p-adic numbers and Igusa zeta function

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Let p be a prime number. We consider the p-adic absolute value over $\mathbb{Q}$:

$$\begin{array}{cccc} |.|_p & : & \mathbb{Q} & \to & \mathbb{R} \\ & & x & \mapsto & p^{-v_p(x)} \end{array}$$

where $v_p(x)$ is the p-adic valuation of x. The field of *p-adic numbers*, denoted by $\mathbb{Q}_p$, is by definition the completion of $(\mathbb{Q}, |.|_p)$. The distance induced by this absolute value is ultrametric, giving to $\mathbb{Q}_p$ very different topological properties then those of the field of real numbers $\mathbb{R}$.

The first objective of this master thesis will be to study the topological properties of $\mathbb{Q}_p$, the notion of p-adic integrals and the notion of p-adic analytic manifolds, see for instance [1], [2].

Then, we will consider a theorem by Igusa [1] (stated here under a weak form)

Theorem. *Let f be a polynomial in $\mathbb{Q}[x_1, .., x_n] \setminus \mathbb{Q}$ and $d\mu$ be an Haar measure over $\mathbb{Q}_p^n$, the* p-adic zeta function

$$\begin{array}{cccc} Z_f & : & \mathbb{C} & \to & \mathbb{C} \\ & & s & \to & \int_{B(0,1)^n} |f(x)|_p^s \, d\mu \end{array}$$

is defined over the open set $\{s \mid Re(s) > 0\}$ and has a meromorphic extension to $\mathbb{C}$ as a rational function.

The main tool of the proof is the theorem of resolution of singularities of Hironaka and the change variable formula for p-adic integrals. This theorem will allow to consider the famous *monodromy conjecture* which relates poles of that rational function to the singularities of f.

At the end, we could consider these questions in the setting of complex formal series $\mathbb{C}[[t]]$, using the theory of *motivic integration* introduced by Kontsevich in the 90's. Unlike $\mathbb{Q}_p$, the ring $\mathbb{C}[[t]]$ which is the t-adic completion of $\mathbb{C}[t]$, is not locally compact and does not have an Haar measure. The motivic integration is then an analogous of the p-adic integration but the values of the integral are not real but *motives*.

Références

[1] Jun-ichi Igusa. *An introduction to the theory of local zeta functions*, volume 14 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI, 2000.

[2] Neal Koblitz. *p-adic numbers, p-adic analysis, and zeta-functions*, volume 58 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1984.