VORONIN's Universality Theorem

ZETAS 2018 Summer School
mini-workshop

The Riemann zeta function $\zeta(s)$ is defined for $\mathrm{Re}(s) > 1$ by

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and is analytically continued to the whole complex plane $\mathbb{C}$ except for a simple pole at $s = 1$.

Theorem (Voronin, '75)

Let $0 < r < 1/4$ and let $f(z)$ be a function which is analytic inside the disc $|z| \leq r$ and continuous up to the boundary of the disc. If $f(z)$ has no zero inside the disc $|z| \leq r$, then for every $\epsilon > 0$ there exists a real number $T = T(\epsilon)$ such that

$$\max_{|z|\leq r}\left\{\left|f(z) - \zeta\left(z + (\frac{3}{4} + iT)\right)\right|\right\} < \epsilon.$$

Voronin added ('75) : the analogous assertion holds for all Dirichlet L-functions.

comments :
... this extraordinary 1975 result received surprinsingly little coverage (it was difficult to find a clear and accurate statement anywhere on the www in March 2004)...

In other words, up to arbitray precision, any function f can be approximated by some vertical translate of the Riemann zeta function.

The “altitude” T exists but its value is noneffective.

Bagchi's Thesis ('81) : ... provided a clear conceptual explanation of this result, as the combination of two independant statements :

- viewing translates of the Riemann zeta function by $t \in [-T, +T]$ as **random variables with values in a space of holomorphic functions on the disc**, Bagchi proves that these random variables converge in law, as $T \to \infty$, to a natural random Dirichlet series, which is also expressed as a **random Euler product**,

- computing the support of the limiting random Dirichlet series, and checking that it contains the space of nowhere vanishing holomorphic functions on the disc, the universality theorem follows easily...

S.M. Voronin, Theorem on the ”Universality” of the Riemann zeta-function, Izv. Akad. Nauk SSSR, Ser. Matem. 39 (1975).

2 books :

- A.A Karatsuba and S.M. Voronin, The Riemann Zeta Function, de Gruyter 1992.
- A. Laurincikas, Limit Theorems for the Riemann Zeta-Function, (1996).

for your own zeta function, L-function, etc : if Euler products exist, trying to find appromate Euler products in the spirit of Bagchi's conceptual approach, and discover a new Universality theorem.

Let λ denote the Lebesgue measure on $\mathbb{R}$.

Theorem (Reich, '77)

Suppose that K is a compact subset of the strip $\{s \in \mathbb{C} : \frac{1}{2} < Re(s) < 1\}$ with a connected complement, $g(s)$ is a continuous function without zeroes in K, and $g(s)$ is analytic inside K. Then for every $\epsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \lambda \left(\sup_{s\in K} |\zeta(s+it) - g(s)| < \epsilon \right) > 0$$

This means that the set of shifts of $\zeta(s)$ that approximate the function $g(s)$ has a positive lower density.

The universality of shifts, in particular of Dirichlet series, has been studied by Gonek ('79), Reich, Bagchi, Laurincikas ('79 -), Matsumoto, Kowalski,...

It turns out that the universality property is possessed by

- Dirichlet L-functions,
- Dedeking zeta functions, by some L-functions related to normed parabolic forms,
- by some more general Dirichlet series.

ex. : Laurincikas, Matsumoto, Steuding '03 : L-functions associated with new forms.

Theorem (Kowalski, '17 - Universality in level aspect)

For q prime ≥ 17, let $S_2(q)^*$ be the nonempty finite set of primitive cusp forms for $\Gamma_0(q)$ with weight 2 and trivial nebentypus. For $f \in S_2(q)^*$, let $L(f,s)$ denote its Hecke L-function

$$L(f,s) = \sum_{n\geq 1} \lambda_f(n) n^{-s},$$

normalized so that the critical line is $Re(s) = 1/2$.
For any real number $r < 1/4$, let D be the open unit disc centered at $3/4$ with radius r. Then for any continuous function $\phi : \overline{D} \to \mathbb{C}$ which is holomorphic and nonvanishing in D and satisfies $\phi(\sigma) > 0$ for $\sigma \cap \mathbb{R}$ we have

$$\liminf_{q\to\infty} \frac{1}{|S_2(q)^*|} |\{f \in S_2(q)^* : \|L(f,\cdot) - \phi\|_\infty < \epsilon\}| > 0$$

for any $\epsilon > 0$, where the L^∞ norm is the norm on $\overline{D}$.

Conjecture (Linnik - Ibragimov)

The universality property is shared by all Dirichlet series which can be analytically continued to the left of their half-plane of absolute convergence.

New effective version :

Y. Lamzouri, S. Lester, M. Radziwill, *An Effective Universality Theorem for the Riemann Zeta Function*