq \bar{\vartheta} < \infty, \]
(2.14)
\[ U \in L^\infty(0,T; L^2(\Omega, R^3)). \]
(2.15)

• A short inspection of the right hand side (2.10) implies that the integrals are well-defined if, for example
\[ \partial_t U \in L^\infty(0,T; L^6(\Omega)), \quad \nabla_x U \in L^\infty(0,T; L^\infty(\Omega, R^3 \times 3)), \]
(2.16)
\[ \partial_t \Theta \in L^\infty(0,T; L^4(\Omega)), \quad \nabla_x \Theta \in L^\infty(0,T; L^\infty(\Omega, R^3)), \]
(2.17)
\[ \partial_t r \in L^\infty(0,T; L^3(\Omega)), \quad \nabla_x r \in L^\infty(0,T; L^6(\Omega)). \]
(2.18)

• Finally,
\[ U|_{\partial \Omega} = 0. \]
(2.19)

Consequently, Theorem 2.3, in particular, the relative entropy inequality (2.10), are valid even if we replace the hypotheses on smoothness and integrability of the test functions \((r, \Theta, U)\) by weaker hypotheses, namely (2.14-2.19). In particular, \(r, \vartheta, U\) may be another (strong) solution emanating from the same initial data \(\varrho_0, \vartheta_0, u_0\).

Remark (Existence of classical solutions)
Existence of classical solutions in class (2.13) with sufficiently smooth initial data on a short time interval \([0, T_{\text{max}})\) that depends on the "size" of the initial data is well known on a large class of domains. The first results in this direction are due to Matsumura and Nishida [34], [35].

3 Weak compactness of the set of weak solutions

In this Section we shall prove Theorem 2.2

3.1 Estimates and weak limits

Step 1 (Estimates due to the Helmholtz free energy)
Let \((\varrho_n, \vartheta_n, u_n)\) be a sequence of weak solution of the problem (1.1–1.8) on \((0,T) \times \Omega\). Any triplet of this sequence satisfies, in particular, the dissipation inequality (2.8). The dissipation inequality will produce most of à priori estimates that are available in this problem. It will be convenient to split \(H_\pi(\varrho, \vartheta) - \partial_\varrho H_\pi(\varrho, \vartheta)(\varrho - \varrho_0) - H_\pi(\varrho, \vartheta)\) in the following way
\[
H_\pi(\varrho, \vartheta) - \partial_\varrho H_\pi(\varrho, \vartheta)(\varrho - \varrho) - H_\pi(\varrho, \vartheta) =
\]
(3.1)
\[
\left[H_\pi(\varrho, \vartheta) - H_\pi(\varrho, \vartheta)\right] + \left[H_\pi(\varrho, \vartheta) - \partial_\varrho H_\pi(\varrho, \vartheta)(\varrho - \varrho) - H_\pi(\varrho, \vartheta)\right].
\]
To continue we recall the equivalence
\[
P'(Z) \sim 1 + Z^{\gamma - 1}, \quad Z > 0
\]
(3.2)
that can be derived from (1.21–1.22). Employing (1.15), (1.18), (1.20) and (3.2), we obtain
\[
H_\vartheta(\varrho, \vartheta) - H_\vartheta(\varrho, \bar{\vartheta}) = \int_\Omega \vartheta \partial_\vartheta e(\varrho) d\vartheta \geq 4a \int_\Omega \vartheta^2 (z - \bar{\vartheta}) dz
\] (3.3)
and
\[
H_\vartheta(\varrho, \vartheta) - \partial_\varrho H_\vartheta(\bar{\varrho}, \vartheta)(\varrho - \bar{\varrho}) - H_\vartheta(\bar{\varrho}, \vartheta) = \int_\Omega \left( \int_\Omega \partial_\varrho^2 H(\varrho, \vartheta) d\varrho \right) d\vartheta - \int_\Omega \left( \int_\Omega \frac{1}{\varrho} \partial_\varrho \rho(w, \vartheta) d\varrho \right) d\vartheta
\] (3.4)
\[
\sim \left[ \varrho \log(\varrho/\bar{\varrho}) - \frac{\varrho}{\bar{\varrho}} \right] + \left[ \varrho^\gamma - \gamma \varrho^{\gamma-1}(\varrho - \bar{\varrho}) \right].
\]
With observations (3.3–3.4) at hand we deduce from (2.8) the following estimates
\[
\text{esssup}_{(0,T)} \int_\Omega \varrho_n u_n^2 dx \leq c,
\] (3.5)
\[
\text{esssup}_{(0,T)} \int_\Omega \varrho_n^2 dx \leq c,
\] (3.6)
\[
\text{esssup}_{(0,T)} \int_\Omega \varrho_n^4 dx \leq c
\] (3.7)
By virtue of (3.5–3.6), we deduce for the momentum,
\[
\|\varrho_n u_n\|_{L^\infty(0,T; L^{2/(\gamma+1)}(\Omega; R^3))} \leq c.
\] (3.8)

**Step 2 (Estimates due to the entropy production rate)**

Here and hereafter we shall need three specific inequalities that may be of general interest. We refer to [17, Chapter 2 and Appendix] for their proofs.

We recall first the following Korn’s type inequality

**Lemma 3.1** Let \( \Omega \) be a bounded Lipschitz domain. There exists \( c > 0 \) such that for all \( \varrho \in W_0^{1,2}(\Omega; R^3) \), there holds
\[
\| \nabla \varrho \|_{L^2(\Omega; R^{3 \times 3})} \leq c \| T(\nabla \varrho) \|_{L^2(\Omega; R^{3 \times 3})}.
\] (3.9)

Another inequality of general interest to be recalled at this place is the following Poincaré’s type inequality.

**Lemma 3.2** Let \( \Omega \) be a bounded Lipschitz domain. For any \( 0 < M < |\Omega| \) there exists \( c(M) \) such that for any \( \varrho \in W^{1,2}(\Omega) \) and \( V \subset \Omega \), \( |V| \geq M \), we have
\[
\| \varrho \|_{L^2(\Omega; R^3)} \leq c(M) \left( \| \nabla \varrho \|_{L^2(\Omega; R^{3 \times 3})} + \int_V w^2 dx \right).
\] (3.10)
The last inequality includes the properties of the entropy function \( S \).

**Lemma 3.3** Let \( \Omega \) be a bounded Lipschitz domain and \( p, \gamma > 1 \). Let \( S \in C(0,\infty) \) be a strictly decreasing function such that \( \lim_{Z \to \infty} S(Z) = 0 \) and
\[
\limsup_{n \to \infty} \int_{\{\varrho_n \leq \varrho_n^1/(\gamma-1)\}} \varrho_n S \left( \frac{\varrho_n}{\varrho_n^1/(\gamma-1)} \right) dx = 0
\]
whenever \( \rho_n \geq 0 \) is bounded in \( L^7(\Omega) \) and \( 0 \leq \vartheta_n \to 0 \) in \( L^p(\Omega) \).

Then for any \( M_0 > 0, \Gamma_0 > 0, \) and \( S_0 > 0 \) there exist \( \alpha = \alpha(M_0, \Gamma_0, S_0) > 0 \) such that for any non-negative functions \( \rho, \vartheta \)

\[
\int_\Omega \rho \, dx \geq M_0, \quad \int_\Omega (\rho^\gamma + \vartheta^p) \, dx \leq \Gamma_0
\]

and

\[
\int_\Omega \rho S \left( \frac{\vartheta}{\vartheta^{(\gamma-1)/2}} \right) \, dx \geq S_0
\]

we have

\[
\left\{ \vartheta \geq \bar{\vartheta} \right\} \geq \alpha.
\]

(3.11)

The "velocity part" of the entropy production yields bounds

\[
\| T(\nabla_x u_n) \|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{3 \times 3}))} \leq c,
\]

\[
T(\nabla_x w) = \nabla_x w + \nabla_x^T w - \frac{2}{3} \text{div} w,
\]

(3.12)

whence employing first Lemma 3.1 and then the standard Poincaré inequality, we get

\[
\| u_n \|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}))} \leq c.
\]

(3.13)

The "temperature part" of the entropy production rate gives

\[
\| \nabla_x \vartheta_n^\alpha \|_{L^2(0,T;L^2(\Omega))} \leq c, \quad \alpha \in [1,3/2]
\]

\[
\| \nabla_x \log \vartheta_n \|_{L^2(0,T;L^2(\Omega))} \leq c,
\]

(3.14)

where \( c > 0 \) is independent of \( n \). In agreement with (1.23–1.24),

\[
|\rho s(\rho, \vartheta)| \leq c(\rho + |\log \rho| + \rho(\log \vartheta)^+ + \vartheta^3).
\]

(3.15)

Consequently, assumptions of Lemma 3.11 are satisfied with some \( 3 < p < 4 \). Therefore, we deduce from (3.14) and Lemma 3.11

\[
\| \vartheta_n^\alpha - \bar{\vartheta}^\alpha \|_{L^2(0,T;W^{1,2}(\Omega))} \leq c, \quad \alpha \in [1,3/2],
\]

\[
\| \log \vartheta_n - \log \bar{\vartheta} \|_{L^2(0,T;W^{1,2}(\Omega))} \leq c.
\]

(3.16)

**Step 3 (Consequence of standard imbeddings)**

We get by the Sobolev imbedding and by interpolation from (3.7) and (3.16) that

\[
\| \vartheta_n \|_{L^3(0,T;L^3(\Omega))} \leq c, \quad \| \vartheta_n \|_{L^{17/4}(0,T \times \Omega)} \leq c.
\]

(3.17)

Recalling (3.7), (3.14), (3.13) and having in mind (1.4), (1.5), (1.21), (1.26) we deduce that

\[
\| S(\vartheta_n, \nabla_x u_n) \|_{L^2(0,T;L^{4/3}(\Omega;\mathbb{R}^{2 \times 3}))} \leq c.
\]

(3.18)
and
\[
\| \mathbf{q}(\varrho_n, \nabla \theta_n) / \varrho_n \|_{L^2(0,T;L^8/7(\Omega;\mathbb{R}^3))} \leq c. \tag{3.19}
\]

Returning to (3.15), we get
\[
\| \varrho_n s(\varrho_n, \theta_n) \|_{L^\infty(0,T;L^q(\Omega))} \leq c \text{ with some } q > 1 \tag{3.20}
\]
as well as
\[
\| \varrho_n s(\varrho_n, \theta_n) u_n \|_{L^q((0,T) \times \Omega;\mathbb{R}^3)} \leq c(n) \text{ with some } q > 1. \tag{3.21}
\]

**Step 4 (Estimates via the Bogovskii operator)**

Under assumptions (1.18–1.22)
\[
| p(\varrho, \theta) | \leq c(\varrho \theta + \varrho^2 + \theta^4). \tag{3.22}
\]

Consequently, we can deduce from (3.6–3.7) only
\[
\| p(\varrho_n, \theta_n) \|_{L^\infty(0,T;L^1(\Omega))} \leq c. \tag{3.23}
\]

We use in the momentum equation (2.5) (written with \((\varrho_n, \theta_n, u_n)\) on \(\Omega\)) the test function \(\varphi = B_\Omega(\varrho_n') \in L^\infty(0,T;W^{1,\pi/\nu}(\Omega))\), where \(\nu > 0\). A straightforward but laborious calculation leads to a conclusion that there exists \(\nu > 0\) such that
\[
\int_0^T \int_\Omega p(\varrho_n, \theta_n) \varphi_d \mathrm{d}x \mathrm{d}t \leq c, \\
\int_0^T \int_\Omega p^{1+\nu}(\varrho_n, \theta_n) \mathrm{d}x \mathrm{d}t \leq c. \tag{3.23}
\]

**Lemma 3.4** Let \(Q\) be a bounded Lipschitz domain. There an operator \(B_Q : L^p(Q) \to W^{1,p}_0(Q;\mathbb{R}^3), 1 < p < \infty\) satisfying,
\[
\text{div}B_Q(g) = g - 1/|Q| \int_Q g \mathrm{d}x, \\
\|B_Q(g)\|_{W^{1,p}(Q)} \leq c(Q) \|g\|_{L^p(Q)}, \\
\|B_Q(g)\|_{L^p(Q)} \leq c(Q) \|f\|_{L^p(Q)}, g = \text{div} f, f \cdot n|_{\partial Q} = 0.
\]

We use in the momentum equation (2.5) (written with \((\varrho_n, \theta_n, u_n)\) on \(\Omega\)) the test function \(\varphi = B_\Omega(\varrho_n') \in L^\infty(0,T;W^{1,\pi/\nu}(\Omega))\), where \(\nu > 0\). A straightforward but laborious calculation leads to a conclusion that there exists \(\nu > 0\) such that
\[
\int_0^T \int_\Omega p(\varrho_n, \theta_n) \varphi_d \mathrm{d}x \mathrm{d}t \leq c, \\
\int_0^T \int_\Omega p^{1+\nu}(\varrho_n, \theta_n) \mathrm{d}x \mathrm{d}t \leq c. \tag{3.23}
\]

See [17, Section 2.2.5, Appendix] for more details and calculations.

**Step 5 (Weak limits)**

Bounds (3.6–3.7), (3.13), (3.16–3.23) imply existence of a subsequence (denoted again \((\varrho_n, \theta_n, u_n)\)) and of a trio \((\varrho, \theta, u)\) such that
\[
\varrho_n \rightharpoonup^* \varrho \text{ in } L^\infty(0,T;L^\gamma(\Omega)), \\
\theta_n \rightharpoonup \vartheta \text{ in } L^\infty(0,T;L^4(\Omega)), \\
u_n \rightharpoonup \nu \text{ in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)), \\
u \rightharpoonup \vartheta \text{ in } L^2(0,T;W^{1,2}(\Omega)). \tag{3.24}
\]
Moreover, if we denote by \( \bar{g}(\varrho, \vartheta, \mathbf{u}) \) a weak limit of the sequence \( g(\varrho_n, \vartheta_n, \mathbf{u}_n) \) in \( L^1((0, T) \times \Omega) \), we have for the non linear quantities,

\[
\log \vartheta_n \rightarrow \log \vartheta \text{ in } L^2(0, T; W^{1,2}(\Omega)),
\]

\[
p(\varrho_n, \vartheta_n) \rightarrow \bar{p}(\varrho, \vartheta) \text{ in } L^q((0, T) \times \Omega_n) \text{ with some } q > 1,
\]

\[
\mathcal{S}(\vartheta_n, \nabla_x \mathbf{u}_n) \rightarrow \mathcal{S}(\vartheta, \nabla_x \mathbf{u}) \text{ in } L^{4/3}((0, T) \times \Omega; \mathbb{R}^{3 \times 3}),
\]

\[
q(\vartheta_n, \nabla_x \vartheta_n) \rightarrow q(\vartheta, \nabla_x \vartheta) \text{ in } L^{6/7}((0, T) \times \Omega; \mathbb{R}^3).
\]

Next, we can use continuity equation (2.4), renormalized continuity equation (2.9) and momentum equation (2.5) to show the equi-continuity of functions

\[
t \mapsto \int_\Omega \varrho_n \varphi \, dx, \quad t \mapsto \int_\Omega h(\varrho_n) \varphi \, dx, \quad t \mapsto \int_\Omega \varrho_n \mathbf{u}_n \varphi \, dx,
\]

on \([0, T]\), where \( \varphi \in C^\infty_c(\Omega) \). This fact makes possible to use the Arzela-Ascoli compactness argument which in combination with the density argument yields convergence of the sequences \( \varrho_n, h(\varrho_n) \) and \( \varrho_n \mathbf{u}_n \) in \( C_{\text{weak}}([0, T]; L^q(\Omega)) \) with some \( q > 6/5 \). Employing moreover the compact imbedding \( L^q(\Omega) \hookrightarrow W^{-1,2}(\Omega) \), we get the convergence of these quantities in \( L^2(0, T; W^{1,2}(\Omega)) \). Summarizing, the above, we have

\[
\varrho_n \rightarrow \varrho \text{ in } C_{\text{weak}}([0, T]; L^\gamma(\Omega)) \text{ and in } L^2(0, T; W^{-1,2}(\Omega)),
\]

\[
h(\varrho_n) \rightarrow h(\varrho) \text{ in } C_{\text{weak}}([0, T]; L^q(\Omega)), \quad q \in [1, \infty) \text{ and in } L^2(0, T; W^{-1,2}(\Omega)),
\]

\[
\varrho_n \mathbf{u}_n \rightarrow \varrho \mathbf{u} \text{ in } C_{\text{weak}}([0, T]; L^{2\gamma/(\gamma+1)}(\Omega; \mathbb{R}^3)) \text{ and in } L^2(0, T; W^{-1,2}(\Omega, R^1)),
\]

\[
\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n \rightarrow \varrho \mathbf{u} \otimes \mathbf{u} \text{ in } L^2(0, T; L^{6\gamma/(4\gamma+3)}(\Omega, \mathbb{R}^{3 \times 3})).
\]

**Step 6 (Limiting equations)**

Now, we are ready to let \( n \rightarrow \infty \) in equations (2.4), (2.5) and (2.9) (written with \((\varrho_n, \vartheta_n, \mathbf{u}_n)\)). We get, in particular,

\[
- \int_\Omega \varrho_0 \varphi(0, \cdot) \, dx = \int_0^T \int_\Omega \left( \varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla \varphi \right) \, dx \, dt
\]

for any \( \varphi \in C^1_c([0, T] \times \Omega) \);

\[
- \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi(0, \cdot) \, dx
\]

\[
\int_0^T \int_\Omega \left( \varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \varphi + \bar{p}(\varrho, \vartheta) \text{div} \varphi - \mathcal{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla \varphi \right) \, dx \, dt
\]

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for any $\varphi \in C^1_c([0, T) \times \Omega; R^3)$, $\varphi|_{\partial \Omega} = 0$;

$$
\int_0^T \int_\Omega \vartheta L_k(\vartheta)(\partial_t \varphi + u \cdot \nabla_x \varphi) \, dx \, dt = \int_0^T \int_\Omega T_k(\vartheta) \text{div}_x \varphi \, dx \, dt - \int_\Omega \vartheta_0 L_k(\vartheta_0) \varphi(0, \cdot) \, dx
$$

(3.29)

and

$$
\int_0^T \int_\Omega \vartheta T_k(\vartheta)(\partial_t \varphi + u \cdot \nabla_x \varphi) \, dx \, dt + \int_0^T \int_\Omega \vartheta_0 T_k(\vartheta - T_k(\vartheta)) \text{div}_x \varphi \, dx \, dt - \int_\Omega T_k(\vartheta_0) \varphi(0, \cdot) \, dx,
$$

(3.30)

where $\varphi \in C^1_c([0, T) \times \Omega)$.

$$
T_k(z) = kT(z/k), \quad L_k(z) = \int_1^z \frac{T_k(w)}{w^2} \, dw,
$$

$$
T(z) = \begin{cases} 
  z & \text{if } z \in [0, 1], \\
  \text{concave on } [0, \infty), \\
  2 & \text{if } z \geq 3.
\end{cases}
$$

3.2 Strong convergence of temperature

Step 1

The entropy inequality (2.6) can be rewritten as identity

$$
\int_\Omega \vartheta_0 s(\vartheta_0, \vartheta_0) \varphi(0, \cdot) \, dx + < \sigma_n, \varphi >
$$

(3.31)

$$
= - \int_0^T \int_\Omega \left( \vartheta_n s(\vartheta_n, \vartheta_n) \partial_t \varphi + \vartheta_n s(\vartheta_n, \vartheta_n) u_n \cdot \nabla_x \varphi + \frac{q(\vartheta_n, \nabla_x \vartheta_n) \cdot \nabla_x \varphi}{\vartheta_n} \right) \, dx \, dt
$$

where $\sigma_n$ is a positive continuous linear functional on the (metric) space $C^1_c([0, T) \times \Omega)$ defined by the above equation. We therefore easily verify by the standard Schwartz argument that

$$
| < \sigma_n, \varphi > | \leq c(K, n) \| \varphi \|_{C(K)} \text{ for any compact } K \subset [0, T) \times \Omega;
$$

whence, in fact, $\sigma_n$ can be extended in a unique way to a positive linear functional on the (metric) space $C_c([0, T) \times \Omega)$. By virtue of the Riesz representation theorem, $\sigma_n$ can be identified with a non negative regular measure $\mu_{\sigma_n}$ on the family of Borel subsets of $[0, T) \times \Omega$ such that,

$$
\mu_{\sigma_n}(K) < \infty, \text{ for any compact } K \subset [0, T) \times \Omega
$$

in the sense that

$$
< \sigma_n, \varphi > = \int_{[0, T) \times \Omega} \varphi \, d\mu_{\sigma_n}, \quad \varphi \in C_c([0, T) \times \Omega).
$$

Moreover, taking for any fixed compact interval $L \subset [0, T)$,

$$
\chi_L \in C^\infty_c([0, T) \times \Omega), \quad 0 \leq \chi_L \leq 1, \chi = 1 \text{ in } L \times \Omega, \quad \partial_t \chi_L \leq 0
$$

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and using (3.31) with test function \( \chi_L(t)1(x) \), we get, due to (3.19), (3.20), (3.21)
\[
| < \sigma_n, \chi_L(\cdot)1(\cdot) > | \leq c,
\]
where \( c \) is independent on \( L \) and \( n \).

Hence finally, the measure \( \mu_{\sigma_n} \) (acting on the family of Borel subsets of \([0, T) \times \Omega]\)) can be extended to the family of Borel subsets of \([0, T] \times \Omega \) by setting
\[
\mu_{\sigma_n}([T] \times \Omega) = 0.
\]
Consequently the linear functional \( \sigma_n \) (continuous on the metric space \( C_c([0, T] \times \Omega) \)) can be extended to a linear functional on the Banach space \( C_c([0, T] \times \Omega) = C([0, T] \times \Omega) \) via the formula
\[
< \sigma_n, \varphi > = \int_{[0,T] \times \Omega} \varphi d\mu_{\sigma_n}, \varphi \in C([0,T] \times \Omega),
\]
and we have
\[
||\sigma_n||_{C([0,T] \times \Omega)} = \mu_{\sigma_n}([0,T] \times \Omega) = \int_{[0,T] \times \Omega} d\mu_{\sigma_n} < c,
\]
where we have used (3.32) to get the last bound.

**Step 2**
We start by recalling the \( \text{Div-Curl} \) lemma, that says:

**Lemma 3.5** Let \( N \) be a positive integer and \( Q \) an open set in \( \mathbb{R}^N \). If \( X_n \to X \) in \( L^p(Q; \mathbb{R}^N) \), \( Y_n \to Y \) in \( L^q(Q; \mathbb{R}^N) \), \( \frac{1}{p} + \frac{1}{q} = \frac{1}{2} < 1 \) and the sequences \( \text{div}X_n, \text{curl}Y_n \) are compact in \( W^{-1,s}(Q) \) resp. \( W^{-1,s}(Q; \mathbb{R}^N \times \mathbb{R}^N) \) with some \( s > 1 \) then
\[
X_n \cdot Y_n \to X \cdot Y \text{ in } L^r(Q).
\]

We may now apply this lemma to the four dimensional vectors
\[
V_n = (\varrho_n s(\varrho_n, \vartheta_n), \varrho_n s(\varrho_n, \vartheta_n), q(\varrho_n, \nabla_x \vartheta_n) \vartheta_n), \quad W_n = (T_k(\vartheta_n), 0, 0, 0)
\]
Since \( \text{div}V_n = \sigma_n \) and since the imbedding \([C([0, T] \times \Omega)]^r \hookrightarrow W^{-1,q}((0, T) \times \Omega)\) is compact for any \( q \in (1, 4/3) \), the assumptions of the lemma on \((0, T) \times \Omega\) are satisfied. Therefore,
\[
\overline{T_k(\vartheta)} s_M(\varrho, \vartheta) + \frac{4}{3} a \overline{T_k(\vartheta)} \vartheta^3 = \overline{T_k(\vartheta)} \overline{s_M(\varrho, \vartheta)} + \frac{4}{3} a \overline{T_k(\vartheta)} \vartheta^3,
\]
where \( s_M(\varrho, \vartheta) = s(\varrho/\vartheta^{1/3}) \).

We shall first prove that
\[
\overline{T_k(\vartheta)} s_M(\varrho, \vartheta) \geq \overline{T_k(\vartheta)} \overline{s_M(\varrho, \vartheta)},
\]
where
\[
T_k(z) = kT(z/k), \quad C[0, \infty) \ni \mathcal{T} = \begin{cases} z \text{ if } z \in [0, 1], \\ \mathcal{T} \text{ strictly increasing on } [0, \infty), \\ \lim_{z \to \infty} \mathcal{T}(z) = 2. \end{cases}
\]
To this end we write
\[ \varrho_s \varrho \left( \varrho_n, \partial_n \right) \left( \mathcal{T}_k(\partial_n) - \mathcal{T}_k(\varrho) \right) = \]
\[ \varrho_n \left[ s_M \left( \varrho_n, \mathcal{T}_k^{-1}(\mathcal{T}_k(\partial_n)) \right) - s_M \left( \varrho_n, \mathcal{T}_k^{-1}(\mathcal{T}_k(\varrho)) \right) + \varrho_n s_M \left( \varrho_n, \mathcal{T}_k^{-1}(\mathcal{T}_k(\varrho)) \right) \mathcal{T}_k(\partial_n) - \mathcal{T}_k(\varrho) \right]. \]

Therefore, inequality (3.34) will be shown if we prove that
\[ \varrho_n s_M \left( \varrho_n, \mathcal{T}_k^{-1}(\mathcal{T}_k(\varrho)) \right) \left( \mathcal{T}_k(\partial_n) - \mathcal{T}_k(\varrho) \right) \rightarrow 0 \text{ weakly in } L^1((0,T) \times \Omega) \text{ as } n \rightarrow \infty. \] (3.35)

The quantity
\[ \varrho_n s_M \left( \varrho_n, \mathcal{T}_k^{-1}(\mathcal{T}_k(\varrho)) \right) \mathcal{L}(t, x, \varrho) \]
can be regarded as a composition of a Carathéodory function with a weakly convergent sequence \( \varrho_n \).

**Step 3**

In what follows, we shall use the fundamental theorem on parametrized Young measures. It is recalled in the following Lemma.

**Lemma 3.6** Let \( Q \in \mathbb{R}^N \) be a measurable set. Then for any bounded sequence \( w_n \) in \( L^1(Q; \mathbb{R}^M) \) there exists a family of parametrized Young measures \( \{ \nu_\varrho \}_{\varrho \in Q} \) such that
\[ \Psi(y, w) = \int_{\mathbb{R}^M} \Psi(y, \lambda) d\nu_\varrho(\lambda), \]
for any Carathéodory function \( \Psi : Q \times \mathbb{R}^M \rightarrow \mathbb{R} \) satisfying
\[ \Psi(y, w_n) \rightharpoonup \Psi(y, w) \text{ weakly in } L^1(Q). \]

Since according to (3.24), (3.26)
\[ h(\partial_n) \rightharpoonup h(\varrho) \text{ in } L^2(0,T; W^{-1,2}(\Omega)), \]
\[ G(\partial_n) \rightharpoonup G(\varrho) \text{ in } L^2(0,T; W^{1,2}(\Omega)), \]
we have
\[ h(\varrho) G(\varrho) = h(\varrho) G(\varrho) \] (3.37)
for any \( h \) belonging to the class (2.9) and \( G \in W^{1,\infty}((0,\infty)) \). This implies (3.35) by virtue of Lemma 3.6.

Indeed, denote \( \nu^\varrho_{(t, x)} \), \( \nu^\varrho_{(x, t)} \) and \( \nu^\varrho(t, x) \) the parametrized Young measures corresponding, in accordance with Lemma 3.6, to the sequences \( (\varrho_n, \partial_n) \), \( \varrho_n \) and \( \partial_n \), respectively. Then we have, due to (3.37) and in agreement with Lemma 3.6,
\[ \int_{\mathbb{R}^2} h(\lambda) G(\mu) d\nu^\varrho_{(t, x)}(\lambda, \mu) = \int_{\mathbb{R}} h(\lambda) d\nu^\varrho(\lambda) \times \int_{\mathbb{R}} G(\mu) d\nu^\varrho(\mu). \]

Consequently,
\[ \psi(t, x, \varrho) \mathcal{G}(\varrho)(t, x) = \int_{\mathbb{R}^2} \psi(t, x, \lambda) G(\mu) d\nu^\varrho_{(t, x)}(\lambda) d\nu^\varrho_{(x, t)}(\mu) = \left( \psi(t, x, \varrho) \mathcal{G}(\varrho) \right)(t, x). \]
This implies (3.35).

**Step 4**
Now, we have to recall several general properties of monotone operators with respect to weak convergence.

**Lemma 3.7** Let $Q \subset \mathbb{R}^N$ be a domain and $(P, G) \in C(R) \times C(R)$ two non decreasing functions. Let $w_n$ be a sequence of functions in $L^1(Q)$ such that

\[
\begin{cases}
  P(w_n) \to P(w), \\
  G(w_n) \to G(w), \\
  P(w_n)G(\vartheta_n) \to P(w)G(\vartheta)
\end{cases}
\]

in $L^1(Q)$.

(i) Then

\[ P(w) G(w) \leq P(w)G(\vartheta). \]

(ii) If $G(z) = z$ and if

\[ P(w) w = P(w)w, \]

then

\[ P(w) = P(w). \]

In the last two formulas $w$ denote weak limit of $w_n$ in $L^1(Q)$.

Lemma 3.7 implies, in particular,

\[ T_k(\vartheta) \vartheta^3 \geq T_k(\vartheta) \vartheta^3 \]

that in turn with (3.33–3.34) yields

\[ T_k(\vartheta) \vartheta^3 = T_k(\vartheta) \vartheta^3 \]

and finally, by monotone convergence, as $k \to \infty$,

\[ \overline{\vartheta^3} = \vartheta \overline{\vartheta^3}. \] (3.38)

The last identity implies

\[ \vartheta_n \to \vartheta \text{ a.e. in } (0, T) \times \Omega. \] (3.39)

**Step 5**
Coming back with (3.39) to the momentum equation (3.28), we obtain

\[ -\int_\Omega \varrho_0 u_0 \cdot \varphi(0, \cdot) \, dx \]

\[ = \int_0^T \int_\Omega \left( \varrho u \cdot \partial_t \varphi + \varrho u \otimes \varrho + \varrho p(\vartheta, \partial_t) \text{div} \varphi - S(\vartheta, \nabla_x u) : \nabla_x \varphi \right) \, dx \, dt \]

for any $\varphi \in C^1([0, T) \times \Omega; \mathbb{R}^3)$, $\varphi|_{\partial \Omega} = 0$.

Moreover, estimate (2.8) yields boundedness of the sequences

\[ \sqrt{\frac{\mu(\vartheta_n)}{\vartheta_n}} (\nabla_x u_n \otimes \vartheta_n)^T = \frac{2}{3} \text{div} u_n, \quad \sqrt{\frac{\eta(\vartheta_n)}{\vartheta_n}} \text{div} u_n, \quad \sqrt{\frac{\kappa(\vartheta_n)}{\vartheta_n}} \nabla_x \vartheta_n \]

(3.41)
in $L^2((0,T)\times \Omega)$; whence by the lower weak continuity combined with (3.39) and (3.24) one gets

$$\int_0^T \int_\Omega \frac{\varphi}{\vartheta} \left( S(\vartheta, \nabla_x u) : \nabla_x u - \frac{q(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) dx \, dt \leq \liminf_{n \to \infty} \int_0^T \int_\Omega \frac{\varphi}{\vartheta} \left( S(\vartheta_n, \nabla_x u_n) : \nabla_x u_n - \frac{q(\vartheta_n, \nabla_x \vartheta_n) \cdot \nabla_x \vartheta_n}{\vartheta_n} \right) dx \, dt,$$

for any $0 \leq \varphi \in C_c((0,T) \times \Omega)$.

Thus effectuating the limit $n \to \infty$ in (2.6) (with $\vartheta_n, \vartheta, u_n$ on place of $\vartheta, \vartheta, u$), we get

$$\int_\Omega \varrho_0 s(\varrho_0, \vartheta_0) \varphi(0,\cdot) \, dx + \int_0^T \int_\Omega \frac{\varphi}{\vartheta} \left( S(\vartheta, \nabla_x u) : \nabla_x u - \frac{q(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) dx \, dt \leq -\int_0^T \int_\Omega \left( \varrho s(\varrho, \vartheta) \partial_t \varphi + \varrho s(\varrho, \vartheta) u \cdot \nabla_x \varphi + \frac{q(\varrho, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \varphi \right) dx \, dt,$$

for any $\varphi \in C_1^1((0,T) \times \Omega)$, $\varphi \geq 0$.

### 3.3 Strong convergence of densities

#### 3.3.1 Effective viscous flux

The main result of this section is the following lemma.

**Lemma 3.8**

$$T_k(\varrho) \text{div}_x u - T_k(\varrho) \text{div}_x u = \frac{1}{3} \mu(\varrho) + \eta(\varrho) \left( \frac{p(\varrho, \vartheta)}{\varrho} T_k(\varrho) - p(\varrho, \vartheta) T_k(\varrho) \right).$$

Lemma 3.8 will be proved in several steps.

**Step 1**

First we introduce operators $A = \nabla_x \Delta^{-1}$ and $\mathcal{R} = \nabla_x \otimes \nabla_x \Delta^{-1}$,

$$(\nabla \Delta^{-1})_i(v) = -\mathcal{F}^{-1} \left[ \frac{\xi_i}{|\xi|^2} \mathcal{F}(v)(\xi) \right], \quad (\nabla \otimes \nabla \Delta^{-1})_{ij}(v) = \mathcal{F}^{-1} \left[ \frac{\xi_i \xi_j}{|\xi|^2} \mathcal{F}(v)(\xi) \right],$$

where $\mathcal{F}$ denotes the Fourier transform

$$[\mathcal{F}(v)](\xi) = \frac{1}{2\pi^3} \int_{\mathbb{R}^3} v(x) \exp(-i\xi \cdot x) \, dx.$$

We recall that these operators have the following properties:

(i) $A$ is a continuous linear operator from $L^1 \cap L^2(\mathbb{R}^3)$ to $L^2 + L^\infty(\mathbb{R}^3; \mathbb{R}^3)$ and from $L^p(\mathbb{R}^3)$ to $L^{3p/(3-p)}(\mathbb{R}^3; \mathbb{R}^3)$ for any $1 < p < 3$.

(ii) $\mathcal{R}$ is a continuous linear operator from $L^p(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3; \mathbb{R}^{3\times 3})$ for any $1 < p < \infty$. 

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(iii) The following formulas hold

$$R(v) = R^T(v), \quad \sum_{j=1}^{3} R_{jj}(v) = v, \quad v \in L^p(R^3),$$

$$\partial_k R_{ij}(v) = R_{ij}(\partial_k v), \quad R_{ij}(\partial_k v) = R_{ik}(\partial_j v), \quad v \in W^{1,p}(R^3),$$

where $1 < p < \infty$;

$$\nabla_x A(v) = R(v), \quad \text{div} A(v) = v, \quad v \in L^p(R^3),$$

where $1 < p < 3$;

$$\int_{R^3} A(v) w \, dx = - \int_{R^3} v A(w) \, dx,$$

with

$$v \in L^p(R^3), \quad w \in L^q(R^3), \quad A(w) \in L^{p'}(R^3), \quad A(v) \in L^{q'}(R^3),$$

where $1 < q, p < 3$;

$$\int_{R^3} R(v) w \, dx = \int_{R^3} v R(w) \, dx, \quad v \in L^p(R^3), \quad w \in L^{p'}(R^3)$$

where $1 < p < \infty$.

**Step 2**

We subtract the limit $n \to \infty$ of the momentum equation (2.5) on the level $n$ tested with $\varphi = \zeta \nabla \Delta^{-1}[T_k(\varphi_n)1_\Omega], \quad \zeta \in C^\infty((0, T) \times \Omega; R^3)$ from the limiting momentum equation (3.40) with the test function $\varphi = \zeta \nabla \Delta^{-1}[T_k(\varphi)1_\Omega]$. This procedure yields

$$\lim_{n \to \infty} \int_0^T \int_{\Omega} \zeta(t, x) \left( p(\varphi_n, \vartheta_n) T_k(\varphi_n) - S(\vartheta_n, u_n) : R[1_\Omega T_k(\varphi_n)] \right) \, dx \, dt \quad \text{and} \quad \int_0^T \int_{\Omega} \zeta(t, x) \left( [p(\varphi, \vartheta) T_k(\varphi) - S(\vartheta, u) : R[1_\Omega T_k(\varphi)]] \right) \, dx \, dt \quad \text{(3.45)}$$

$$\lim_{n \to \infty} \int_0^T \int_{\Omega} \left( T_k(\varphi_n) u_n \cdot \nabla \zeta(\varphi_n) u_n - \zeta \varphi_n (u_n \otimes u_n) : R[1_\Omega T_k(\varphi_n)] \right) \, dx \, dt$$

$$- \int_0^T \int_{\Omega} \left( T_k(\varphi) u \cdot \nabla \zeta(\varphi) u - \zeta \varphi (u \otimes u) : R[1_\Omega T_k(\varphi)] \right) \, dx \, dt,$$

where $(R[z])_j = \sum_{k=1}^{3} R_{jk}[z_k]$. In fact, other terms that should eventually appear at the right hand side are zero by virtue of standard compactness arguments.

**Step 3**

Next, we shall need the following lemma in the spirit of Meyer [37], see [17, Theorem 10.27]
Lemma 3.9 (Commutators I) \( \text{Let } U_n \rightarrow U \text{ in } L^p(R^3; R^3), \ v_n \rightarrow v \text{ in } L^q(R^3) \), where \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1 \).

Then \( v_n R[U_n] - R[v_n]U_n \rightarrow v R[U] - R[v]U \) in \( L^r(R^3; R^3) \).

Lemma 3.9 is a consequence of the Dic-curl lemma 3.5. Indeed, realizing that
\[
U \cdot R(V) - V \cdot R(U) = X \cdot R(V) + Y \cdot R(U),
\]
where
\[
X = U - R(U), \quad Y = R(V) - V.
\]
Seeing that \( \text{div} X = \text{div} Y = 0 \) and that \( \text{curl} R(U) = \text{curl} R(V) = 0 \), we obtain employing Lemma 3.5.

\[
V_n \cdot R[U_n] - U_n \cdot R[V_n] \rightarrow V \cdot R[U] - U \cdot R[V]
\]
provided \( V_n \rightarrow V \) in \( L^q(R^3) \), \( U_n \rightarrow U \) in \( L^p(R^3; R^3) \). Taking in the last formula subsequently \( V_n = (v_n, 0, 0), (0, v_n, 0), \) and \( (0, 0, v_n) \) we get Lemma 3.9.

**Step 4**
Combining this lemma with the convergence established in the first two lines of (3.26), we get
\[
\left( T_k(\varrho_n) R[\varrho \omega_n u_n] - \varrho \omega_n u_n \cdot R[1_\Omega T_k(\varrho_n)] \right) (t) \rightarrow \left( T_k(\varrho) R[\varrho \omega u] - \varrho \omega u \cdot R[1_\Omega T_k(\varrho)] \right) (t)
\]
weakly in \( L^r(\Omega; R^3) \) with some \( r > 6/5 \) for all \( t \in [0, T] \); whence by the compact imbedding \( L^r(\Omega) \hookrightarrow W^{-1,2}(\Omega) \) and the Lebesgue dominated convergence theorem used over \( (0, T) \) we conclude
\[
\int_0^T \int_\Omega u_n \cdot \left( T_k(\varrho_n) R[\varrho \omega_n u_n] - \varrho \omega_n u_n \cdot R[1_\Omega T_k(\varrho_n)] \right) dx dt \\
\rightarrow \int_0^T \int_\Omega u \cdot \left( T_k(\varrho) R[\varrho \omega u] - \varrho \omega u \cdot R[1_\Omega T_k(\varrho)] \right) dx dt.
\]
Thus (3.45) yields
\[
\int_0^T \int_\Omega \zeta(t, x) \left( p(\varrho, \vartheta) T_k(\varrho) - p(\varrho, \vartheta) T_k(\varrho) \right) dx dt
\]
\[
= \int_0^T \int_\Omega \zeta(t, x) \left( S(\varrho, \vartheta) : R[1_\Omega T_k(\varrho)] - S(\varrho, \vartheta) : R[1_\Omega T_k(\varrho)] \right) dx dt.
\]

**Step 5**
Hereafter we will need another another commutator lemma in the spirit of Meyer [37], see [17, Theorem 10.28]
Lemma 3.10 (Commutators II) Let \( w \in W^{1,r}(R^3) \), \( z \in L^p(R^3; R^3) \), \( 1 < r < 3 \), \( \frac{1}{r} + \frac{1}{p} < \frac{1}{2} < 1 \).

Then for all such \( s \) we have

\[
||R[wz] - wR[z]||_{W^{s,\beta}(R^3; R^3)} \leq C||w||_{W^{1,r}(R^3)}||z||_{L^p(R^3; R^3)},
\]

where \( \frac{\beta}{s} = \frac{1}{r} - \frac{1}{6} - \frac{1}{2} \). Here, \( ||\cdot||_{W^{s,\beta}(R^3)} \) denotes the norm in the Sobolev–Slobodetskii space \( W^{s,\beta}(R^3) \).

We can write

\[
\int_0^T \int_\Omega \zeta(t, x) \mathbb{S}(\vartheta, u) : R[1_\Omega T_k(\vartheta)] \, dx \, dt = \int_0^T \int_\Omega \zeta(t, x) \left( -\frac{2}{3}\mu(\vartheta) + \eta(\vartheta) \right) T_k(\vartheta) \mathbb{D}_u \, dx \, dt
\]

\[
+ \int_0^T \int_\Omega T_k(\vartheta) \left\{ \mathcal{R} : \left[ \zeta(\vartheta) \left( \nabla_x u + (\nabla_x u)^T \right) \right] - \zeta(\vartheta) \mathcal{R} : \left[ \nabla_x u + (\nabla_x u)^T \right] \right\} \, dx \, dt
\]

\[
+ \int_0^T \int_\Omega T_k(\vartheta) \zeta(\vartheta) \mathcal{R} : \left[ \nabla_x u + (\nabla_x u)^T \right] \, dx \, dt,
\]

where \( \mathcal{R} : (Z) = \sum_{i,j=1}^3 \mathcal{R}_{ij}(Z_{ij}) \). Whence

\[
\int_0^T \int_\Omega \zeta(t, x) \mathbb{S}(\vartheta, u) : R[1_\Omega T_k(\vartheta)] \, dx \, dt = \lim_{n \to \infty} \int_0^T \int_\Omega \zeta(t, x) \left( \frac{4}{3}\mu(\vartheta_n) + \eta(\vartheta_n) \right) T_k(\vartheta_n) \mathbb{D}_u \, dx \, dt
\]

\[
+ \lim_{n \to \infty} \int_0^T \int_\Omega T_k(\vartheta_n) \omega(\vartheta_n, u_n) \, dx \, dt
\]

and

\[
\int_0^T \int_\Omega \zeta(t, x) \mathbb{S}(\vartheta, u) : R[1_\Omega T_k(\vartheta)] \, dx \, dt = \int_0^T \int_\Omega \zeta(t, x) \left( \frac{4}{3}\mu(\vartheta) + \eta(\vartheta) \right) T_k(\vartheta) \mathbb{D}_u \, dx \, dt
\]

\[
+ \int_0^T \int_\Omega T_k(\vartheta) \omega(\vartheta, u) \, dx \, dt,
\]

where

\[
\omega(\vartheta_n, u_n) = \left( \mathcal{R} : \left[ \zeta(t, x) \mu(\vartheta_n) \left( \nabla u_n + (\nabla u_n)^T \right) \right] - \zeta(t, x) \mu(\vartheta_n) \mathcal{R} : \left[ \nabla u_n + (\nabla u_n)^T \right] \right).
\]

Thanks to Lemma 3.10, the sequence \( \omega(\vartheta_n, u_n) \) is bounded in \( L^1(0, T; W^{3,\beta}(\Omega; R^3)) \) with some \( \beta \in (0, 1) \), \( q > 1 \); whence \( \text{curl} U_n \) is compact in \( W^{-1,r}(0, T) \times \Omega; R^{3 \times 3} \) (and of course \( \text{div} V_n \) is compact in \( W^{-1,r}(0, T) \times \Omega; R^{3 \times 3} \), cf. (2.9)) with some \( r > 1 \), where

\[
V_n \equiv [T_k(\vartheta_n), T_k(\vartheta_n) u_n], \quad U_n \equiv [\omega(\vartheta_n, u_n), 0, 0, 0].
\]

We may thus apply a convenient version of Div-curl lemma (see e.g. [17, Theorem 10.21]) to these 4-dimensional vector fields to get

\[
\omega(\vartheta_n, u_n) T_k(\vartheta_n) \rightarrow \omega(\vartheta, u) T_k(\vartheta),
\]

where, due to (3.39),

\[
\omega(\vartheta, u) = \omega(\vartheta, u).
\]

This result in combination with (3.46) and (3.47) yields the effective viscous flux identity (3.44).
3.3.2 Oscillations defect measure and renormalized continuity equation

Going back to (3.2), we deduce employing the hypotheses (1.18–1.22) that

\[ p(\varrho, \vartheta) = d\vartheta^\gamma + p_m(\varrho, \vartheta), \quad \text{for some } d \geq 0, \tag{3.49} \]

where \( \partial_\varrho p_m(\varrho, \vartheta) \geq 0. \)

We calculate

\[
d \limsup_{n \to \infty} \int_0^T \int_\Omega \frac{\zeta}{1+\vartheta} |T_k(\varrho_n) - T_k(\varrho)|^{\gamma+1} \, dx \, dt \leq \tag{3.50}
\]

\[
d \limsup_{n \to \infty} \int_0^T \int_\Omega \frac{\zeta}{1+\vartheta} |(T_k(\varrho_n) - T_k(\varrho))(\varrho_n - \varrho^\gamma)| \, dx \, dt \leq
\]

\[
\int_0^T \int_\Omega \frac{\zeta}{1+\vartheta} (\varrho^\gamma T_k(\varrho) - \vartheta \cdot \vartheta^\gamma T_k(\varrho))^2 \, dx \, dt \leq \int_0^T \int_\Omega \frac{\zeta}{1+\vartheta} \left( p(\varrho, \vartheta) T_k(\varrho) - \vartheta \int_\Omega \frac{\partial \varphi}{\partial \varrho} T_k(\varrho) \right)^2 \, dx \, dt,
\]

where \( \zeta \in C^\infty([0,T] \times \Omega), \varrho \geq 0. \) To get the first inequality we have used \((b-a)^\gamma \leq b^\gamma - a^\gamma\) and \(T_k(b) - T_k(a) \leq b - a, \) with any \( b \geq a \geq 0. \) To derive the second one, we have used the inequalities \( \vartheta \geq \varrho^\gamma \) and \( T_k(\varrho) \leq T_k(\vartheta) \) that hold due to the convexity of functions \( \varrho \to \vartheta^\gamma \) and the concavity of functions \( \varrho \to T_k(\varrho). \) Finally, to derive the third one, we have employed (3.49), the fact that \( \overline{p(\varrho, \vartheta) T_k(\varrho)} = \overline{\varrho} \overline{p(\varrho, \vartheta)} T_k(\varrho) \) for any bounded function \( \varrho. \) This result holds since the sequence \( \vartheta_s \) converges almost everywhere to \( \vartheta, \) see (3.39)), and due to the well known relation between the weak limits of monotone functions

\[
\overline{p_m(\cdot, \varrho) T_k(\cdot)} - \overline{p_m(\cdot, \vartheta)} T_k(\vartheta) \geq 0, \tag{3.51}
\]

cf. Lemma 3.7.

Next, we verify that

\[
\int_0^T \int_\Omega |T_k(\varrho_n) - T_k(\vartheta)|^q \, dx \, dt \leq \int_0^T \int_\Omega \frac{1}{(1+\vartheta)^\beta} |T_k(\varrho_n) - T_k(\varrho)|^q \frac{1}{(1+\vartheta)^\beta} \, dx \, dt
\]

\[
\leq c \left[ \int_0^T \int_\Omega \frac{1}{(1+\vartheta)^\beta} |T_k(\varrho_n) - T_k(\varrho)|^{\gamma+1} \, dx \, dt \right]^{q/(\gamma+1)},
\]

where \( q > 2, \) provided \( \beta(\gamma+1) = q \) and \( \beta(\gamma+1)/(\gamma+1-q) \leq 17/3, \) cf. (3.17).

The expression

\[
\int_0^T \int_\Omega \frac{1}{1+\vartheta} |T_k(\varrho_n) - T_k(\varrho)|^{\gamma+1} \, dx \, dt
\]

that stays at the right hand side of the last inequality can be calculated from the effective viscous flux identity (3.44); using (3.13) we find that

\[
\int_0^T \int_\Omega \frac{1}{1+\vartheta} |T_k(\varrho_n) - T_k(\varrho)|^{\gamma+1} \, dx \, dt \leq c(n) \|T_k(\varrho_n) - T_k(\varrho)\|_{L^2((0,T) \times \Omega)}
\]

\[
\leq \|T_k(\varrho_n) - T_k(\vartheta)\|^q_{L^2((0,T) \times \Omega)} \|T_k(\varrho_n) - T_k(\varrho)\|^{q/(2q-2)}_{L^2((0,T) \times \Omega)}.
\]

Consequently, one concludes that

\[
\text{osc}_q[\varrho_n \to \varrho][(0,T) \times \Omega] \equiv \sup_{k>0} \limsup_{n \to 0} \int_0^T \int_\Omega |T_k(\varrho_n) - T_k(\varrho)|^q \, dx \, dt \leq \text{ with some } q > 2, \tag{3.52}
\]

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where the expression at the left hand side is called oscillations defect measure, see [17, Chapter 3, Section 3.7.5].

On the other hand, relation (3.52) implies that the limit quantities $\varrho, u$ satisfy the renormalized equation of continuity (2.9), see [17, Lemma 3.8], that reads

**Lemma 3.11 (Renormalized continuity equation)** Let $Q \subset \mathbb{R}^3$ be open set and let

$$
\varrho_n \to \varrho \quad \text{in } L^1((0, T) \times Q), \\
u_n \to \mathbf{u} \quad \text{in } L^r((0, T) \times Q; \mathbb{R}^3), \\
abla u_n \to \nabla u \quad \text{in } L^r((0, T) \times Q; \mathbb{R}^{3 \times 3}), \quad r > 1.
$$

Let

$$\text{osc}_{Q}[\varrho_n \to \varrho](0, T) < \infty$$

(3.53)

for $\frac{1}{q} < 1 - \frac{1}{r}$, where $(\varrho_n, u_n)$ solve the renormalized continuity equation (2.9) (with $\Omega$ replaced by $Q$).

Then the limit functions $\varrho, u$ solve (2.9) with ($\Omega$ replaced by $Q$) as well.

We deduce using Lemma 3.11 that

$$
\int_0^T \int_{\Omega} p(\varrho, \vartheta) \left( \vartheta \partial_t \varphi + \mathbf{u} \cdot \nabla \varphi \right) d\mathbf{x} dt = \int_0^T \int_{\Omega} T_k(\varrho) \text{div} \mathbf{u} \varphi d\mathbf{x} dt - \int_{\Omega} \varrho_0 L_k(\varrho_0) \varphi(0, \cdot) d\mathbf{x},
$$

(3.54)

where $L_k(\varrho)$ is defined in (3.30).

Now, we write (3.29) and (3.54) with test function $\varphi = 1$ and deduce

$$
\int_{\Omega} \left( p(\varrho, \vartheta) T_k(\varrho) - p(\varrho, \vartheta) T_k(\varrho) \right)(\tau) d\mathbf{x} = \int_0^T \int_{\Omega} \left( T_k(\varrho) \text{div} \mathbf{u} - T_k(\varrho) \text{div} \mathbf{u} \right) d\mathbf{x} dt + \int_0^T \int_{\Omega} \left( T_k(\varrho) \text{div} \mathbf{u} - T_k(\varrho) \text{div} \mathbf{u} \right) d\mathbf{x} dt
$$

for a.a. $\tau \in (0, T)$, where the second term at the right hand side tends to 0 as $k \to \infty$. Reasoning as in (3.51) we verify that

$$
\frac{1}{\varrho} \log \varrho - \varrho \log \varrho > 0 \quad \text{a.e. in } (0, T) \times \Omega.
$$

Consequently, there holds

$$
\int_{\Omega} \left( \frac{1}{\varrho} \log \varrho - \varrho \log \varrho \right)(\tau) d\mathbf{x} \leq 0.
$$

Finally, since the function $z \to z \log z$ is strictly convex on $(0, \infty)$, we necessarily have

$$
\frac{1}{\varrho} \log \varrho - \varrho \log \varrho = 0 \quad \text{a.e. in } (0, T) \times \Omega.
$$

$$
\varrho_s \to \varrho \quad \text{a.e. in } (0, T) \times \Omega.
$$

(3.55)

With (3.39) and (3.55) at hand, momentum equation (3.28) turns into (2.5) and entropy inequality (3.43) turns into (2.6). Moreover, clearly

$$
\int_0^T \eta'(t) \int_{\Omega} \left( \frac{1}{2} \varrho_u \mathbf{u}_n^2 + \varrho_e(\varrho_n, \vartheta_n) \right) d\mathbf{x} dt \to \int_0^T \eta'(t) \int_{\Omega} \left( \frac{1}{2} \varrho \mathbf{u}^2 + \varrho_e(\varrho, \vartheta) \right) d\mathbf{x} dt,
$$

cf. last line in (3.26), (3.23) and (3.39), (3.55). The latter limit yields (2.7). Theorem 2.3 is proved.
4 Proof of Theorem 2.3. Relative entropy inequality.

We deduce a relative entropy inequality satisfied by any weak solution to the Navier-Stokes-Fourier system. To this end, consider a trio \((r, \Theta, \textbf{U})\) of smooth functions, \(r\) and \(\Theta\) bounded below away from zero in \([0, T] \times \Omega\), and \(\textbf{U}|_{\partial \Omega} = 0\).

Taking \(\varphi = \frac{1}{2}|\textbf{U}|^2\) as a test function in (2.4), we get

\[
\int_\Omega \frac{1}{2} \partial_t |\textbf{U}|^2 (\tau, \cdot) \, dx = \int_\Omega \frac{1}{2} \partial_t |\textbf{U}(0, \cdot)|^2 \, dx + \int_0^T \int_\Omega \left( \textbf{e} \cdot \partial_t \textbf{U} + \textbf{g} \cdot \nabla_x \textbf{U} \right) \, dx \, dt. \tag{4.1}
\]

Similarly, the choice \(\varphi = \textbf{U}\) in (2.5) gives rise to

\[
\int_\Omega \textbf{g} \cdot \textbf{U} (\tau, \cdot) \, dx - \int_\Omega \varphi_0 \cdot \textbf{U}(0, \cdot) \, dx \tag{4.2}
\]

\[
= \int_0^T \int_\Omega \left( \textbf{g} \cdot \partial_t \textbf{U} + \textbf{g} \cdot \nabla_x \textbf{U} : \nabla_x \textbf{U} - p(\varphi, \varrho) \text{div}_x \textbf{U} - S(\varphi, \nabla_x \textbf{U} : \nabla_x \textbf{U}) \right) \, dx \, dt.
\]

Combining relations (4.1), (4.2) with the total energy balance (2.7) we may infer that

\[
\int_\Omega \left( \frac{1}{2} \varphi |\textbf{u} - \textbf{u}_0|^2 + \varphi \varrho (\varphi, \varrho) \right) (\tau, \cdot) \, dx = \int_\Omega \left( \frac{1}{2} \varphi_0 |\textbf{u}_0 - \textbf{U}(0, \cdot)|^2 + \varphi_0 \varrho(\varphi, \varrho_0) \right) \, dx \tag{4.3}
\]

\[
+ \int_0^T \int_\Omega \left( \left( \varphi \partial_t \textbf{U} + \varphi \textbf{u} \cdot \nabla_x \textbf{U} \right) \cdot (\textbf{U} - \textbf{u}) - p(\varphi, \varrho) \text{div}_x \textbf{U} + S(\varphi, \nabla_x \textbf{U} : \nabla_x \textbf{U}) \right) \, dx \, dt.
\]

Now, take \(\varphi = \Theta > 0\) as a test function in the entropy inequality (2.6) to obtain

\[
\int_\Omega \varphi_0 s(\varphi_0, \varrho_0) \Theta (0, \cdot) \, dx - \int_\Omega \varphi s(\varphi, \varrho) \Theta (\tau, \cdot) \, dx \tag{4.4}
\]

\[
+ \int_0^T \int_\Omega \frac{\Theta}{\varrho} \left( S(\varphi, \nabla_x \textbf{u}) : \nabla_x \textbf{u} - \frac{q(\varphi, \nabla_x \varphi) \cdot \nabla_x \varphi}{\varrho} \right) \, dx \, dt
\]

\[
\leq - \int_0^T \int_\Omega \left( \varphi s(\varphi, \varrho) \partial_t \Theta + \varrho s(\varphi, \varrho) \textbf{u} \cdot \nabla \Theta + \frac{q(\varphi, \nabla_x \varphi) \cdot \nabla_x \Theta}{\varrho} \right) \, dx \, dt.
\]

Thus, the sum of (4.3), (4.4) reads

\[
\int_\Omega \left( \frac{1}{2} \varphi |\textbf{u} - \textbf{U}|^2 + \varphi \varrho (\varphi, \varrho) - \Theta \varrho s(\varphi, \varrho) \right) (\tau, \cdot) \, dx \tag{4.5}
\]

\[
+ \int_0^T \int_\Omega \frac{\Theta}{\varrho} \left( S(\varphi, \nabla_x \textbf{u}) : \nabla_x \textbf{u} - \frac{q(\varphi, \nabla_x \varphi) \cdot \nabla_x \varphi}{\varrho} \right) \, dx \, dt
\]

\[
= \int_\Omega \left( \frac{1}{2} \varphi_0 |\textbf{u}_0 - \textbf{U}(0, \cdot)|^2 + \varphi_0 \varrho(\varphi_0, \varrho_0) - \Theta(0, \cdot) \varrho_0 s(\varphi_0, \varrho_0) \right) \, dx
\]

\[
+ \int_0^T \int_\Omega \left( \left( \varphi \partial_t \textbf{U} + \varphi \textbf{u} \cdot \nabla_x \textbf{U} \right) \cdot (\textbf{U} - \textbf{u}) - p(\varphi, \varrho) \text{div}_x \textbf{U} + S(\varphi, \nabla_x \textbf{U} : \nabla_x \textbf{U}) \right) \, dx \, dt
\]
Next, we take \( \varphi = \partial_t H_\Theta(r, \Theta) \) as a test function in (2.4) to deduce that

\[
\int_\Omega \varphi \partial_t H_\Theta(r, \Theta)(\tau, \cdot) \, dx = \int_\Omega \varphi_0 \partial_t H_{\Theta(0, \cdot)}(r(0, \cdot), \Theta(0, \cdot)) \, dx \\
+ \int_0^\tau \int_\Omega \left( \varphi \partial_t \left( \partial_x H_\Theta(r, \Theta) \right) + \varphi \nabla_x \left( \partial_x H_\Theta(r, \Theta) \right) \right) \, dx \, dt,
\]

which, combined with (4.5), gives rise to

\[
\int_\Omega \left( \frac{1}{2} | \varphi |^2 + H_\Theta(\varphi, \varphi) - \partial_\varphi (H_\Theta)(r(\varphi) - r - H_\Theta) \right)(\tau, \cdot) \, dx +
\]

\[
\int_0^\tau \int_\Omega \frac{\Theta}{\varphi} \left( S(\varphi, \nabla_x \varphi) : \nabla_x \varphi - \frac{q(\varphi, \nabla_x \varphi)}{\varphi} \cdot \nabla_x \varphi \right) \, dx \, dt
\]

\[
\leq \int_\Omega \frac{1}{2} \varphi_0 | \varphi_0 - U(0, \cdot) |^2 \, dx
\]

\[
+ \int_\Omega \left( H_{\Theta(0, \cdot)}(\varphi_0, \varphi_0) - \partial_\varphi (H_{\Theta(0, \cdot)})(r(0, \cdot), \Theta(0, \cdot)) \right)(\varphi_0 - r(0, \cdot)) - H_{\Theta(0, \cdot)}(r(0, \cdot), \Theta(0, \cdot)) \, dx
\]

\[
+ \int_0^\tau \int_\Omega \left( \varphi \partial_t \left( \partial_x H_\Theta(r, \Theta) \right) + \varphi \nabla_x \left( \partial_x H_\Theta(r, \Theta) \right) \right) \, dx \, dt
\]

\[
+ \int_0^\tau \int_\Omega \partial_t \left( r \partial_\varphi(H_\Theta)(r, \Theta) - H_\Theta(r, \Theta) \right) \, dx \, dt
\]

Furthermore, seeing that

\[
\partial_y \left( \partial_y H_\Theta(r, \Theta) \right) = -r \partial_\varphi \partial_y \Theta - r \partial_y s(r, \Theta) \partial_y \Theta + \partial_y^2 \partial_\varphi H_\Theta(r, \Theta) \partial_y \varphi + \partial_y^2 \partial_\varphi H_\Theta(r, \Theta) \partial_y \Theta
\]

for \( y = t, x \), we may rewrite (4.6) in the form

\[
\int_\Omega \left( \frac{1}{2} | \varphi |^2 + H_\Theta(\varphi, \varphi) - \partial_\varphi (H_\Theta)(r(\varphi) - r - H_\Theta) \right)(\tau, \cdot) \, dx +
\]

\[
\int_0^\tau \int_\Omega \frac{\Theta}{\varphi} \left( S(\varphi, \nabla_x \varphi) : \nabla_x \varphi - \frac{q(\varphi, \nabla_x \varphi)}{\varphi} \cdot \nabla_x \varphi \right) \, dx \, dt
\]

\[
\leq \int_\Omega \frac{1}{2} \varphi_0 | \varphi_0 - U(0, \cdot) |^2 \, dx
\]

\[
+ \int_\Omega \left( H_{\Theta(0, \cdot)}(\varphi_0, \varphi_0) - \partial_\varphi (H_{\Theta(0, \cdot)})(r(0, \cdot), \Theta(0, \cdot)) \right)(\varphi_0 - r(0, \cdot)) - H_{\Theta(0, \cdot)}(r(0, \cdot), \Theta(0, \cdot)) \, dx
\]
Thus relation (4.7) can be finally written in a more concise form

\[
+ \int_0^T \int_{\Omega} \left( (\varrho \partial_t U + \varrho u \cdot \nabla_x U) \cdot (U - u) - p(\vartheta, \vartheta) \text{div}_x U + S(\vartheta, \nabla_x u) : \nabla_x U \right) \, dx \, dt \\
- \int_0^T \int_{\Omega} \left( \varrho \left( s(\vartheta, \vartheta) - s(r, \Theta) \right) \partial_t \Theta + \varrho \left( s(\vartheta, \vartheta) - s(r, \Theta) \right) u \cdot \nabla_x \Theta + \frac{q(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \right) \, dx \, dt.
\]

In order to simplify (4.7), we recall several useful identities that follow directly from Gibbs’ relation (1.13):

\[
\partial^2_{\varepsilon, \varepsilon}(H_\Theta)(r, \Theta) = \frac{1}{r} \partial_\Theta p(r, \Theta),
\]

\[
r \partial_\varepsilon s(r, \Theta) = -\frac{1}{r} \partial_\Theta p(r, \Theta),
\]

and

\[
\partial^2_{\varepsilon, \varepsilon}(H_\Theta)(r, \Theta) = \partial_\varepsilon (\varrho (\vartheta - \Theta) \partial_\Theta s(r, \Theta)) = (\vartheta - \Theta) \partial_\Theta \left( \varrho \partial_\Theta \left( \varrho \vartheta \right) \right)(r, \Theta) = 0,
\]

Thus relation (4.7) can be finally written in a more concise form

\[
\int_\Omega \left( \frac{1}{2} \varrho \| u - U \|^2 + E(\vartheta, \vartheta) \right) \, \Theta(\cdot, \cdot) \, dx + \int_0^T \int_{\Omega} \frac{\Theta}{\vartheta} \left( S(\vartheta, \nabla_x u) : \nabla_x u - \frac{q(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \, dx \, dt \leq \int_{\Omega} \frac{1}{2} \varrho_0 |u_0 - U(0, \cdot)|^2 + E(\vartheta_0, \vartheta_0) \, \Theta(0, \cdot) \, dx
\]

\[
= \int_0^T \int_{\Omega} \varrho \left( u - U \right) \cdot \nabla_x U \left( U - u \right) \, dx \, dt + \int_0^T \int_{\Omega} \varrho \left( s(\vartheta, \vartheta) - s(r, \Theta) \right) \left( U - u \right) \cdot \nabla_x \Theta \, dx \, dt
\]

\[
+ \int_0^T \int_{\Omega} \varrho \left( \partial_t U + U \cdot \nabla_x U \right) \left( U - u \right) - p(\vartheta, \vartheta) \text{div}_x U + S(\vartheta, \nabla_x u) : \nabla_x U \, dx \, dt.
\]

where \( \mathcal{E} \) was introduced in (2.12).
On the other hand, we observe that

\[- \int_{\Omega} \varrho \mathbf{u} \cdot \nabla p(r, \Theta) \, dx = \int_{\Omega} \left( g(\mathbf{U} - \mathbf{u}) \cdot \frac{\nabla p(r, \Theta)}{r} - \frac{\varrho}{r} \mathbf{U} \cdot \nabla p(r, \Theta) \right) \, dx \]

\[\int_{\Omega} \left( g(\mathbf{U} - \mathbf{u}) \cdot \frac{\nabla p(r, \Theta)}{r} + \left( 1 - \frac{\varrho}{r} \right) \mathbf{U} \cdot \nabla p(r, \Theta) + \varrho \nabla p(r, \Theta) \text{div} \mathbf{U} \right) \, dx\]

Using the latter identity in the formula (4.9), we get the relative entropy inequality in the form (2.10). Theorem 2.3 is proved.

5 Weak-strong uniqueness

In this section we prove Theorem 2.4 by applying the relative entropy inequality (2.10) to \( \varrho = \tilde{\varrho}, \Theta = \tilde{\Theta}, \) and \( \mathbf{U} = \tilde{\mathbf{u}}, \) where \( (\tilde{\varrho}, \tilde{\Theta}, \tilde{\mathbf{u}}) \) is a classical (smooth) solution of the Navier-Stokes-Fourier system such that

\[\tilde{\varrho}(0, \cdot) = \varrho_0, \tilde{\mathbf{u}}(0, \cdot) = \mathbf{u}_0, \tilde{\Theta}(0, \cdot) = \Theta_0.\]

Accordingly, the integrals depending on the initial values on the right-hand side of (2.10) vanish, and we apply a Gronwall type argument to deduce the desired result, namely,

\[\varrho \equiv \tilde{\varrho}, \Theta \equiv \tilde{\Theta}, \text{ and } \mathbf{u} \equiv \tilde{\mathbf{u}}.\]

Here, the hypothesis of thermodynamic stability formulated in (1.14) will play a crucial role.

5.1 Preliminaries, notation

Following [17, Chapters 4,5] we introduce essential and residual component of each quantity appearing in (2.10). To begin, we choose positive constants \( \underline{\varrho}, \overline{\varrho}, \underline{\vartheta}, \overline{\vartheta} \) in such a way that

\[0 < \underline{\varrho} \leq \frac{1}{2} \min_{(t,x) \in [0,T] \times \overline{\Omega}} \tilde{\varrho}(t,x) \leq \max_{(t,x) \in [0,T] \times \overline{\Omega}} \tilde{\varrho}(t,x) \leq \overline{\varrho},\]

\[0 < \vartheta \leq \frac{1}{2} \min_{(t,x) \in [0,T] \times \overline{\Omega}} \tilde{\theta}(t,x) \leq \max_{(t,x) \in [0,T] \times \overline{\Omega}} \tilde{\theta}(t,x) \leq \overline{\vartheta}.\]

In can be shown, as a consequence of the hypothesis of thermodynamic stability (1.14), or, more specifically, of (1.16), (1.17), that

\[\mathcal{E}(\varrho, \vartheta | \tilde{\varrho}, \tilde{\vartheta}) \geq c \left\{ \begin{array}{ll} |\varrho - \tilde{\varrho}|^2 + |\vartheta - \tilde{\vartheta}|^2 & \text{if } (\varrho, \vartheta) \in [\underline{\varrho}, \overline{\varrho}] \times [\underline{\vartheta}, \overline{\vartheta}] \\ 1 + |\rho s(\varrho, \vartheta)| + \rho e(\varrho, \vartheta) & \text{otherwise,} \end{array} \right. \] (5.1)

whenever \([\tilde{\varrho}, \tilde{\vartheta}] \in [\underline{\varrho}, \overline{\varrho}] \times [\underline{\vartheta}, \overline{\vartheta}]\), where the constant \( c \) depends only on \( \underline{\varrho}, \overline{\varrho}, \underline{\vartheta}, \overline{\vartheta} \) and the structural properties of the thermodynamic functions \( c, s, e \), see [17, Chapter 3, Proposition 3.2].

In the sequel, it will be convenient to decompose each measurable function \( h \) as

\[h = h_{\text{ess}} + h_{\text{res}},\]
where
\[
h_{\text{res}}(t, x) = \begin{cases} h(t, x) & \text{if } (\varrho(t, x), \vartheta(t, x)) \in \overline{[\varrho, \overline{\varrho}] \times [\vartheta, \overline{\vartheta}]} \\ 0 & \text{otherwise} \end{cases}, \quad h_{\text{res}} = h - h_{\text{ess}}.
\]

With this notation at hand, and thanks to (3.15), (3.22), we can write
\[
\mathcal{E}(\varrho, \vartheta | \bar{\varrho}, \bar{\vartheta}) \geq c \left( |\varrho|_{r_{\text{res}}}^2 + |\vartheta|_{r_{\text{res}}}^2 + |\varrho - \bar{\varrho}|_{r_{\text{ess}}}^2 + |\vartheta - \bar{\vartheta}|_{r_{\text{ess}}}^2 \right),
\]
where by virtue of (5.2),
\[
\mathcal{E}(\varrho, \vartheta | \bar{\varrho}, \bar{\vartheta}) \in L^\infty(0, T; L^1(\Omega)).
\]

5.2 Relative entropy balance

Taking \( r = \bar{\varrho}, \Theta = \bar{\vartheta}, U = \bar{u} \) in (2.10) and using the fact that the initial values coincide, we obtain
\[
\int_\Omega \left( \frac{1}{2}\varrho |u - \bar{u}|^2 + \mathcal{E}(\varrho, \vartheta | \bar{\varrho}, \bar{\vartheta}) \right) (\tau, \cdot) \, dx + \int_0^\tau \int_\Omega \left( \mathcal{S}(\vartheta, \nabla_x u) : \nabla_x u - \frac{q(\varrho, \nabla_x u) \cdot \nabla_x \vartheta}{\varrho} \right) \, dx \, dt \leq \int_0^\tau \int_\Omega \left( \varrho (s(\varrho, \vartheta) - s(\bar{\varrho}, \bar{\vartheta})) (\bar{u} - u) \cdot \nabla_x \vartheta \right) \, dx \, dt
\]

\[
+ \int_0^\tau \int_\Omega \left( \varrho (\partial_t \bar{u} + \bar{u} \cdot \nabla_x \bar{u}) \right) \cdot (\bar{u} - u) - p(\varrho, \vartheta) \text{div}_x \bar{u} + \mathcal{S}(\vartheta, \nabla_x u) : \nabla_x \bar{u} \right) \, dx \, dt
\]

\[
- \int_0^\tau \int_\Omega \left( \varrho (s(\varrho, \vartheta) - s(\bar{\varrho}, \bar{\vartheta})) \partial_t \bar{\vartheta} + \varrho (s(\varrho, \vartheta) - s(\bar{\varrho}, \bar{\vartheta})) \bar{u} \cdot \nabla_x \bar{\vartheta} + \frac{q(\varrho, \nabla_x \vartheta)}{\varrho} \cdot \nabla_x \bar{\vartheta} \right) \, dx \, dt,
\]

In order to handle the integrals on the right-hand side of (5.4), we proceed by several steps:

**Step 1:**
We have
\[
\int_\Omega \varrho |u - \bar{u}|^2 |\nabla_x \bar{u}| \, dx \, dt \leq 2 ||\nabla_x \bar{u}||_{L^\infty(\Omega; R^3)} \int_\Omega \frac{1}{2}\varrho |u - \bar{u}|^2 \, dx.
\]

**Step 2:**
\[
||\nabla_x \bar{\vartheta}||_{L^\infty(\Omega; R^3)} \left[ 2 \varrho \int_\Omega \left| \left( s(\varrho, \vartheta) - s(\bar{\varrho}, \bar{\vartheta}) \right) \bar{u} \cdot \nabla_x \bar{\vartheta} \right| \, dx \right] \leq c(\delta) \int_\Omega \mathcal{E}(\varrho, \vartheta | \bar{\varrho}, \bar{\vartheta}) \, dx
\]

where, by virtue of (5.2),
\[
\int_\Omega \left| s(\varrho, \vartheta) - s(\bar{\varrho}, \bar{\vartheta}) \right| \, dx \leq \delta ||u - \bar{u}||_{L^2(\Omega; R^3)} + c(\delta) \int_\Omega \mathcal{E}(\varrho, \vartheta | \bar{\varrho}, \bar{\vartheta}) \, dx
\]
for any $\delta > 0$.

Similarly, by interpolation inequality,

$$
\int_{\Omega} \left\| \left[ \varrho \left( s(\rho, \vartheta) - s(\tilde\rho, \tilde\vartheta) \right) - \varrho \right] \right\|_{L^{5/3}(\Omega; R^3)}^2 \, |u - \tilde u| \, dx 
\leq \delta \|u - \tilde u\|_{L^{5/3}(\Omega; R^3)}^2 + c(\delta) \left( \int_{\Omega} \mathcal{E}(\rho, \vartheta|\tilde\rho, \tilde\vartheta) \, dx \right)^{5/3},
$$

for any $\delta > 0$, where, again by (5.2)

$$
\left\| \left[ \varrho \left( s(\rho, \vartheta) - s(\tilde\rho, \tilde\vartheta) \right) \right] \right\|_{L^{5/3}(\Omega)}^2 \leq c \left( \int_{\Omega} \mathcal{E}(\rho, \vartheta|\tilde\rho, \tilde\vartheta) \, dx \right)^{5/3}.
$$

Combining (5.6), (5.7) and (5.3), we conclude that

$$
\left| \int_{\Omega} \varrho \left( s(\rho, \vartheta) - s(\tilde\rho, \tilde\vartheta) \right) (\tilde u - u) \cdot \nabla_x \tilde\vartheta \, dx \right| 
\leq \|\nabla_x \tilde\vartheta\|_{L^{\infty}(\Omega; R^3)} \left[ \delta \|u - \tilde u\|_{W^{1,2}_0(\Omega; R^3)}^2 + c(\delta) \int_{\Omega} \mathcal{E}(\rho, \vartheta|\tilde\rho, \tilde\vartheta) \, dx \right] 
\leq K(\delta, \cdot) \int_{\Omega} \mathcal{E}(\rho, \vartheta|\tilde\rho, \tilde\vartheta) \, dx
$$

for any $\delta > 0$. Here and hereafter, $K(\delta, \cdot)$ is a generic constant depending on $\delta$, $\tilde\rho$, $\tilde u$, $\tilde\vartheta$ through the norms induced by (2.13), and $\rho$, $\vartheta$, while $K(\cdot)$ is independent of $\delta$ but depends on $\tilde\rho$, $\tilde u$, $\tilde\vartheta$, $\rho$, $\vartheta$ through the norms (2.13).

**Step 3:**

Writing

$$
\int_{\Omega} \varrho (\partial_t \tilde u + \tilde u \cdot \nabla_x \tilde u) \cdot (\tilde u - u) \, dx = \int_{\Omega} \varrho (\tilde u - u) \cdot \left( \nabla_x S(\tilde\rho, \nabla_x \tilde u) - \nabla_x p(\tilde\rho, \tilde\vartheta) \right) \, dx
$$

we observe that the first integral on the right-hand side can be handled in the same way as in Step 2, namely,

$$
\left\| \left[ \varrho \left( s(\rho, \vartheta) - s(\tilde\rho, \tilde\vartheta) \right) - \varrho \right] \right\|_{W^{1,2}_0(\Omega; R^3)}^2 \leq K(\delta, \cdot) \|\varrho - \varrho\|_{L^2(\Omega)}^2 + \delta \|u - \tilde u\|_{L^2(\Omega, R^3)}^2,
$$

while, integrating by parts,

$$
\int_{\Omega} (\tilde u - u) \cdot \left( \nabla_x S(\tilde\rho, \nabla_x \tilde u) - \nabla_x p(\tilde\rho, \tilde\vartheta) \right) \, dx
$$
Thus using again (5.1), (5.3) and continuous imbedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ we arrive at

\[
\left| \int_{\Omega} \varrho \left( \partial_t \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} \right) \cdot (\hat{u} - u) \, dx \right| \leq \int_{\Omega} \left( S(\tilde{\varrho}, \nabla \tilde{u}) : \nabla \tilde{u}(u - \hat{u}) + p(\tilde{\varrho}, \tilde{\vartheta}) \text{div}_x(\tilde{u} - u) \right) \, dx
\]

\[\leq c \left( \| \nabla \tilde{u} \|_{L^1(\Omega; R^n)} + c(\delta) \int_{\Omega} E(\varrho, \vartheta|\tilde{\varrho}, \tilde{\vartheta}) \, dx \right) \leq \int_{\Omega} \left( S(\tilde{\varrho}, \nabla \tilde{u}) : \nabla \tilde{u}(u - \hat{u}) + p(\tilde{\varrho}, \tilde{\vartheta}) \text{div}_x(\tilde{u} - u) \right) \, dx
\]

\[+ \delta \| u - \tilde{u} \|^2_{W^{1,2}(\Omega; R^n)} + K(\delta, \cdot) \int_{\Omega} E(\varrho, \vartheta|\tilde{\varrho}, \tilde{\vartheta}) \, dx
\]

for any $\delta > 0$.

**Step 4:**

Next, we get

\[
\int_{\Omega} \varrho \left( s(\varrho, \vartheta) - s(\tilde{\varrho}, \tilde{\vartheta}) \right) \partial_t \tilde{\vartheta} \, dx =
\]

\[
\int_{\Omega} \varrho \left[ s(\varrho, \vartheta) - s(\tilde{\varrho}, \tilde{\vartheta}) \right] \partial_t \tilde{\vartheta} \, dx + \int_{\Omega} \varrho \left[ s(\varrho, \vartheta) - s(\tilde{\varrho}, \tilde{\vartheta}) \right] \vartheta \partial_t \tilde{\vartheta} \, dx,
\]

where

\[
\leq \| \partial_t \tilde{\vartheta} \|_{L^\infty(\Omega)} \left( \int_{\Omega} [s(\varrho, \vartheta)]_{\text{res}} \, dx + \| s(\tilde{\varrho}, \tilde{\vartheta}) \|_{L^\infty(\Omega)} \int_{\Omega} [\vartheta]_{\text{res}} \, dx \right) \leq K(\cdot) \int_{\Omega} E(\varrho, \vartheta|\tilde{\varrho}, \tilde{\vartheta}) \, dx,
\]

while

\[
= \int_{\Omega} (\varrho - \tilde{\varrho}) \left[ s(\varrho, \vartheta) - s(\tilde{\varrho}, \tilde{\vartheta}) \right] \partial_t \tilde{\vartheta} \, dx + \int_{\Omega} \tilde{\varrho} \left[ s(\varrho, \vartheta) - s(\tilde{\varrho}, \tilde{\vartheta}) \right] \vartheta \partial_t \tilde{\vartheta} \, dx,
\]

where, with help of Taylor-Lagrange formula,

\[
\leq \left( \sup_{(\varrho, \vartheta) \in [\varrho, \tilde{\varrho}] \times [\vartheta, \tilde{\vartheta}]} |\partial_\varrho s(\varrho, \vartheta)| + \sup_{(\varrho, \vartheta) \in [\varrho, \tilde{\varrho}] \times [\vartheta, \tilde{\vartheta}]} |\partial_\vartheta s(\varrho, \vartheta)| \right) \| \partial_t \tilde{\vartheta} \|_{L^\infty(\Omega)} \times
\]

\[
\times \int_{\Omega} \left[ \| \varrho - \tilde{\varrho} \|_{L^1(\Omega)} + \| \vartheta - \tilde{\vartheta} \|_{L^1(\Omega)} \right] \, dx \leq K(\cdot) \int_{\Omega} E(\varrho, \vartheta|\tilde{\varrho}, \tilde{\vartheta}) \, dx.
\]

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Finally, we write
\[
\int_\Omega \tilde{\varrho} \left[ (s(\varrho, \tilde{\vartheta}) - s(\tilde{\varrho}, \tilde{\vartheta})) \right]_{\text{ess}} \partial_t \tilde{\vartheta} \, dx = \int_\Omega \tilde{\varrho} \left[ (s(\varrho, \tilde{\vartheta}) - \partial_\varrho s(\tilde{\varrho}, \tilde{\vartheta})(\varrho - \tilde{\varrho}) - \partial_{\tilde{\varrho}} s(\tilde{\varrho}, \tilde{\vartheta})(\tilde{\varrho} - \varrho) - s(\tilde{\varrho}, \tilde{\vartheta})) \right]_{\text{ess}} \partial_t \tilde{\vartheta} \, dx
\]
\[
- \int_\Omega \tilde{\varrho} \left[ \partial_\varrho s(\tilde{\varrho}, \tilde{\vartheta})(\varrho - \tilde{\varrho}) + \partial_{\tilde{\varrho}} s(\tilde{\varrho}, \tilde{\vartheta})(\tilde{\varrho} - \varrho) \right]_{\text{ess}} \partial_t \tilde{\vartheta} \, dx + \int_\Omega \tilde{\varrho} \left[ \partial_\varrho s(\tilde{\varrho}, \tilde{\vartheta})(\varrho - \tilde{\varrho}) + \partial_{\tilde{\varrho}} s(\tilde{\varrho}, \tilde{\vartheta})(\tilde{\varrho} - \varrho) \right] \partial_t \tilde{\vartheta} \, dx,
\]
where the first two integrals on the right-hand side can be estimated exactly as in (5.10), (5.11). Thus we conclude that
\[
- \int_\Omega \tilde{\varrho} (s(\varrho, \tilde{\vartheta}) - s(\tilde{\varrho}, \tilde{\vartheta})) \partial_t \tilde{\vartheta} \, dx \leq K(\cdot) \int_\Omega \mathcal{E}(\varrho, \tilde{\varrho}) \, dx \tag{5.12}
\]
\[
- \int_\Omega \tilde{\varrho} \left[ \partial_\varrho s(\tilde{\varrho}, \tilde{\vartheta})(\varrho - \tilde{\varrho}) + \partial_{\tilde{\varrho}} s(\tilde{\varrho}, \tilde{\vartheta})(\tilde{\varrho} - \varrho) \right] \partial_t \tilde{\vartheta} \, dx.
\]

**Step 5:**
Similarly to Step 4, we get
\[
- \int_\Omega \tilde{\varrho} (s(\varrho, \tilde{\vartheta}) - s(\tilde{\varrho}, \tilde{\vartheta})) \tilde{\vartheta} \cdot \nabla_x \tilde{\vartheta} \, dx \leq K(\cdot) \int_\Omega \mathcal{E}(\varrho, \tilde{\varrho}) \, dx \tag{5.13}
\]
\[
- \int_\Omega \tilde{\varrho} \left[ \partial_\varrho s(\tilde{\varrho}, \tilde{\vartheta})(\varrho - \tilde{\varrho}) + \partial_{\tilde{\varrho}} s(\tilde{\varrho}, \tilde{\vartheta})(\tilde{\varrho} - \varrho) \right] \tilde{\vartheta} \cdot \nabla_x \tilde{\vartheta} \, dx.
\]

**Step 6:**
Finally, we have
\[
\int_\Omega \left( \left( 1 - \frac{\varrho}{\tilde{\varrho}} \right) \partial_t p(\tilde{\varrho}, \tilde{\vartheta}) - \frac{\varrho}{\tilde{\varrho}} \tilde{\vartheta} \cdot \nabla_x p(\tilde{\varrho}, \tilde{\vartheta}) \right) \, dx \tag{5.14}
\]
\[
= \int_\Omega \left( \varrho - \tilde{\varrho} \right) \frac{1}{\tilde{\varrho}} \left[ \partial_t p(\tilde{\varrho}, \tilde{\vartheta}) + \tilde{\vartheta} \cdot \nabla_x p(\tilde{\varrho}, \tilde{\vartheta}) \right] \, dx + \int_\Omega p(\tilde{\varrho}, \tilde{\vartheta}) \text{div}_x u \, dx
\]
\[
+ \int_\Omega \left( \varrho - \tilde{\varrho} \right) \frac{1}{\tilde{\varrho}} \nabla_x p(\tilde{\varrho}, \tilde{\vartheta}) \cdot (u - \bar{u}) \, dx,
\]
where, by means of the same arguments as in Step 2,
\[
\left| \int_\Omega \left( \varrho - \tilde{\varrho} \right) \frac{1}{\tilde{\varrho}} \nabla_x p(\tilde{\varrho}, \tilde{\vartheta}) \cdot (u - \bar{u}) \, dx \right| \leq \delta \|u - \bar{u}\|_{W^{1,\infty}(\Omega;R^3)} + \int_\Omega \mathcal{E}(\varrho, \tilde{\varrho}) \, dx
\]
for any \( \delta > 0 \). Resuming this step, we have
\[
\int_\Omega \left( \left( 1 - \frac{\varrho}{\tilde{\varrho}} \right) \partial_t p(\tilde{\varrho}, \tilde{\vartheta}) - \frac{\varrho}{\tilde{\varrho}} \tilde{\vartheta} \cdot \nabla_x p(\tilde{\varrho}, \tilde{\vartheta}) \right) \, dx \leq \delta \|u - \bar{u}\|_{W^{1,\infty}(\Omega;R^3)} + \int_\Omega \mathcal{E}(\varrho, \tilde{\varrho}) \, dx + \int_\Omega p(\tilde{\varrho}, \tilde{\vartheta}) \text{div}_x u \, dx
\]
for any \( \delta > 0 \).
\[ \delta \| \mathbf{u} - \tilde{\mathbf{u}} \|^2_{W^{1,2}(\Omega, \mathbb{R}^d)} + K(\delta, \cdot) \int_\Omega \mathcal{E}(\vartheta, \varrho|\tilde{\varrho}, \tilde{\vartheta}) \, dx. \]

**Step 7:**
Summing up the estimates (5.5), (5.8), (5.9), (5.12 - 5.15), we can rewrite the relative entropy inequality (5.4) in the form

\[ \int_\Omega \left( \frac{1}{2} \mathcal{E}(\vartheta, \varrho|\tilde{\varrho}, \tilde{\vartheta}) \right) (\tau, \cdot) \, dx \]

\[ + \int_0^T \int_\Omega \left( \frac{\tilde{\varrho}}{\varrho} \mathcal{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \mathcal{S}((\tilde{\varrho}, \nabla_x \tilde{\mathbf{u}}) : (\nabla_x \mathbf{u} - \nabla_x \tilde{\mathbf{u}}) - \mathcal{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \tilde{\mathbf{u}}) \right) \, dx \, dt \]

\[ + \int_0^T \int_\Omega \left( \mathbf{q}(\vartheta, \nabla_x \varrho) \cdot \nabla_x \tilde{\vartheta} - \frac{\tilde{\varrho}}{\varrho} \mathbf{q}(\vartheta, \nabla_x \varrho) \cdot \nabla_x \tilde{\vartheta} \right) \, dx \, dt \leq \]

\[ \int_0^T \left[ \delta \| \mathbf{u} - \tilde{\mathbf{u}} \|^2_{W^{1,2}(\Omega, \mathbb{R}^d)} + K(\delta, \cdot) \int_\Omega \left( \frac{1}{2} \mathcal{E}(\vartheta, \varrho|\tilde{\varrho}, \tilde{\vartheta}) + \mathcal{E}(\vartheta, \varrho|\tilde{\varrho}, \tilde{\vartheta}) \right) \, dx \right] \, dt \]

\[ + \int_0^T \int_\Omega \left( p(\tilde{\varrho}, \tilde{\vartheta}) - p(\vartheta, \varrho) \right) \text{div}_x \tilde{\mathbf{u}} \, dx \, dt \]

\[ + \int_0^T \int_\Omega \left( \tilde{\varrho} - \varrho \right) \frac{1}{\tilde{\varrho}} \left[ \partial_\vartheta s(\tilde{\varrho}, \tilde{\vartheta}) + \tilde{\mathbf{u}} \cdot \nabla_x p(\tilde{\varrho}, \tilde{\vartheta}) \right] \, dx \, dt \]

\[ - \int_0^T \int_\Omega \tilde{\varrho} \left( \partial_\vartheta s(\tilde{\varrho}, \tilde{\vartheta})(\varrho - \tilde{\varrho}) + \partial_\varrho s(\tilde{\varrho}, \tilde{\vartheta})(\vartheta - \tilde{\vartheta}) \right) \left[ \partial_\vartheta \tilde{\vartheta} + \tilde{\mathbf{u}} \cdot \nabla_x \tilde{\vartheta} \right] \, dx \, dt \]

for any \( \delta > 0. \)

**Step 8:**
Our next goal is to control the last three integrals on the right-hand side of (5.16). To this end, we use (4.8) to obtain

\[ \int_\Omega (\tilde{\varrho} - \varrho) \frac{1}{\tilde{\varrho}} \left[ \partial_\vartheta p(\tilde{\varrho}, \tilde{\vartheta}) + \tilde{\mathbf{u}} \cdot \nabla_x p(\tilde{\varrho}, \tilde{\vartheta}) \right] \, dx \]

\[ - \int_\Omega \tilde{\varrho} \left( \partial_\vartheta s(\tilde{\varrho}, \tilde{\vartheta})(\varrho - \tilde{\varrho}) + \partial_\varrho s(\tilde{\varrho}, \tilde{\vartheta})(\vartheta - \tilde{\vartheta}) \right) \left[ \partial_\vartheta \tilde{\vartheta} + \tilde{\mathbf{u}} \cdot \nabla_x \tilde{\vartheta} \right] \, dx \]

\[ = \int_\Omega (\tilde{\varrho} - \varrho) \partial_\vartheta s(\tilde{\varrho}, \tilde{\vartheta}) \left[ \partial_\vartheta \tilde{\vartheta} + \tilde{\mathbf{u}} \cdot \nabla_x \tilde{\vartheta} \right] \, dx + \int_\Omega (\tilde{\varrho} - \varrho) \frac{1}{\tilde{\varrho}} \partial_\varrho p(\tilde{\varrho}, \tilde{\vartheta}) \left[ \partial_\vartheta \tilde{\varrho} + \tilde{\mathbf{u}} \cdot \nabla_x \tilde{\vartheta} \right] \, dx, \]

where, as \( \tilde{\varrho}, \tilde{\mathbf{u}} \) satisfy the equation of continuity (1.1),

\[ \int_\Omega (\tilde{\varrho} - \varrho) \frac{1}{\tilde{\varrho}} \partial_\varrho p(\tilde{\varrho}, \tilde{\vartheta}) \left[ \partial_\vartheta \tilde{\varrho} + \tilde{\mathbf{u}} \cdot \nabla_x \tilde{\vartheta} \right] \, dx = - \int_\Omega (\tilde{\varrho} - \varrho) \partial_\varrho p(\tilde{\varrho}, \tilde{\vartheta}) \text{div}_x \tilde{\mathbf{u}} \, dx. \] (5.17)

Finally, using (4.8) once more, we deduce that

\[ \int_\Omega (\tilde{\varrho} - \varrho) \partial_\vartheta s(\tilde{\varrho}, \tilde{\vartheta}) \left[ \partial_\vartheta \tilde{\vartheta} + \tilde{\mathbf{u}} \cdot \nabla_x \tilde{\vartheta} \right] \, dx \]

(5.18)
\[
\int_{\Omega} \tilde{g}(\tilde{\vartheta} - \vartheta) \left[ \partial_t s(\tilde{\vartheta}, \tilde{\vartheta}) + \tilde{u} \cdot \nabla_x s(\tilde{\vartheta}, \tilde{\vartheta}) \right] \, dx - \int_{\Omega} (\tilde{\vartheta} - \vartheta) \partial_\vartheta p(\tilde{\vartheta}, \tilde{\vartheta}) \text{div}_x \tilde{u} \, dx
\]

\[
= \int_{\Omega} (\tilde{\vartheta} - \vartheta) \left[ \frac{1}{\vartheta} \left( S(\tilde{\vartheta}, \nabla_x \tilde{u}) : \nabla_x \tilde{u} - \frac{q(\tilde{\vartheta}, \nabla_x \tilde{\vartheta}) \cdot \nabla_x \tilde{\vartheta}}{\vartheta} \right) - \text{div}_x \left( \frac{q(\tilde{\vartheta}, \nabla_x \tilde{\vartheta})}{\vartheta} \right) \right] \, dx 
- \int_{\Omega} (\tilde{\vartheta} - \vartheta) \partial_\vartheta p(\tilde{\vartheta}, \tilde{\vartheta}) \text{div}_x \tilde{u} \, dx
\]

Seeing that

\[
\left| \int_{\Omega} \left( p(\tilde{\vartheta}, \tilde{\vartheta}) - \partial_\vartheta p(\tilde{\vartheta}, \tilde{\vartheta})(\tilde{\vartheta} - \vartheta) - \partial_\vartheta p(\tilde{\vartheta}, \tilde{\vartheta})(\tilde{\vartheta} - \vartheta) - p(\vartheta, \vartheta) \right) \text{div}_x \tilde{u} \, dx \right| 
\leq c\|\text{div}_x \tilde{u}\|_{L^\infty(\Omega)} \int_{\Omega} E(\vartheta, \vartheta | \tilde{\vartheta}, \tilde{\vartheta}) \, dx
\]

we may use the previous relations to rewrite (5.16) in the form

\[
\int_{\Omega} \left( \frac{1}{2} \vartheta |\tilde{u} - \tilde{u}|^2 + E(\vartheta, \vartheta | \tilde{\vartheta}, \tilde{\vartheta}) \right)(\tau, \cdot) \, dx 
+ \int_0^\tau \int_{\Omega} \left( \frac{\tilde{\vartheta}}{\vartheta} S(\tilde{\vartheta}, \nabla_x \tilde{u}) : \nabla_x \tilde{u} - S(\tilde{\vartheta}, \nabla_x \tilde{u}) \right) \, dx \, dt 
- \int_0^\tau \int_{\Omega} \left( \frac{q(\tilde{\vartheta}, \nabla_x \tilde{\vartheta}) \cdot \nabla_x \tilde{\vartheta}}{\vartheta} - \frac{\tilde{\vartheta}}{\vartheta} q(\tilde{\vartheta}, \nabla_x \tilde{\vartheta}) \cdot \nabla_x \tilde{\vartheta} \right) 
+ (\tilde{\vartheta} - \vartheta) \frac{q(\tilde{\vartheta}, \nabla_x \tilde{\vartheta}) \cdot \nabla_x \tilde{\vartheta}}{\vartheta^2} + \frac{q(\tilde{\vartheta}, \nabla_x \tilde{\vartheta})}{\vartheta} \cdot \nabla_x (\tilde{\vartheta} - \vartheta) \right) \, dx \, dt 
\leq \int_0^\tau \left[ \delta \|\tilde{u} - \tilde{u}\|_{W^{1,2}(\Omega; R^3)}^2 + K(\delta, \cdot) \int_{\Omega} \left( \frac{1}{2} \vartheta |\tilde{u} - \tilde{u}|^2 + E(\vartheta, \vartheta | \tilde{\vartheta}, \tilde{\vartheta}) \right) \, dx \right] \, dt
\]

for any \( \delta > 0 \).

### 5.3 Viscous terms and terms related to the heat conductivity

**Step 1**

Concerning the viscous term, we shall investigate separately the cases \( 0 < \vartheta < \tilde{\vartheta} \) and \( \vartheta \geq \tilde{\vartheta} \).

In the first case, we have

\[
1_{\{0 < \vartheta < \tilde{\vartheta}\}} \left( \frac{\tilde{\vartheta}}{\vartheta} S(\vartheta, \nabla u) : \nabla u - S(\tilde{\vartheta}, \nabla u) : \nabla \tilde{u} + S(\tilde{\vartheta}, \nabla \tilde{u}) : \nabla (\tilde{u} - u) + \frac{\vartheta - \tilde{\vartheta}}{\vartheta} S(\tilde{\vartheta}, \nabla \tilde{u}) : \nabla \tilde{u} \right)
\]

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\[
= 1_{\{0 < \vartheta < \tilde{\vartheta}\}} \left( \frac{\partial - \vartheta}{\partial} S(\vartheta, \nabla_x u) : \nabla_x u + \left( S(\vartheta, \nabla_x u) - S(\tilde{\vartheta}, \nabla \tilde{u}) \right) : \nabla_x (u - \tilde{u}) + \frac{\vartheta - \tilde{\vartheta}}{\vartheta} S(\tilde{\vartheta}, \nabla_x \tilde{u}) : \nabla_x \tilde{u} \right)
\]

\[
\geq 1_{\{0 < \vartheta < \tilde{\vartheta}\}} \left[ \left( S(\vartheta, \nabla u) - S(\vartheta, \nabla u) \right) : \nabla_x (u - \tilde{u}) + \frac{\tilde{\vartheta} - \vartheta}{\vartheta} \left( S(\vartheta, \nabla u) : \nabla u - S(\vartheta, \nabla u) : \nabla \tilde{u} \right) \right]
\]

In the second case we write

\[
1_{\{\vartheta \geq \tilde{\vartheta}\}} \left( \frac{\partial}{\partial} S(\vartheta, \nabla u) : \nabla u - S(\vartheta, \nabla u) : \nabla \tilde{u} + S(\tilde{\vartheta}, \nabla \tilde{u}) : \nabla (u - \tilde{u}) + \frac{\vartheta - \tilde{\vartheta}}{\vartheta} S(\tilde{\vartheta}, \nabla \tilde{u}) : \nabla \tilde{u} \right)
\]

Putting together both two inequalities, taking into account, (1.4), (1.25), (5.2), (5.3), Young and Hölder inequalities, and employing Lemmas 3.1, 3.2, we finally deduce that

\[
\mathcal{L}_{\vartheta, \tilde{\vartheta}} \equiv L_{\vartheta, \tilde{\vartheta}} \left( \mathbb{P} \left( \vartheta \varphi, \vartheta \right) \right) = \mathcal{L}_{\vartheta, \tilde{\vartheta}} \left( \mathbb{P} \left( \vartheta \varphi, \vartheta \right) \right)
\]

where \( \alpha \) and \( c \) are convenient positive constants.

**Step 2**

For the heat conducting term, we have similarly,

\[
- \frac{\partial}{\partial} \varphi(\vartheta, \nabla \vartheta) \cdot \nabla \vartheta + \frac{\partial}{\partial} \varphi(\vartheta, \nabla \vartheta) \cdot \nabla \vartheta - \frac{\partial}{\partial} \varphi(\vartheta, \nabla \vartheta) \cdot \nabla \vartheta - \vartheta \frac{\partial}{\partial} q(\vartheta, \nabla \vartheta) \cdot \nabla \vartheta
\]

\[
= \frac{\partial}{\partial} \varphi(\vartheta, \nabla \vartheta) \cdot \nabla \vartheta - \vartheta (q(\vartheta, \nabla \vartheta) - \vartheta (q(\vartheta, \nabla \vartheta)) \cdot \nabla (\vartheta - \tilde{\vartheta})
\]

\[
+ (\vartheta - \tilde{\vartheta}) q(\vartheta, \nabla \vartheta) \cdot \nabla (\vartheta - \tilde{\vartheta})
\]

Whence,

\[
- \int_0^T \int_{\Omega} \left( \frac{\partial}{\partial} \varphi(\vartheta, \nabla \vartheta) \cdot \nabla \vartheta - \frac{\partial}{\partial} \varphi(\vartheta, \nabla \vartheta) \cdot \nabla \vartheta + \frac{\partial}{\partial} \varphi(\vartheta, \nabla \vartheta) \cdot \nabla (\vartheta - \tilde{\vartheta}) + \frac{\vartheta - \tilde{\vartheta}}{\vartheta} \varphi(\vartheta, \nabla \vartheta) \cdot \nabla \vartheta \right) \cdot \nabla \vartheta \, dx \, dt
\]

\[
\geq \alpha \| \nabla (\vartheta - \tilde{\vartheta}) \|_{L^2(0,T;W^{1,2}(\Omega, \mathbb{R}^3))} - c \int_0^T \int_{\Omega} \mathcal{E}(\vartheta, \vartheta | \tilde{\vartheta}, \tilde{\vartheta}) \, dx \, dt.
\]
where $\alpha$ and $c$ are convenient positive constants.

Finally, putting together (5.19), (5.20–5.21) we get

\[
\int_\Omega \left( \frac{1}{2} \rho |u - \tilde{u}|^2 + E(\rho, \vartheta | \tilde{\rho}, \tilde{\vartheta}) \right) \tau, \cdot \, d\Omega \\
+ \alpha \left( \left\| \sqrt{\kappa(\vartheta)} \left( \nabla_x \log \vartheta - \nabla_x \log \tilde{\vartheta} \right) \right\|_{L^2((0,T) \times \Omega)}^2 + \| u - \tilde{u} \|_{L^2(0,T; W^{1,2}(\Omega; R^3))}^2 \right) \\
\leq c \int_0^\tau \int_\Omega \left( \frac{1}{2} \rho |u - \tilde{u}|^2 + E(\rho, \vartheta | \tilde{\rho}, \tilde{\vartheta}) \right) \, dx \, dt
\]

for a. a. $\tau \in (0, T)$. We conclude by applying the Gronwall lemma to this inequality. Theorem 2.4 is proved.

References


